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# TILINGS BY $(0.5, n)$-CROSSES AND PERFECT CODES*

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**Abstract.** The existence question for tiling of the $n$-dimensional Euclidian space by crosses is well known. A few existence and nonexistence results are known in the literature. Of special interest are tilings of the Euclidian space by crosses with arms of length one, also known as Lee spheres with radius one. Such a tiling forms a perfect code. In this paper crosses with arms of length half are considered. These crosses are scaled by two to form a discrete shape. A tiling with this shape is also known as a perfect dominating set. We prove that an integer tiling for such a shape exists if and only if $n = 2^t - 1$ or $n = 3^t - 1$, where $t > 0$. A strong connection of these tilings to binary and ternary perfect codes in the Hamming scheme is shown.

**Key words.** cross, perfect code, perfect dominating set, semicross, tiling

**AMS subject classifications.** 05B07, 11H31, 11H71, 52C22

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**1. Introduction.** Packing and covering are two fundamental concepts in combinatorics. Tiling is a concept which combines both packing and covering and hence it attracts a substantial interest. Tiling of the Euclidian space with specific shapes is one of the main interests in this respect. Two of the shapes in this context are the semicross and the cross. A $(k, n)$-*semicross* is an $n$-dimensional shape whose center is an $n$-dimensional unit cube from which $n$ *arms* consisting of $k$ $n$-dimensional unit cubes are spanned in the $n$ positive directions. A $(k, n)$-*cross* is an $n$-dimensional shape whose center is an $n$-dimensional unit cube from which $2n$ *arms* consisting of $k$ $n$-dimensional unit cubes are spanned in the $n$ directions (one for the positive and one for the negative). Examples of a $(2, 3)$-cross and a $(2, 3)$-semicross are given in Figure 1.1. Packing and tiling with semicrosses and crosses is a well-studied topic (see [20, 21] and references therein).

As mentioned in [21] the origins of the study of the cross and semicross are in several independent sources [8, 11, 17, 25], some of which are pure mathematics and some connected to coding theory. Semicross and cross are two types of "error spheres" as explained in [7]. Golomb and Welch [8] proved that the $(1, n)$-cross tiles the $n$-dimensional Euclidian space for all $n \geq 1$. Such a tiling is a perfect code in the Manhattan metric, and if the tiling is periodic, then it is also a perfect code in the Lee metric. Their work inspired work (see [5] and references therein) on perfect codes in the Lee (and Manhattan) metric.

As said before, packing and tiling with semicrosses and crosses are well-studied topics [3, 6, 8, 9, 10, 13, 17, 18, 19, 22, 23, 24]. The results in these research works include bounds on the size of the arms, constructions for such packings and tilings, parameters for which such tilings cannot exist, and lattice and nonlattice tilings. Recently, the topic has gained new interest since the $(k, n)$-semicross is the error sphere of the *asymmetric error model* associated with flash memories [2, 12], the most advanced type of storage currently used. Schwartz [15] investigated lattice tilings

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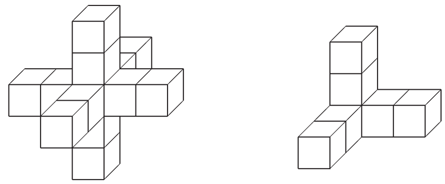

FIG. 1.1. *A* $(2,3)$*-cross and a* $(2,3)$*-semicross.*

with generalized crosses and semicrosses in the connection of an *unbalanced limited magnitude error model* for multilevel flash memories.

Not much is known about tiling of crosses with arms which are not of integer length. Moreover, most tilings considered in the literature are integer lattice tilings. In this paper we study the existence of tiling of the $n$-dimensional Euclidian space with a $(0.5,n)$-cross. The $(0.5,n)$-cross consists of one complete (nonfractional) unit cube and $2n$ halves unit cubes. Usually, it is more convenient to handle tiling with complete unit cubes. Hence, we scale the $(0.5,n)$-cross by two to obtain a new shape, which will be denoted in the sequel by $\Upsilon_n$. The shape $\Upsilon_n$ consists of $2^n(n+1)$ complete unit cubes. For the shape $\Upsilon_n$ we will discuss only integer tiling (also known as $\mathbb{Z}$-tiling), which is a tiling in which the centers of the unit cubes are placed on points of $\mathbb{Z}^n$. We prove that such a tiling exists if and only if $n = 2^t - 1$ or $n = 3^t - 1$, where $t > 0$. The related tiling with a $(0.5,n)$-cross (obtained after scaling by 0.5) will be called a $\frac{1}{2}\mathbb{Z}$-tiling. We present an analysis of the structure obtained from such a tiling. The tiling which is considered for the $(0.5,n)$-cross is usually not an integer tiling. Moreover, we discuss general tilings, and not just lattice tilings, as done in most literature.

Dejter [4] has brought to our attention that a tiling with $\Upsilon_n$ is a perfect dominating set in $\mathbb{Z}^n$. This problem was considered by several authors, e.g., [1, 26] and references therein. The problem that we consider in this paper is one of the main open problems on this topic.

The rest of this paper is organized as follows. In section 2 we present the basic concepts used throughout this paper. We define a tiling, a lattice tiling, an integer tiling, and a periodic tiling. We discuss how to handle a tiling with a $(0.5,n)$-cross. We discuss three distance measures which are used in our discussion: the well-known Hamming distance, the Manhattan distance which is used for codes in $\mathbb{Z}^n$, and a new distance measure needed for the $(0.5,n)$-crosses, the cross distance. We also discuss how a tiling with a $(0.5,n)$-cross can be analyzed. In section 3 we make an analysis of a tiling with a $(0.5,n)$-cross and prove that such a $\frac{1}{2}\mathbb{Z}$-tiling can exist only if $n = 2^t - 1$ or $n = 3^t - 1$, where $t > 0$. Some necessary conditions for the existence of such a tiling are given. In section 4 we show how we can construct a tiling with a $(0.5,n)$-cross from a binary perfect single-error-correcting code of length $n = 2^t - 1$ and vice versa. Finally, we show how to construct a tiling with a $(0.5,n)$-cross from a ternary perfect single-error-correcting code of length $\frac{n}{2} = \frac{3^t-1}{2}$.

**2. Basic concepts.** For two vectors $X = (x_1, x_2, \ldots, x_n), Y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$ and a scalar $\alpha \in \mathbb{R}$, the *vector addition* $X+Y$ is defined by $X+Y \stackrel{\text{def}}{=} (x_1+y_1, x_2+y_2, \ldots, x_n+y_n)$ and the *scalar multiplication* is defined by $\alpha X \stackrel{\text{def}}{=} (\alpha x_1, \alpha x_2, \ldots, \alpha x_n)$. For two sets $\mathcal{S}_1 \subseteq \mathbb{R}^n$ and $\mathcal{S}_2 \subseteq \mathbb{R}^n$ the *set addition* $\mathcal{S}_1 + \mathcal{S}_2$ is defined by $\mathcal{S}_1 + \mathcal{S}_2 \stackrel{\text{def}}{=} \{X + Y \ : \ X \in \mathcal{S}_1,\ Y \in \mathcal{S}_2\}$. For a set $\mathcal{S} \in \mathbb{R}^n$ and a vector $U \in \mathbb{R}^n$ the *translation* of $\mathcal{S}$ by $U$ is $U + \mathcal{S} \stackrel{\text{def}}{=} \{U + X \ : \ X \in \mathcal{S}\}$. The *multiplication* of $\mathcal{S}$ by a scalar $\alpha \in \mathbb{R}$ is defined by $\alpha\mathcal{S} \stackrel{\text{def}}{=} \{\alpha X \ : \ X \in \mathcal{S}\}$.

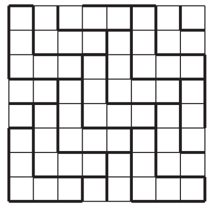

FIG. 2.1. *Tiling of* $\mathbb{R}^2$ *with a* $(2,2)$*-semicross.*

Let $\mathcal{S}$ be an $n$-dimensional shape in the $n$-dimensional Euclidian space ($\mathbb{R}^n$). We say that two translations of $\mathcal{S}$, $\mathcal{S}_1$ and $\mathcal{S}_2$, are *disjoint* if their intersection is contained in at most an $(n-1)$-dimensional space. A *tiling* $\mathcal{T}$ of the $n$-dimensional Euclidian space with the shape $\mathcal{S}$ is a set of disjoint translations of $\mathcal{S}$ such that each point $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ is contained in at least one translation of $\mathcal{S}$. In what follows a tiling $\mathcal{T}$ will be defined by a set of points $\mathbb{T}$ in $\mathbb{R}^n$ and a shape $\mathcal{S}$. The point $X \in \mathbb{T}$ if and only if the translation $X + \mathcal{S} \in \mathcal{T}$. Henceforth, $\mathbb{T}$ will be called a tiling if the shape $\mathcal{S}$ is known. For example, the set of points $\mathbb{T} = \{(i, i+5j) \ : \ i, j \in \mathbb{Z}\}$ and the $(2,2)$-semicross define a tiling of $\mathbb{R}^2$. (Part of the tiling is presented in Figure 2.1.) A tiling $\mathbb{T}$ with a shape $\mathcal{S}$ is called an *integer tiling* (also a $\mathbb{Z}$*-tiling*) if $\mathbb{T} \subseteq \mathbb{Z}^n$. An $n$-dimensional *unit cube* centered at a point $(c_1, c_2, \ldots, c_n) \in \mathbb{R}^n$ consists of the points $\{(x_1, x_2, \ldots, x_n) \ : \ |x_i - c_i| \leq 0.5, \ 1 \leq i \leq n\}$. The $n$-dimensional shape $\mathcal{S}$ is a *discrete shape* if $\mathcal{S}$ is a union of $n$-dimensional unit cubes, whose centers are in $\mathbb{Z}^n$. Hence, a discrete $n$-dimensional shape $\mathcal{S}$ can be defined by a set of points from $\mathbb{Z}^n$. Therefore, in an integer tiling with a discrete shape $\mathcal{S}$, each point of $\mathbb{Z}^n$ is contained in exactly one translation of $\mathcal{S}$.

LEMMA 2.1. *If* $\mathbb{T}$ *is a tiling with a shape* $\mathcal{S}$ *and* $U \in \mathbb{R}^n$*, then* $U + \mathbb{T}$ *is also a tiling with* $\mathcal{S}$*.*

By Lemma 2.1 we can assume that the origin, denoted by $\mathbf{0}$, is always a point in the tiling. Therefore, given a tiling $\mathbb{T}$ we assume without loss of generality that the origin is an element in $\mathbb{T}$. For a set $\mathcal{S} \subseteq \mathbb{R}^n$ and a permutation $\sigma$ of $\{1, 2, \ldots, n\}$, let $\sigma(\mathcal{S}) \stackrel{\text{def}}{=} \{\sigma(X) \ : \ X \in \mathcal{S}\}$.

LEMMA 2.2. *If* $\mathbb{T}$ *is a tiling with an* $n$*-dimensional shape* $\mathcal{S}$ *and* $\sigma$ *is a permutation of* $\{1, 2, \ldots, n\}$*, then* $\sigma(\mathbb{T})$ *is a tiling with the* $n$*-dimensional shape* $\sigma(\mathcal{S})$*.*

The vector $(x_1, x_2, \ldots, x_n)$ is called the $r$th unit vector and will be denoted by $e_r$ if $x_r = 1$ and for all $i \neq r$, $x_i = 0$. A set $\mathcal{S}$ is called *periodic* with period $p$ if $X \in \mathcal{S}$ implies that $X + \alpha \cdot p \cdot e_i \in \mathcal{S}$ for all $\alpha \in \mathbb{Z}$ and $1 \leq i \leq n$. A tiling $\mathbb{T}$ with the shape $\mathcal{S}$ is a periodic tiling if it is a periodic set. The following simple lemma is left for the reader.

LEMMA 2.3. *A tiling* $\mathbb{T}$ *is a periodic with period* $p$ *if and only if* $X \in \mathbb{T}$ *implies that* $X + p \cdot e_i \in \mathbb{T}$ *for all* $i$*,* $1 \leq i \leq n$*.*

A *lattice* $\Lambda$ is a discrete, additive subgroup of the real $n$-space $\mathbb{R}^n$,

$$\Lambda \stackrel{\text{def}}{=} \{u_1 v_1 + u_2 v_2 + \cdots + u_n v_n \ : \ u_1, u_2, \ldots, u_n \in \mathbb{Z}\},$$

where $\{v_1, v_2, \ldots, v_n\}$ is a set of linearly independent vectors in $\mathbb{R}^n$, i.e., the lattice has rank $n$. The set of vectors $\{v_1, v_2, \ldots, v_n\}$ is called *the basis* for $\Lambda$, and the $n \times n$ matrix

$$\mathbf{G} \stackrel{\text{def}}{=} \begin{bmatrix} v_{11} & v_{12} & \dots & v_{1n} \\ v_{21} & v_{22} & \dots & v_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ v_{n1} & v_{n2} & \dots & v_{nn} \end{bmatrix}$$

having these vectors as its rows is said to be the *generator matrix* for $\Lambda$.

The *volume* of a lattice $\Lambda$, denoted by $V(\Lambda)$, is inversely proportional to the number of lattice points per a unit volume. More precisely, $V(\Lambda)$ may be defined as the volume of the *fundamental parallelogram* $\Pi(\Lambda)$, which is given by

$$\Pi(\Lambda) \stackrel{\text{def}}{=} \{\xi_1 v_1 + \xi_2 v_2 + \cdots + \xi_n v_n \ : \ 0 \leq \xi_i < 1, \ 1 \leq i \leq n\} \ .$$

There is a simple expression for the volume of $\Lambda$, namely, $V(\Lambda) = |\det \mathbf{G}|$.

A lattice $\Lambda$ is a *lattice tiling* with $\mathcal{S}$ if $\mathbb{T} \stackrel{\text{def}}{=} \Lambda$ forms a tiling with $\mathcal{S}$. A lattice tiling $\Lambda$ is an *integer lattice tiling* with $\mathcal{S}$ if all entries of $\mathbf{G}$ are integers. The following lemma is well known.

LEMMA 2.4. *If $\Lambda$ defines a lattice tiling with the shape $\mathcal{S}$, then $V(\Lambda) = |\mathcal{S}|$, where $|\mathcal{S}|$ denote the volume of $\mathcal{S}$.*

A *code* $\mathcal{C}$ of length $n$ over $\mathbb{Z}_q$ (respectively, $\mathbb{Z}$) is a subset of $\mathbb{Z}_q^n$ (respectively, $\mathbb{Z}^n$). Let $\Lambda_n$ be the lattice generated by the basis $\{q \cdot e_i \ : \ 1 \leq i \leq n\}$. A code $\mathcal{C} \subseteq \mathbb{Z}_q^n$ can be viewed also as a subset of $\mathbb{Z}^n$. The code $E(\mathcal{C}) = \mathcal{C} + \Lambda_n$ is the *expanded code* of $\mathcal{C}$. If $E(\mathcal{C})$ is a tiling of $\mathbb{Z}^n$ with the shape $\mathcal{S}$, then we also call $\mathcal{C}$ a tiling of $\mathbb{Z}_q^n$ with the shape $\mathcal{S}$. A tiling $\mathbb{T} \subseteq \mathbb{Z}^n$ with a period $p$ can be viewed as an expanded code $E(\mathcal{C})$ of a code $\mathcal{C}$ of length $n$ over $\mathbb{Z}_p$, where $\mathcal{C} = \mathbb{T} \cap \{0, 1, \dots, p-1\}^n$. In what follows we denote the set of integers in $\mathbb{Z}_p$ without the group structure by $\tilde{\mathbb{Z}}_p \stackrel{\text{def}}{=} \{0, 1, \dots, p-1\}$; we will also refer to $\mathbb{T}$ as a code and to its elements as codewords.

To handle a tiling with a $(0.5, n)$-cross we will need to use three distance measures, the well-known Hamming distance, the Manhattan distance, and the new defined cross distance.

For every two given words $X, Y \in \mathbb{Z}_q^n$ the *Hamming distance* $d_H(X, Y)$ is the number of positions in which $X$ and $Y$ differ, i.e.,

$$d_H(X, Y) \stackrel{\text{def}}{=} |\{i \ : \ x_i \neq y_i, \ 1 \leq i \leq n\}| \ .$$

For every two given points $X, Y \in \mathbb{Z}^n$ the *Manhattan distance* $d_M(X, Y)$ is defined by

$$d_M(X, Y) \stackrel{\text{def}}{=} \sum_{i=1}^{n} |x_i - y_i| \ .$$

For every two given points $X, Y \in \mathbb{Z}^n$ we defined the *cross distance* $d_C(X, Y)$ as follows:

$$d_C(X, Y) \stackrel{\text{def}}{=} \sum_{i=1}^{n} \max\{0, |y_i - x_i| - 1\}.$$

*Remark* 1. The cross distance can be generalized for two points $X, Y \in \mathbb{R}^n$. We will use this generalization only in this section.

*Remark* 2. The Hamming distance is an association scheme, while the Manhattan distance is only a metric distance and not an association scheme. (See [14] for the

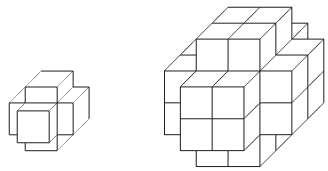

FIG. 2.2. *A* $(0.5, 3)$*-cross and an* $\Upsilon_3$.

definition of an association scheme.) The cross distance is not a metric, but it will be most important in the discussion of tilings with a $(0.5, n)$-cross.

For each distance measure we defined the *weight* of a point (word) $X$ to be the distance between $X$ and the point $\mathbf{0}$. The cross weight of a point $X$ will be denoted by $w_C(X)$.

A unit cube centered at $(c_1, c_2, \ldots, c_n) \in \mathbb{R}^n$ is a union of two disjoint half unit cubes in one of the $n$ directions. For the $r$th direction one *half unit cube* is defined by the set of points $\{(x_1, x_2, \ldots, x_n) \;:\; 0 \leq x_r - c_r \leq 0.5,\ |x_i - c_i| \leq 0.5,\ 1 \leq i \leq n,\ i \neq r\}$ and a second *half unit cube* is defined by the set of points $\{(x_1, x_2, \ldots, x_n) \;:\; -0.5 \leq x_r - c_r \leq 0,\ |x_i - c_i| \leq 0.5,\ 1 \leq i \leq n,\ i \neq r\}$. A $(0.5, n)$-cross is a unit cube to which two half unit cubes are attached in the $r$th direction for each $1 \leq r \leq n$, one in its negative direction and one in its positive direction. It is more convenient to handle shapes with complete unit cubes (discrete shapes) and therefore we will scale the $(0.5, n)$-cross by two to obtain a new shape which will be called $\Upsilon_n$. An example of a $(0.5, 3)$-cross and an $\Upsilon_3$ is given in Figure 2.2. The complete unit cube in the $(0.5, n)$-cross is transferred into an $n$-dimensional cube with sides of length two in $\Upsilon_n$. This cube in $\Upsilon_n$ will be called the *core* of $\Upsilon_n$; the core consists of $2^n$ unit cubes. In what follows we will be interested only in integer tilings with $\Upsilon_n$. In such an integer tiling $\Upsilon_n$ can be represented by $2^n(n+1)$ points of $\mathbb{Z}^n$ which are the centers of its $2^n(n+1)$ unit cubes. Let $\mathbb{T}$ be a tiling with $\Upsilon_n$. We assume that if $X = (x_1, x_2, \ldots, x_n) \in \mathbb{T}$, then the set $\{(c_1, c_2, \ldots, c_n) \;:\; c_i \in \{x_i - 1, x_i\},\ 1 \leq i \leq n\}$ is the related core of the translation $X + \Upsilon_n$. The core of $\Upsilon_n$ is $\{-1, 0\}^n$ and $\Upsilon_n \stackrel{\text{def}}{=} \{U \;:\; d_M(X, U) = 1,\ X \in \{-1, 0\}^n\}$. Even so we represent $\Upsilon_n$ as a set of points in $\mathbb{Z}^n$ the integer tiling which we discussed is also for the shape in the Euclidian space. If $\mathbb{T}$ is a tiling with $\Upsilon_n$, then $0.5\mathbb{T}$ is a tiling with a $(0.5, n)$-cross. Clearly, if for each $(x_1, x_2, \ldots, x_n) \in \mathbb{T}$, $x_i$ is even for all $1 \leq i \leq n$, then also $0.5\mathbb{T}$ is an integer tiling. However, if there exists a point $(x_1, x_2, \ldots, x_n) \in \mathbb{T}$ such that for at least one $j$ we have that $x_j$ is odd, then $0.5\mathbb{T}$ is not an integer tiling. To this end we define a $\frac{1}{2}\mathbb{Z}$-tiling. A tiling $\mathbb{T}$ is a $\frac{1}{2}\mathbb{Z}$*-tiling* if $\mathbb{T} \subseteq 0.5\mathbb{Z}^n$.

LEMMA 2.5. *The tiling* $\mathbb{T}$ *is an integer tiling with* $\Upsilon_n$ *if and only if* $0.5\mathbb{T}$ *is a* $\frac{1}{2}\mathbb{Z}$*-tiling with a* $(0.5, n)$*-cross.*

Given a set $\mathbb{T} \subset \mathbb{Z}^n$, we would like to know whether $\mathbb{T}$ is a tiling with $\Upsilon_n$. To show that $\mathbb{T}$ is a tiling we have to prove the following:

($\mathcal{P}$.1) For each point $Y \in \mathbb{Z}^n$ there exists a translation $\mathcal{S}_1$ of $\Upsilon_n$ in the tiling such that $\mathcal{S}_1$ contains $Y$.

($\mathcal{P}$.2) A point $Y \in \mathbb{Z}^n$ is not contained in more than one translation of $\Upsilon_n$ in the tiling, i.e., for each two translations $\mathcal{S}_1$ and $\mathcal{S}_2$ of $\Upsilon_n$ in the tiling we have $\mathcal{S}_1 \cap \mathcal{S}_2 = \varnothing$.

A set $\mathbb{T} \subset \mathbb{Z}^n$ is a *covering* with $\Upsilon_n$ if it satisfies property ($\mathcal{P}$.1) and it is a *packing* with $\Upsilon_n$ if it satisfies property ($\mathcal{P}$.2). A tiling is clearly both a covering and a packing.

The following two lemmas are immediate results from the definition of $\Upsilon_n$.

LEMMA 2.6. *If $\mathcal{S}$ is a translation of $\Upsilon_n$ and $X \in \mathcal{S}$ is not a core point of $\mathcal{S}$, then there exists a core point $Y \in \mathcal{S}$ such that $d_M(X,Y) = 1$.*

LEMMA 2.7. *If $\mathcal{S}_1$ and $\mathcal{S}_2$ are two translations of $\Upsilon_n$ for which $\mathcal{S}_1 \cap \mathcal{S}_2 \neq \varnothing$, then there exists a point $X \in \mathcal{S}_1 \cap \mathcal{S}_2$ which is not in the core of $\mathcal{S}_1$.*

COROLLARY 2.8. *If $\mathcal{S}_1$ and $\mathcal{S}_2$ are two translations of $\Upsilon_n$ for which $\mathcal{S}_1 \cap \mathcal{S}_2 \neq \varnothing$, then there exist two core points $X_1 \in \mathcal{S}_1$ and $X_2 \in \mathcal{S}_2$ such that $d_M(X_1, X_2) \leq 2$.*

LEMMA 2.9. *If $\mathcal{S}_1$ and $\mathcal{S}_2$ are two translations of $\Upsilon_n$ for which there exist two core points $X_1 \in \mathcal{S}_1$ and $X_2 \in \mathcal{S}_2$ such that $d_M(X_1, X_2) \leq 2$, then $\mathcal{S}_1 \cap \mathcal{S}_2 \neq \varnothing$.*

*Proof.* If $d_M(X_1, X_2) \leq 2$, then there exists a point $Y \in \mathbb{Z}^n$ such that $d_M(X_1, Y) \leq 1$ and $d_M(X_2, Y) \leq 1$. By definition $Y \in \mathcal{S}_1 \cap \mathcal{S}_2$. ◻

COROLLARY 2.10. *Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be two translations of $\Upsilon_n$. Then $\mathcal{S}_1 \cap \mathcal{S}_2 = \varnothing$ if and only if for every two core points $X_1 \in \mathcal{S}_1$ and $X_2 \in \mathcal{S}_2$ we have $d_M(X_1, X_2) \geq 3$.*

THEOREM 2.11. *Let $\mathcal{S}_1 = X + \Upsilon_n$ and $\mathcal{S}_2 = Y + \Upsilon_n$, where $X, Y \in \mathbb{Z}^n$, be two translations of $\Upsilon_n$. Then $\mathcal{S}_1 \cap \mathcal{S}_2 = \varnothing$ if and only if $d_C(X,Y) \geq 3$.*

*Proof.* Let $\tilde{X} = (\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_n)$ and $\tilde{Y} = (\tilde{y}_1, \tilde{y}_2, \ldots, \tilde{y}_n)$ be the centers of mass of $\mathcal{S}_1$ and $\mathcal{S}_2$, respectively. Clearly, $\tilde{X}$ and $\tilde{Y}$ are in $(0.5, 0.5, \ldots, 0.5) + \mathbb{Z}^n$. The core points of $\mathcal{S}_1$ are $\{(c_1, c_2, \ldots, c_n) \colon c_i \in \{\tilde{x}_i - 0.5, \tilde{x}_i + 0.5\}\}$ and the core points of $\mathcal{S}_2$ are $\{(c_1, c_2, \ldots, c_n) \colon c_i \in \{\tilde{y}_i - 0.5, \tilde{y}_i + 0.5\}\}$. Let $X' = (x'_1, x'_2, \ldots, x'_n)$ and $Y' = (y'_1, y'_2, \ldots, y'_n)$ be the two core points of $\mathcal{S}_1$ and $\mathcal{S}_2$, respectively, defined as follows. If $\tilde{x}_i = \tilde{y}_i$, then $x'_i \overset{\text{def}}{=} \tilde{x}_i + 0.5$ and $y'_i \overset{\text{def}}{=} \tilde{y}_i + 0.5$. If $\tilde{x}_i < \tilde{y}_i$, then $x'_i \overset{\text{def}}{=} \tilde{x}_i + 0.5$ and $y'_i \overset{\text{def}}{=} \tilde{y}_i - 0.5$. If $\tilde{x}_i > \tilde{y}_i$, then $x'_i \overset{\text{def}}{=} \tilde{x}_i - 0.5$ and $y'_i \overset{\text{def}}{=} \tilde{y}_i + 0.5$. Clearly, $d_C(X,Y) = d_C(\tilde{X}, \tilde{Y}) = d_M(X', Y')$ and for any two core points $\hat{X} \in \mathcal{S}_1$ and $\hat{Y} \in \mathcal{S}_2$ we have that $d_M(\hat{X}, \hat{Y}) \geq d_M(X', Y')$. Now, by Corollary 2.10 we have that $\mathcal{S}_1 \cap \mathcal{S}_2 = \varnothing$ if and only if $d_C(X,Y) \geq 3$. ◻

COROLLARY 2.12. *The set $\mathbb{T}$ induces a packing of the $n$-dimensional Euclidian space with $\Upsilon_n$ if and only if for every two elements $X, Y \in \mathbb{T}$, we have $d_C(X,Y) \geq 3$.*

To prove that a set is a tiling with $\Upsilon_n$ we will have to show that it satisfies properties ($\mathcal{P}$.1) and ($\mathcal{P}$.2). For this purpose we will have to show that each point of $\mathbb{Z}^n$ is contained (covered) in exactly one translation $\mathcal{S}$ of $\Upsilon_n$ in the tiling. A point $A \in \mathbb{Z}^n$ is *covered* by a codeword $X$ in a tiling $\mathbb{T}$ if $A$ is contained in the translation $X + \Upsilon_n$. In this case we say that $X$ *covers* $A$.

Given a tiling $\mathbb{T}$ with $\Upsilon_n$ it has to satisfy properties ($\mathcal{P}$.1) and ($\mathcal{P}$.2). By considering how each point $A \in \mathbb{Z}^n$ is covered by a codeword $X \in \mathbb{T}$ (property ($\mathcal{P}$.1)) we will discover the structure of $\mathbb{T}$. To this end we will also use property ($\mathcal{P}$.2), i.e., for each two codewords $X, Y \in \mathbb{T}$ we have that $d_C(X,Y) \geq 3$ (by Corollary 2.12).

**3. The nonexistence of other integer tilings.** In this section we will prove that an integer tiling $\mathbb{T}$ with $\Upsilon_n$ exists only if $n = 2^t - 1$ or $n = 3^t - 1$ for some $t > 0$. In subsection 3.1 we prove this claim for odd $n$ if $\mathbb{T}$ is an integer tiling and for all $n$ if $\mathbb{T}$ is a lattice tiling. In subsection 3.2 we complete the proof for even $n$. We will obtain this goal by proving that given a tiling $\mathbb{T}$ with $\Upsilon_n$, certain elements of $\mathbb{Z}^n$ must be contained in $\mathbb{T}$. It will be proved by considering how elements with a small cross weight are covered. For the rest of this section let $\mathbb{T}$ be a tiling with $\Upsilon_n$. We recall that without loss of generality we assumed that $\mathbf{0} \in \mathbb{T}$ and hence by Corollary 2.12, if $X, Y \in \mathbb{T} \setminus \{\mathbf{0}\}$, where $X \neq Y$, then $w_C(X) \geq 3$, $w_C(Y) \geq 3$, and $d_C(X,Y) \geq 3$.

**3.1. Tiling for odd $n$ and lattice tiling.** In this subsection we first prove that for every $r$, $1 \leq r \leq n$, the point $2e_r$ is covered by either $4e_r$ or $3e_r + 2e_s$ for some $s \neq r$. Then we prove that if $3e_r + 2e_s$ is a codeword, then $n$ is even, which will imply that if $n$ is odd, then $n = 2^t - 1$ for some $t > 0$. Finally, we prove that two codewords of the form $3e_r + 2e_s$ are disjoint, i.e., have a disjoint set of nonzero coordinates. This will lead to the main result only for a lattice tiling. The first lemma is an immediate result from the definition of $\Upsilon_n$.

LEMMA 3.1. *Let $X \in \mathbb{T}$ and $A = (a_1, a_2, \ldots, a_n) \in \mathbb{Z}^n$. The point $A$ is covered by $X$ if and only if $x_i \in \{a_i - 1, a_i, a_i + 1, a_i + 2\}$, for $1 \le i \le n$, and for at most one $i$ we have $x_i \in \{a_i - 1, a_i + 2\}$.*

Let $\mathcal{D}_1$ be the set of points from $\{0, 1, 2, 3\}^n$ in which 2 and 3 appear exactly once.

LEMMA 3.2. *If $X \in \mathcal{D}_1 \cap \mathbb{T}$, then $X = 3e_r + 2e_s$ for some $r \neq s$.*

*Proof.* Assume without loss of generality that $X = (3, 2, 1, x_4, \ldots, x_n)$, where $x_i \in \{0, 1\}$, for $4 \le i \le n$. The point $A = (1, 1, -1, 0, \ldots, 0)$ is covered by a codeword $Y \in \mathbb{T}$. By Lemma 3.1 we have that $Y \notin \{X, \mathbf{0}\}$ and we can distinguish between three cases:

*Case* 1. If $y_i \in \{a_i, a_i + 1\}$ for all $i$, $1 \le i \le n$, then $w_C(Y) \le 2$, a contradiction.

*Case* 2. There exists a $j$ such that $y_j = a_j - 1$ and $y_i \in \{a_i, a_i + 1\}$ for all $i \neq j$. Since $w_C(Y) \ge 3$ it follows that $j = 3$ and hence $Y = (2, 2, -2, y_4, \ldots, y_n)$, where $y_i \in \{0, 1\}$, for $4 \le i \le n$. This implies that $d_C(X, Y) = 2$, a contradiction.

*Case* 3. There exists a $j$ such that $y_j = a_j + 2$ and $y_i \in \{a_i, a_i + 1\}$ for all $i \neq j$.

Since $w_C(Y) \ge 3$ it follows that $j \neq 3$. Without loss of generality it implies that $Y$ can take one of the following forms:

- $Y = (3, 2, y_3, y_4, \ldots, y_n)$ or $Y = (2, 3, y_3, y_4, \ldots, y_n)$, where $y_3 \in \{-1, 0\}$ and $y_i \in \{0, 1\}$, for $4 \le i \le n$.
- $Y = (2, 2, y_3, 2, y_5, \ldots, y_n)$, where $y_3 \in \{-1, 0\}$ and $y_i \in \{0, 1\}$, for $5 \le i \le n$.

Both forms imply that $d_C(X, Y) \le 2$, a contradiction.

Therefore, there is no codeword $Y \in \mathbb{T}$ which covers $A$, a contradiction. Thus, if $X \in \mathcal{D}_1 \cap \mathbb{T}$, then $X = 3e_r + 2e_s$ for some $r \neq s$. ∎

Let $\mathcal{D}_2$ be the set of points from $\{0, 1, 4\}^n$ in which 4 appears exactly once.

LEMMA 3.3. *If $X \in \mathcal{D}_2 \cap \mathbb{T}$, then $X = 4e_r$ for some $1 \le r \le n$.*

*Proof.* Assume without loss of generality that $X = (4, 1, x_3, \ldots, x_n)$, where $x_i \in \{0, 1\}$, for $3 \le i \le n$. The point $A = (1, 1, 0, \ldots, 0)$ is covered by a codeword $Y \in \mathbb{T}$. By Lemma 3.1 we have that $Y \notin \{X, \mathbf{0}\}$ and we can distinguish between two cases:

*Case* 1. If $y_i \in \{a_i, a_i + 1\}$ for all $i$, $1 \le i \le n$, with a possible exception for at most one $j$, for which $y_j = a_j - 1$, then $w_C(Y) \le 2$, a contradiction.

*Case* 2. There exists a $j$ such that $y_j = a_j + 2$ and $y_i \in \{a_i, a_i + 1\}$ for all $i \neq j$. Without loss of generality it implies that $Y$ can take one of the following forms:

- $Y = (3, 2, y_3, \ldots, y_n)$ or $Y = (2, 3, y_3, \ldots, y_n)$, where $y_i \in \{0, 1\}$ for $3 \le i \le n$.
- $Y = (2, 2, 2, y_4, \ldots, y_n)$, where $y_i \in \{0, 1\}$ for $4 \le i \le n$.

Hence, $d_C(X, Y) \le 2$, a contradiction.

Therefore, there is no codeword $Y \in \mathbb{T}$ which covers $A$, a contradiction. Thus, if $X \in \mathcal{D}_2 \cap \mathbb{T}$, then $X = 4e_r$ for some $1 \le r \le n$. ∎

COROLLARY 3.4. *For each $r$, $1 \le r \le n$, the point $2e_r$ is covered by a codeword $X \in \mathbb{T}$, where either $X = 4e_r$ or $X = 3e_r + 2e_s$ for some $s \neq r$.*

*Proof.* By Lemma 3.1, $X$ is not the all-zero codeword. Moreover, since $w_C(X) \ge 3$ it can be easily verified that either $X \in \mathcal{D}_1$ or $X \in \mathcal{D}_2$. It follows from Lemmas 3.2 and 3.3 that either $X = 4e_r$ or $X = 3e_r + 2e_s$ for some $s \neq r$. ∎

Let $\mathcal{D}_3$ be the set of points from $\{0, 1, 2\}^n$ in which 2 appears exactly three times.

LEMMA 3.5. *If $X = 3e_r + 2e_s \in \mathbb{T}$, then for every $k \notin \{r, s\}$ there exists a unique $j \notin \{r, s, k\}$ and a codeword $Y \in \mathcal{D}_3 \cap \mathbb{T}$ such that $y_r = 1, y_s = y_k = y_j = 2$.*

*Proof.* Let $k \notin \{r, s\}$ and consider the point $A = e_r + e_s + e_k$. Without loss of generality we assume that $r = 1$, $s = 2$, and $k = 3$, i.e., $X = (3, 2, 0, \ldots, 0)$ and $A = (1, 1, 1, 0 \ldots, 0)$. The point $A$ is covered by a codeword $Y \in \mathbb{T}$. By Lemma 3.1 we have that $Y \notin \{X, \mathbf{0}\}$ and we can distinguish between three cases:

*Case* 1. If $y_i \in \{a_i, a_i + 1\}$ for $1 \leq i \leq n$, then since $w_C(Y) \geq 3$ it follows that $Y = (2, 2, 2, y_4, \ldots, y_n)$, where $y_i \in \{0, 1\}$, for $4 \leq i \leq n$. Hence, $d_C(X, Y) = 1$, a contradiction.

*Case* 2. There exists a $j$ such that $y_j = a_j - 1$ and $y_i \in \{a_i, a_i + 1\}$ for all $i \neq j$. If $j \leq 3$, then $w_C(Y) \leq 2$, a contradiction. If $j > 3$, then since $w_C(Y) \geq 3$ it follows that $Y = (2, 2, 2, y_4, \ldots, y_n)$, where $y_i \in \{-1, 0, 1\}$ for $4 \leq i \leq n$, and hence $d_C(X, Y) = 1$, a contradiction.

*Case* 3. There exists a $j$ such that $y_j = a_j + 2$ and $y_i \in \{a_i, a_i + 1\}$ for all $i \neq j$. If $j \leq 3$, then since $w_C(Y) \geq 3$ and $d_C(X, Y) \geq 3$ it follows that $Y = (1, 2, 3, y_4, \ldots, y_n)$, where $y_i \in \{0, 1\}$, for $4 \leq i \leq n$, a contradiction to Lemma 3.2.

Therefore, there exists a $j > 3$ such that $y_j = a_j + 2$ and $y_i \in \{a_i, a_i + 1\}$ for all $i \neq j$. Without loss of generality we assume that $j = 4$. Since $w_C(Y) \geq 3$ and $d_C(X, Y) \geq 3$ it follows that $Y = (1, 2, 2, 2, y_5, \ldots, y_n)$, where $y_i \in \{0, 1\}$, for $5 \leq i \leq n$. The uniqueness of $j$ follows from the fact that if there exists another $j$ and a related codeword $Y'$, then $d_C(Y, Y') \leq 2$. ∎

Corollary 3.6. *If $3e_r + 2e_s \in \mathbb{T}$, then $n$ is even.*

*Proof.* By Lemma 3.5 all coordinates except for $r$ and $s$ should be paired in disjoint pairs. (Such a pair $\{k, j\}$ induces a codeword of the form $Y = (y_1, y_2, \ldots, y_n) \in \mathcal{D}_3 \cap \mathbb{T}$, where $y_r = 1, y_s = y_k = y_j = 2$.) Thus, $n$ is even. ∎

From Corollaries 3.4 and 3.6 we infer the following.

Corollary 3.7. *If $n$ is odd, then for all $X \in \mathbb{T}$ and $1 \leq r \leq n$ we have $X + 4e_r \in \mathbb{T}$, i.e., $\mathbb{T}$ is a periodic tiling with period 4.*

Theorem 3.8. *If $\mathbb{T}$ is an integer tiling with $\Upsilon_n$, where $n$ is an odd integer, then $n = 2^t - 1$ for some $t > 0$.*

*Proof.* By Corollary 3.7 we have that $\mathbb{T}$ is a periodic tiling with period 4. Therefore, the size of $\Upsilon_n$ divides $4^n$. The size of $\Upsilon_n$ is $2^n(n + 1)$ and hence $n = 2^t - 1$ for some $t > 0$. ∎

Lemma 3.9. *If there exist two distinct codewords $X = 3e_i + 2e_j$ and $X' = 3e_r + 2e_s$ in $\mathbb{T}$, then $\{i, j\} \cap \{r, s\} = \varnothing$.*

*Proof.* Without loss of generality we assume that $i = 1$ and $j = 2$. Since $d_C(X, X') \geq 3$ it follows that $r \neq 1$ and $X' \neq 3e_2 + 2e_1$. If $r = 2$ or $s = 2$, then without loss of generality we assume that $X' = (0, 3, 2, 0, \ldots, 0)$ or $X' = (0, 2, 3, 0, \ldots, 0)$. By Lemma 3.5 we have a codeword $Y = (1, 2, 2, y_4, \ldots, y_n) \in \mathcal{D}_3 \cap \mathbb{T}$. It implies that $d_C(X', Y) = 1$, a contradiction. The case where $s = 1$ and $r > 2$ is symmetric to the case where $r = 2$ and $s > 2$. ∎

From Corollary 3.4 and Lemma 3.9 we have the next corollary.

Corollary 3.10. *If $3e_r + 2e_s \in \mathbb{T}$, then $4e_s \in \mathbb{T}$.*

Theorem 3.11. *If $\mathbb{T}$ is an integer lattice tiling with $\Upsilon_n$, then either $n = 2^t - 1$ or $n = 3^t - 1$ for some $t > 0$.*

*Proof.* Assume that there are exactly $k$ codewords of the form $3e_i + 2e_j$ in $\mathbb{T}$. From Corollaries 3.4 and 3.10 and by Lemma 3.9 the lattice $\mathbb{T}$ contains a sublattice defined by these $k$ codewords and $n - k$ codewords of the form $4e_s$. The generator matrix of this sublattice is a block-diagonal matrix with $k$ $2 \times 2$ blocks of the form $\begin{bmatrix} 3 & 2 \\ 0 & 4 \end{bmatrix}$ and $n - 2k$ $1 \times 1$ blocks of the form $[\ 4\ ]$. The volume of this sublattice is divided by the volume of the lattice $\mathbb{T}$. The volume of the sublattice is $3^k 4^{n-k}$ and therefore the volume of the lattice $\mathbb{T}$ is of the form $3^\ell 2^m$ for some $\ell \geq 0$ and $m \geq 0$. On the other hand the volume of the lattice $\mathbb{T}$ is the volume of the shape $\Upsilon_n$, i.e., $2^n(n+1)$. By Theorem 3.8 we have that if $n$ is odd, then $n = 2^t - 1$ for some $t > 0$. If $n$ is even, then $n+1$ is odd and since $3^\ell 2^m = 2^n(n+1)$ we must have that $n = 3^\ell - 1$ for some $\ell > 0$. $\square$

**3.2. Tiling for even $n$.** In this subsection we will use the concept of a packing triple system to prove that if $n$ is even, then $\mathbb{T}$ contains exactly $\frac{n}{2}$ codewords of the form $3e_r + 2e_s$, where the union of their nonzero coordinates is the set of all $n$ coordinates. The structure of the codewords in $\mathbb{T}$ which was proved in subsection 3.1 and will be proved in this subsection, combined with arguments based on reflections and translations of the tiling, will imply a period 12 for the tiling when $n$ is even. As a consequence we infer that if $n$ is even, then $n = 3^t - 1$ for some $t > 0$.

A *packing triple system* of order $n$ is a pair $(Q, \mathcal{B})$, where $Q$ is an $n$-set and $\mathcal{B}$ is a collection of 3-subsets of $Q$, called *blocks*, such that each 2-subset of $Q$ is contained in at most one block of $\mathcal{B}$. Spencer [16] proved that if $n \not\equiv 5 \pmod 6$, then

$$|\mathcal{B}| \leq \left\lfloor \frac{n}{3} \left\lfloor \frac{n-1}{2} \right\rfloor \right\rfloor . \tag{3.1}$$

LEMMA 3.12. *For each $1 \leq i < j \leq n$, the point $e_i + e_j$ is covered by a codeword $X \in \mathbb{T}$, where $X = 3e_i + 2e_j$, or $X = 3e_j + 2e_i$, or $X \in \mathcal{D}_3$, where $x_i = x_j = 2$.*

*Proof.* The proof follows from Lemmas 3.1 and 3.2 and the fact that for each nonzero codeword $X \in \mathbb{T}$ we have $w_C(X) \geq 3$. $\square$

Let

$$\mathcal{F}_1 \stackrel{\text{def}}{=} \{\{i, j\} \ : \ 3e_i + 2e_j \in \mathbb{T}\}$$

and

$$\mathcal{F}_2 \stackrel{\text{def}}{=} \left\{ \{i, j, k\} \ : \ 2e_i + 2e_j + 2e_k + \sum_{m \notin \{i,j,k\}} \alpha_m e_m \in \mathbb{T}, \ \alpha_m \in \{0, 1\} \right\}.$$

Since $\mathbb{T}$ is a tiling it follows that each point $e_i + e_j$, $i \neq j$, is covered by exactly one codeword of $\mathbb{T}$. As a consequence of Lemma 3.12, we have that each pair $\{r, s\}$ is a subset of exactly one element from $\mathcal{F}_1 \cup \mathcal{F}_2$. Therefore, $\mathcal{F}_2$ is a packing triple system of order $n$.

THEOREM 3.13. *If $\mathbb{T}$ is an integer tiling with $\Upsilon_n$, then $n \not\equiv 4$ (mod 6).*

*Proof.* Assume $\mathbb{T}$ is an integer tiling with $\Upsilon_n$, $n \equiv 4 \pmod 6$. By (3.1) we have that

$$|\mathcal{F}_2| \leq \frac{n^2 - 2n - 2}{6} .$$

Since each pair $\{i, j\} \subset \{1, 2, \ldots, n\}$ is contained in either $\mathcal{F}_1$ or $\mathcal{F}_2$ it follows that

$$|\mathcal{F}_1| + 3|\mathcal{F}_2| = \binom{n}{2} .$$

Hence, $|\mathcal{F}_1| \geq \frac{n}{2} + 1$. Lemma 3.9 implies that $|\mathcal{F}_1| \leq \frac{n}{2}$, a contradiction. $\square$

By using the same arguments as in the proof of Theorem 3.13 we have that if $n \equiv 0$ or $2 \pmod 6$, then $|\mathcal{F}_1| \geq \frac{n}{2}$. Hence, by Lemma 3.9 we infer the next lemma.

Lemma 3.14. *If $n$ is even and $\mathbb{T}$ is an integer tiling with $\Upsilon_n$, then there are exactly $\frac{n}{2}$ codewords of the form $3e_r + 2e_s$.*

Combing Lemmas 3.9 and 3.14 we infer the following.

Corollary 3.15. *If $n$ is even and $\mathbb{T}$ is an integer tiling with $\Upsilon_n$, then there are exactly $\frac{n}{2}$ codewords of the form $3e_r + 2e_s$ and the set $\{i \ : \ 3e_i + 2e_j \in \mathbb{T}$ or $3e_j + 2e_i \in \mathbb{T}\}$ contains all the integers between 1 and $n$.*

Let $\mathbb{T}'$ be the tiling of $\mathbb{Z}^n$ with $\Upsilon_n$ defined by $\mathbb{T}' \stackrel{\text{def}}{=} \{X \ : \ -X \in \mathbb{T}\}$. Since $\mathbb{T}'$ is a tiling of $\mathbb{Z}^n$ with $\Upsilon_n$, it follows that the lemmas and the corollaries of section 3 hold also for $\mathbb{T}'$. They imply new lemmas and corollaries for $\mathbb{T}$. For example, we have the next corollary.

Corollary 3.16. *For each $r$, $1 \leq r \leq n$, the point $-2e_r$ is covered by a codeword $X \in \mathbb{T}$, where either $X = -4e_r$ or $X = -3e_r - 2e_s$ for some $s \neq r$.*

In a similar way we can define $2^n$ tilings of $\mathbb{Z}^n$ with $\Upsilon_n$. For $A = (a_1, a_2, \dots, a_n)$, where $a_i \in \{-1, 1\}$, let $\mathbb{T}_A$ be the tiling of $\mathbb{Z}^n$ with $\Upsilon_n$ defined by $\mathbb{T}_A \stackrel{\text{def}}{=} \{(x_1, x_2, \dots, x_n) : (a_1x_1, a_2x_2, \dots, a_nx_n) \in \mathbb{T}\}$. As for $\mathbb{T}' = \mathbb{T}_{(-1,-1,\dots,-1)}$, each lemma and each corollary of section 3 holds for $\mathbb{T}_A$ and thus implies new claims on $\mathbb{T}$. Without loss of generality we assume (based on Lemma 2.2 and Corollaries 3.10 and 3.15) that $3e_{2i-1} + 2e_{2i} \in \mathbb{T}$ and $4e_{2i} \in \mathbb{T}$ for all $1 \leq i \leq \frac{n}{2}$.

Lemma 3.17. *If $X = 3e_r + 2e_s \in \mathbb{T}$, then $-4e_s \in \mathbb{T}$.*

*Proof.* Without loss of generality we will prove the claim for $r = 1$ and $s = 2$; let $A = (1, -1, 1, \dots, 1)$. Since $3e_{2i-1} + 2e_{2i} \in \mathbb{T}$ for all $2 \leq i \leq \frac{n}{2}$, it follows that $3e_{2i-1} + 2e_{2i} \in \mathbb{T}_A$, for all $2 \leq i \leq \frac{n}{2}$, and by Corollary 3.15 we have that either $3e_1 + 2e_2 \in \mathbb{T}_A$ or $2e_1 + 3e_2 \in \mathbb{T}_A$. If $2e_1 + 3e_2 \in \mathbb{T}_A$, then Corollary 3.10 implies that $Y = 4e_1 \in \mathbb{T}_A$. Therefore, $Y = 4e_1 \in \mathbb{T}$, and since $d_C(X, Y) = 1$ we have a contradiction. Hence, $3e_1 + 2e_2 \in \mathbb{T}_A$, and therefore by Corollary 3.10 we have that $4e_2 \in \mathbb{T}_A$, i.e., $-4e_2 \in \mathbb{T}$. $\square$

Corollary 3.18. $4e_s \in \mathbb{T}$ *if and only if* $-4e_s \in \mathbb{T}$.

Lemma 3.19. *If $X = 3e_r + 2e_s \in \mathbb{T}$, then $-3e_r - 2e_s \in \mathbb{T}$.*

*Proof.* Without loss of generality we will prove the claim for $r = 1$ and $s = 2$; let $A = (-1, -1, 1, \dots, 1)$. Since $3e_{2i-1} + 2e_{2i} \in \mathbb{T}$ for all $2 \leq i \leq \frac{n}{2}$, it follows that $3e_{2i-1} + 2e_{2i} \in \mathbb{T}_A$ for all $2 \leq i \leq \frac{n}{2}$, and by Corollary 3.15 we have that either $3e_1 + 2e_2 \in \mathbb{T}_A$ or $2e_1 + 3e_2 \in \mathbb{T}_A$. If $2e_1 + 3e_2 \in \mathbb{T}_A$, then Lemma 3.17 implies that $-4e_1 \in \mathbb{T}_A$. Therefore, $Y = 4e_1 \in \mathbb{T}$, and since $d_C(X, Y) = 1$ we have a contradiction. Hence, $3e_1 + 2e_2 \in \mathbb{T}_A$, and therefore we have that $-3e_1 - 2e_2 \in \mathbb{T}$. $\square$

Corollary 3.20. $3e_r + 2e_s \in \mathbb{T}$ *if and only if* $-3e_r - 2e_s \in \mathbb{T}$.

Lemma 3.21. *If $3e_r + 2e_s \in \mathbb{T}$, then $12e_r, 12e_s \in \mathbb{T}$.*

*Proof.* By Corollary 3.10 we have that $4e_s \in \mathbb{T}$. The translation $\mathbb{T}_1 = -4e_s + \mathbb{T}$ is a tiling with $\Upsilon_n$ for which $\mathbf{0}, -4e_s \in \mathbb{T}_1$. It follows by Corollary 3.18 that $4e_s \in \mathbb{T}_1$ and hence $8e_s \in \mathbb{T}$. Similarly, $12e_s \in \mathbb{T}$.

Similarly, by Corollary 3.20 we have that $\mathbf{0}, 3e_r + 2e_s \in \mathbb{T}$ implies that $6e_r + 4e_s, 9e_r + 6e_s, 12e_r + 8e_s \in \mathbb{T}$. The translation $\mathbb{T}_1 = -12e_r - 8e_s + \mathbb{T}$ is a tiling with $\Upsilon_n$ for which $\mathbf{0}, -3e_r - 2e_s \in \mathbb{T}_1$. By Corollary 3.20 and Lemma 3.17 we have that $-4e_s \in \mathbb{T}_1$, and hence $12e_r + 4e_s \in \mathbb{T}$. Similarly, by Corollary 3.18 we have $12e_r + 4e_s, 12e_r + 8e_s \in \mathbb{T}$, which implies that $12e_r \in \mathbb{T}$. $\square$

Corollary 3.22. *If $n$ is even and $\mathbb{T}$ is an integer tiling with $\Upsilon_n$, then $\mathbb{T}$ is a periodic tiling with period* 12.

Theorem 3.23. *If $\mathbb{T}$ is an integer tiling with $\Upsilon_n$, where $n$ is an even integer, then $n = 3^t - 1$ for some $t > 0$.*

*Proof.* By Corollary 3.22 we have that $\mathbb{T}$ is a periodic tiling with period 12. Therefore, the size of $\Upsilon_n$ divides $12^n$. The size of $\Upsilon_n$ is $2^n(n+1)$ and hence $n+1$ divides $2^n 3^n$. Since $n$ is even it follows that $n+1$ is odd and thus $n = 3^t - 1$ for some $t > 0$. ∎

Theorems 3.8 and 3.23 are combined to obtain the next corollary.

COROLLARY 3.24. *If $\mathbb{T}$ is an integer tiling with $\Upsilon_n$, then either $n = 2^t - 1$ or $n = 3^t - 1$ for some $t > 0$.*

COROLLARY 3.25. *If $\mathbb{T}$ is a $\frac{1}{2}\mathbb{Z}$-tiling with a $(0.5, n)$-cross, then either $n = 2^t - 1$ or $n = 3^t - 1$ for some $t > 0$.*

**4. Tilings based on perfect codes.** In section 3 we proved that a $\frac{1}{2}\mathbb{Z}$-tiling with $(0.5, n)$-cross exists only if $n = 2^t - 1$ or $n = 3^t - 1$ for some $t > 0$. In this section we will prove that this necessary condition is also sufficient. Surprisingly, two constructions which produce the related tilings are based on perfect codes in the Hamming scheme. If $n = 2^t - 1$, then the perfect code is binary of length $n$ and the construction of the tiling is very simple. If $n = 3^t - 1$, then the perfect code is ternary of length $\frac{n}{2}$.

We will refer only to perfect codes with minimum Hamming distance 3. A code $\mathcal{C}$ has *minimum Hamming distance* $d$ if for every two distinct codewords $X, Y \in \mathcal{C}$ we have $d_H(X, Y) \geq d$. The minimum Hamming distance of $\mathcal{C}$ will be denoted by $d_H(\mathcal{C})$. Similarly, we define the *minimum cross distance* of a code. A code $\mathcal{C}$ of length $n$ over $\mathbb{Z}_q$, with minimum Hamming distance 3, is called *perfect* if for each word $A \in \mathbb{Z}_q^n$ there exists a codeword $X \in \mathcal{C}$ such that $d_H(A, X) \leq 1$. Such a code is also called a single-error-correcting perfect code since it is capable of correcting a single transmission error [14]. The *sphere* of radius $\rho$ centered at $A = (a_1, a_2, \ldots, a_n)$ is the set $\{B \in \mathbb{Z}_q^n \ : \ d_H(A, B) \leq \rho\}$. The code $\mathcal{C}$ is a single-error-correcting perfect code if and only if $\mathcal{C}$ is a tiling of $\mathbb{Z}_q^n$ with a sphere of radius one. Binary $(q = 2)$ perfect codes exists if and only if $n = 2^t - 1$, where $t > 0$. Ternary $(q = 3)$ perfect codes exists if and only if $n = \frac{3^t - 1}{2}$, where $t > 0$. These are the only perfect codes which are of interest in this section. Finally, we note that a perfect code is identified by its size, its minimum distance, and the fact that each element of $\mathbb{Z}_q^n$ is covered by at least one codeword. One can easily verify that given any two of these parameters one can determine whether the code is perfect or not perfect. This fact will be used throughout this section.

*Remark* 3. A perfect code $\mathcal{C}$ of length $n$ over $\mathbb{Z}_q$ is known to exist if $q$ is a power of a prime and $n = \frac{q^t - 1}{q - 1}$, where $t > 0$. The related sphere of radius one can be viewed as a $(q-1, n)$-semicross or as a $(\frac{q-1}{2}, n)$-cross. Thus, these perfect codes form tilings with the related semicrosses and crosses. Only if $q$ is a prime then some of the known tilings are lattice tilings (they are related to linear perfect codes). If $q$ is not a prime, then the tiling of $\mathbb{Z}^n$ is done first by using any one-to-one mapping between GF$(q)$ (on which the codes are defined) and $\mathbb{Z}_q$. Tilings of this type have applications in flash memories [15]. If $q = 2$, then $\mathcal{C}$ is a tiling of $\mathbb{Z}_2^n$ with $(0.5, n)$-cross and $E(\mathcal{C})$ forms a tiling of $\mathbb{Z}^n$ with $(0.5, n)$-cross.

**4.1. Binary perfect codes.** Since the size of a sphere with radius one in $\mathbb{Z}_2^n$ is $n+1$, it follows that a binary perfect code of length $n = 2^t - 1$ has $2^{n-t}$ codewords.

THEOREM 4.1. *There exists a one-to-one correspondence between the set of binary perfect codes of length $n = 2^t - 1$ and the set of integer tilings with $\Upsilon_n$ in which each codeword has only even entries.*

*Proof.* Note first that by Corollary 3.7 a tiling $\mathbb{T}$ of $\mathbb{Z}^n$ with $\Upsilon_n$ is periodic with period 4 and hence it can be reduced to a tiling of $\mathbb{Z}_4^n$ with $\Upsilon_n$.

The size of a $(1,n)$-semicross is equal to the size of a $(0.5,n)$-cross. It implies that the number of codewords in a binary single-error-correcting perfect code $\mathcal{C}$ of length $n = 2^t - 1$ is equal to the number of codewords in a tiling $\mathbb{T}$ of $\mathbb{Z}_4^n$ with $\Upsilon_n$. If $X, Y \in \{0,2\}^n$, then $0.5X$ and $0.5Y$ are binary words and it is easy to verify that $d_C(X,Y) = d_H(0.5X, 0.5Y)$.

Therefore, if $\mathcal{C}$ is a binary perfect code of length $n = 2^t - 1$, then $2E(\mathcal{C})$ is a tiling of $\mathbb{Z}^n$ with $\Upsilon_n$ in which each codeword has only even entries. Similarly, if $\mathbb{T}$ is a tiling of $\mathbb{Z}^n$ with $\Upsilon_n$, in which each codeword has only even entries, then $0.5\mathbb{T} \cap \{0,1\}^n$ is a binary perfect code. $\square$

COROLLARY 4.2. *There exists a one-to-one correspondence between the set of binary perfect codes of length* $n = 2^t - 1$ *and the set of integer tilings with* $(0.5, n)$*-cross.*

Do there exist any integer tilings with $\Upsilon_n$, where $n = 2^t - 1$, except for those implied by Theorem 4.1? The answer is that there exist many such tilings. Let $\mathcal{C}$ be a binary code of length $n$. Its *punctured* code $\mathcal{C}'$ of length $n-1$ is defined by $\mathcal{C}' \stackrel{\text{def}}{=} \{c \ : \ (c,x) \in \mathcal{C}, \ x \in \{0,1\}\}$.

CONSTRUCTION 1. *Let* $\mathcal{C}$ *be a binary perfect code of length* $n$ *and* $\mathcal{C}'$ *its punctured code. Let* $\mathcal{C}'_e$ *and* $\mathcal{C}'_o$ *be the set of codewords from* $\mathcal{C}'$ *with even weight and odd weight, respectively. We define a code* $\mathcal{C}^* \stackrel{\text{def}}{=} \mathcal{C}_1^* \cup \mathcal{C}_2^*$ *over* $\mathbb{Z}_4^n$*, where*

$$\mathcal{C}_1^* \stackrel{\text{def}}{=} \{(2c, 2x) \ : \ c \in \mathcal{C}'_e, \ (c,x) \in \mathcal{C}\} \ \text{and} \quad \mathcal{C}_2^* \stackrel{\text{def}}{=} \{(2c, 2x+1) \ : \ c \in \mathcal{C}'_o, \ (c,x) \in \mathcal{C}\} \ .$$

THEOREM 4.3. $E(\mathcal{C}^*)$ *defines a tiling of* $\mathbb{Z}^n$ *with* $\Upsilon_n$*, in which not all entries are even.*

*Proof.* Since $d_H(\mathcal{C}) = 3$ it follows that $d_H(\mathcal{C}') = d_H(\mathcal{C}'_e) = d_H(\mathcal{C}'_o) = 2$ and $d_C(\mathcal{C}_1^*) = d_C(\mathcal{C}_2^*) = 3$. If $\tilde{c}_1 \in \mathcal{C}'_e$ and $\tilde{c}_2 \in \mathcal{C}'_o$, then $d_H(\tilde{c}_1, \tilde{c}_2)$ is an odd integer. Hence, since $d_H(\mathcal{C}') = 2$, it follows that $d_H(\tilde{c}_1, \tilde{c}_2) \geq 3$. Therefore, if $\tilde{c}_1^* \in \mathcal{C}_1$ and $\tilde{c}_2^* \in \mathcal{C}_2$, then $d_C(\tilde{c}_1^*, \tilde{c}_2^*) \geq 3$ and thus $d_C(\mathcal{C}^*) \geq 3$. The minimum distance of the code $\mathcal{C}^*$ and its number of codewords implies that $\mathcal{C}^*$ is a tiling of $\mathbb{Z}_4^n$ with $\Upsilon_n$. It is easy to verify that $\mathcal{C}'_o$ has at least one codeword (in fact it can be proved that it contains exactly half the codewords) and hence the last entry in at least one of the codewords of $\mathcal{C}^*$ is 1 or 3. $\square$

*Example* 1. The following code forms a tiling of $\mathbb{Z}_4^7$ with $\Upsilon_7$:

| | | | |
|---|---|---|---|
| 0000000 | 0000222 | 2222000 | 2222222 |
| 2200201 | 2200023 | 0022201 | 0022023 |
| 2020021 | 2020203 | 0202021 | 0202203 |
| 2002002 | 2002220 | 0220002 | 0220220 |

*Remark* 4. Let $\xi$ be a mapping from $\mathbb{Z}_4$ to $\mathbb{Z}_2$ defined by $\xi(0) = \xi(1) = 0$, $\xi(2) = \xi(3) = 1$. If $\mathbb{T}$ forms a tiling of $\mathbb{Z}_4^n$ with $\Upsilon_n$, then the code $\mathcal{C} = \{\xi(X) \ : \ X \in \mathbb{T}\}$, where $\xi(x_1, x_2, \ldots, x_n) = (\xi(x_1), \xi(x_2), \ldots, \xi(x_n))$ is a binary perfect code of length $n$.

*Remark* 5. By Corollary 3.7 an integer tiling with $\Upsilon_n$, where $n$ is odd, has period 4. Hence, the related $\frac{1}{2}\mathbb{Z}$-tiling $\mathbb{T}$ with $(0.5,n)$-cross has period 2. It implies that this tiling is also a tiling with the $(1,n)$-semicross (even if $\mathbb{T}$ is not a $\mathbb{Z}$-tiling).

**4.2. Ternary perfect codes.** Let $n = 3^t - 1$, where $t > 0$, and $\nu = \frac{n}{2}$. Since the size of a sphere with radius one in $\mathbb{Z}_3^\nu$ is $2\nu + 1$, it follows that a ternary perfect code of length $\nu$ has $3^{\nu - t}$ codewords. Let $\Lambda_n$ be the lattice generated by the basis

$\{3e_{2i-1}+2e_{2i} : 1 \leq i \leq \nu\} \cup \{4e_{2i} : 1 \leq i \leq \nu\}$. Let $G_n$ be the quotient group $\mathbb{Z}^n/\Lambda_n$. Recall that we denote the set of integers in $\mathbb{Z}_p$ without the group structure by $\tilde{\mathbb{Z}}_p \overset{\text{def}}{=} \{0,1,\ldots,p-1\}$. The following lemma can be readily verified.

LEMMA 4.4. *The group $G_2$ has size 12, and the 12 representatives of elements from $G_2$ (the cosets of $\Lambda_2$ in $\mathbb{Z}^2$) can be taken as $\tilde{\mathbb{Z}}_3 \times \tilde{\mathbb{Z}}_4$.*

Let $(\tilde{\mathbb{Z}}_3 \times \tilde{\mathbb{Z}}_4)^m \overset{\text{def}}{=} \underbrace{(\tilde{\mathbb{Z}}_3 \times \tilde{\mathbb{Z}}_4) \times (\tilde{\mathbb{Z}}_3 \times \tilde{\mathbb{Z}}_4) \times \ \cdots \ \times (\tilde{\mathbb{Z}}_3 \times \tilde{\mathbb{Z}}_4)}_{m \text{ times}}$.

COROLLARY 4.5. *The group $G_n$ has size $12^\nu$ and the $12^\nu$ representatives of elements from $G_n$ (the cosets of $\Lambda_n$ in $\mathbb{Z}^n$) can be taken as the elements of $(\tilde{\mathbb{Z}}_3 \times \tilde{\mathbb{Z}}_4)^\nu$.*

Consider the mapping $\Phi : \mathbb{Z}_3^\nu \to G_n$ defined by

$$\Phi(x_1,x_2,\ldots,x_\nu) = (\phi(x_1),\phi(x_2),\ldots,\phi(x_\nu)) \ ,$$

where $\phi : \mathbb{Z}_3 \to G_2$ is a mapping defined by

$$\phi(x) = \begin{cases} (0,0) & \text{if } x = 0, \\ (1,2) & \text{if } x = 1, \\ (2,0) & \text{if } x = 2. \end{cases}$$

It is easy to verify that both $\phi$ and $\Phi$ are injective group homomorphisms.

Let $\mathcal{C}$ be a ternary perfect code of length $\nu$ with $3^{\nu-t}$ codewords, and let $\Phi(\mathcal{C}) \overset{\text{def}}{=} \{\Phi(\tilde{c}) : \tilde{c} \in \mathcal{C}\}$. Since the elements of $\Phi(\mathcal{C})$ are representatives of elements of $G_n$ (see Corollary 4.5) it follows that the elements of $\Phi(\mathcal{C})$ can be considered as elements in $\mathbb{Z}^n$. Let $\mathbb{T}_n \overset{\text{def}}{=} \Phi(\mathcal{C}) + \Lambda_n$.

THEOREM 4.6. *The set $\mathbb{T}_n$ is a tiling of $\mathbb{Z}^n$ with $\Upsilon_n$.*

*Proof.* Clearly, $\Lambda_n$ is a lattice with period 12 and hence $\mathbb{T}_n$ is a periodic code of $\mathbb{Z}^n$ with period 12. Therefore, without loss of generality we can restrict our discussion to $\mathbb{Z}_{12}^n$, i.e., codewords of $\mathbb{T}_n \cap \tilde{\mathbb{Z}}_{12}^n$. Since $|\Upsilon_n| = 2^{2\nu}3^t$ it follows that the size of the tiling $\mathbb{T}_n$ in $\tilde{\mathbb{Z}}_{12}^n$, $|\mathbb{T}_n \cap \tilde{\mathbb{Z}}_{12}^n|$, should be $2^{2\nu}3^{2\nu-t}$. To prove that $\mathbb{T}_n$ is a tiling of $\mathbb{Z}^n$ with $\Upsilon_n$ we will show that the size of $\mathbb{T}_n \cap \tilde{\mathbb{Z}}_{12}^n$ is $2^{2\nu}3^{2\nu-t}$ and we will prove that each point of $\mathbb{Z}^n$ is covered by an element of $\mathbb{T}_n$.

*Claim.* For any two codewords $\tilde{c}_1, \tilde{c}_2 \in \mathcal{C}$ and two lattice points $Y_1, Y_2 \in \Lambda_n$, we have $\Phi(\tilde{c}_1) + Y_1 \neq \Phi(\tilde{c}_2) + Y_2$, unless $\tilde{c}_1 = \tilde{c}_2$ and $Y_1 = Y_2$.

*Proof.* Assume that $\Phi(\tilde{c}_1)+Y_1 = \Phi(\tilde{c}_2)+Y_2$, i.e., $\Phi(\tilde{c}_1)-\Phi(\tilde{c}_2) = Y_2-Y_1$, $\tilde{c}_1,\tilde{c}_2 \in \mathcal{C}$, and $Y_1, Y_2 \in \Lambda_n$. Hence, $Y_2 - Y_1 = (\alpha_1,\ldots,\alpha_n)$ is a lattice point and unless $Y_1 = Y_2$ we have that for at least one $i$, $|\alpha_i| > 2$. Denote $\Phi(\tilde{c}_1) - \Phi(\tilde{c}_2) = (\beta_1,\ldots,\beta_n)$. By the definition of $\Phi$, for each $i$, $1 \leq i \leq n$, we have $|\beta_i| \leq 2$. Therefore, $Y_1 = Y_2$ and $\Phi(\tilde{c}_1) = \Phi(\tilde{c}_2)$ and since $\Phi$ is an injective mapping it implies that $\tilde{c}_1 = \tilde{c}_2$ and the claim is proved.

The claim implies that $|\mathbb{T}_n \cap \tilde{\mathbb{Z}}_{12}^n| = |\Phi(\mathcal{C})| \cdot |\Lambda_n \cap \tilde{\mathbb{Z}}_{12}^n|$. Since $\Phi$ is an injective mapping we also have that $|\Phi(\mathcal{C})| = |\mathcal{C}|$. Since $\Lambda_n$ has period 12 and $V(\Lambda_n) = 12^\nu$ it follows that $|\Lambda_n \cap \tilde{\mathbb{Z}}_{12}^n| = 12^\nu$. Therefore,

$$|\mathbb{T}_n \cap \tilde{\mathbb{Z}}_{12}^n| = |\Phi(\mathcal{C})| \cdot |\Lambda_n \cap \tilde{\mathbb{Z}}_{12}^n| = |\mathcal{C}| \cdot |\Lambda_n \cap \tilde{\mathbb{Z}}_{12}^n| = 3^{\nu-t}12^\nu = 2^{2\nu}3^{2\nu-t}$$

as required.

To show that every point of $\mathbb{Z}^n$ is covered by an element of $\mathbb{T}_n$ we first partition the elements of $\tilde{\mathbb{Z}}_3 \times \tilde{\mathbb{Z}}_4$ into three classes:

$$\begin{aligned} [(0,0)] &= \{(0,0),(0,3),(2,2),(2,1)\}, \\ [(1,2)] &= \{(1,2),(1,1),(0,1),(0,2)\}, \\ [(2,0)] &= \{(2,0),(1,3),(2,3),(1,0)\}. \end{aligned}$$

Table 4.1.

| | | | | | | |
|---|---|---|---|---|---|---|
| | Class [(0,0)] | (0,0) | (0,3) | (2,2) | (2,1) | $\longleftarrow (x_1, x_2)$ |
| (P.1) | $[(y_1,y_2)] = [(0,0)]$ | (0,0) | (0,4) | (3,2) | (3,2) | $\longleftarrow (u_1,u_2)+(y_1,y_2)$ |
| (P.2) | $[(y_1,y_2)] = [(1,2)]$ | (1,2) | (1,2) | (1,2) | (1,2) | $\longleftarrow (u_1,u_2)+(y_1,y_2)$ |
| (P.2) | $[(y_1,y_2)] = [(2,0)]$ | (2,0) | (2,4) | (2,4) | (2,0) | $\longleftarrow (u_1,u_2)+(y_1,y_2)$ |
| | Class [(1,2)] | (1,2) | (1,1) | (0,1) | (0,2) | $\longleftarrow (x_1, x_2)$ |
| (P.2) | $[(y_1,y_2)] = [(0,0)]$ | (3,2) | (3,2) | (0,0) | (0,4) | $\longleftarrow (u_1,u_2)+(y_1,y_2)$ |
| (P.1) | $[(y_1,y_2)] = [(1,2)]$ | (1,2) | (1,2) | (1,2) | (1,2) | $\longleftarrow (u_1,u_2)+(y_1,y_2)$ |
| (P.2) | $[(y_1,y_2)] = [(2,0)]$ | (2,4) | (2,0) | (-1,2) | (-1,2) | $\longleftarrow (u_1,u_2)+(y_1,y_2)$ |
| | Class [(2,0)] | (2,0) | (1,3) | (2,3) | (1,0) | $\longleftarrow (x_1, x_2)$ |
| (P.2) | $[(y_1,y_2)] = [(0,0)]$ | (3,2) | (0,4) | (3,2) | (0,0) | $\longleftarrow (u_1,u_2)+(y_1,y_2)$ |
| (P.2) | $[(y_1,y_2)] = [(1,2)]$ | (4,0) | (1,2) | (4,4) | (1,2) | $\longleftarrow (u_1,u_2)+(y_1,y_2)$ |
| (P.1) | $[(y_1,y_2)] = [(2,0)]$ | (2,0) | (2,4) | (2,4) | (2,0) | $\longleftarrow (u_1,u_2)+(y_1,y_2)$ |

The following two properties are readily verified (as can be verified from Table 4.1.):

(P.1) For each element $(x_1, x_2)$ in a class $[(y_1, y_2)]$ there exists an element $(u_1, u_2) \in \Lambda_2$ such that $u_i + y_i \in \{x_i, x_i + 1\}$ for $i \in \{1, 2\}$.

(P.2) For each element $(x_1, x_2) \in \tilde{\mathbb{Z}}_3 \times \tilde{\mathbb{Z}}_4$ and each class $[(y_1, y_2)]$ there exists an element $(u_1, u_2) \in \Lambda_2$ such that $u_i + y_i \in \{x_i - 1, x_i, x_i + 1, x_i + 2\}$ for $i \in \{1, 2\}$, and for at most one $i$ we have $u_i + y_i \in \{x_i - 1, x_i + 2\}$.

Consider the mapping $\Psi : (\tilde{\mathbb{Z}}_3 \times \tilde{\mathbb{Z}}_4)^\nu \to \mathbb{Z}_3^\nu$ defined by

$$\Psi(x_1, x_2, \dots, x_n) = (\psi(x_1, x_2), \psi(x_3, x_4), \dots, \psi(x_{n-1}, x_n)) \ ,$$

where $\psi : \tilde{\mathbb{Z}}_3 \times \tilde{\mathbb{Z}}_4 \to \mathbb{Z}_3$ is a mapping defined by

$$\psi(x_1, x_2) = \begin{cases} 0 & \text{if } (x_1, x_2) \in [(0, 0)], \\ 1 & \text{if } (x_1, x_2) \in [(1, 2)], \\ 2 & \text{if } (x_1, x_2) \in [(2, 0)]. \end{cases}$$

For a given point $A = (a_1, a_2, \dots, a_n) \in \mathbb{Z}^n$ we will exhibit a point $X \in \mathbb{T}_n$ which covers $A$. By Corollary 4.5 we have that there exists an element $Y \in \Lambda_n$ such that $A+Y \in (\tilde{\mathbb{Z}}_3 \times \tilde{\mathbb{Z}}_4)^\nu$. Let $B = A + Y = (b_1, b_2, \dots, b_n)$ and let $\Psi(B) = (\alpha_1, \alpha_2, \dots, \alpha_\nu) \in \mathbb{Z}_3^\nu$. Since $\mathcal{C}$ is a perfect code of length $\nu$ over $\mathbb{Z}_3$ it follows that there exists a codeword $(c_1, c_2, \dots, c_\nu) \in \mathcal{C}$ such that $d_H((\alpha_1, \alpha_2, \dots, \alpha_\nu), (c_1, c_2, \dots, c_\nu)) \leq 1$. Let $(\gamma_1, \gamma_2, \dots, \gamma_n) = \Phi(c_1, c_2, \dots, c_\nu)$. Note that by the definitions of $\Phi$ and $\Psi$ it follows that $(b_{2i-1}, b_{2i})$ and $\phi(\alpha_i)$ are in the same class for all $1 \leq i \leq \nu$. Now, we distinguish between two cases:

*Case* 1. If $(\alpha_1, \alpha_2, \dots, \alpha_\nu) = (c_1, c_2, \dots, c_\nu)$, then by property (P.1) there exists an element $(u_1, u_2, \dots u_n) \in \Lambda_n$ such that $u_i + \gamma_i \in \{b_i, b_i + 1\}$ for $1 \leq i \leq n$. Therefore, by Lemma 3.1 we have that $(u_1, u_2, \dots, u_n) + (\gamma_1, \gamma_2, \dots, \gamma_n)$ covers $B$ and hence the required $X$ is $(u_1, u_2, \dots u_n) + (\gamma_1, \gamma_2, \dots, \gamma_n) - Y$.

*Case* 2. If $(\alpha_1, \alpha_2, \dots, \alpha_\nu) \neq (c_1, c_2, \dots, c_\nu)$, then $d_H((\alpha_1, \alpha_2, \dots, \alpha_\nu), (c_1, c_2, \dots, c_\nu)) = 1$, and hence there exists exactly one coordinate $s$ such that $\alpha_s \neq c_s$. By properties (P.1) and (P.2) there exists an element $(u_1, u_2, \dots, u_n) \in \Lambda_n$ such that $u_i + \gamma_i \in \{b_i - 1, b_i, b_i + 1, b_i + 2\}$ for $1 \leq i \leq n$, and for at most one $i$ we have $u_i + \gamma_i \in \{b_i - 1, b_i + 2\}$. Therefore, by Lemma 3.1 we have that $(u_1, u_2, \dots, u_n) + (\gamma_1, \gamma_2, \dots, \gamma_n)$ covers $B$ and hence the required $X$ is $(u_1, u_2, \dots, u_n) + (\gamma_1, \gamma_2, \dots, \gamma_n) - Y$.

Since we proved that the size of $\mathbb{T}_n \cap \tilde{\mathbb{Z}}_{12}^n$ is $2^{2\nu} 3^{2\nu - t}$ and each point of $\mathbb{Z}^n$ is covered by an element of $\mathbb{T}_n$, the theorem is proved. $\square$

Theorem 4.7. *If $\mathcal{C}$ is a linear code, then $\mathbb{T}_n$ is a lattice tiling.*

*Proof.* The proof follows immediately from Theorem 4.6 and the facts that $\mathcal{C}$ is a linear code and $\Phi$ is a group homomorphism. ∎

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