SIAM J. DISCRETE MATH.
Vol. 18, No. 4, pp. 844–878

ON THE OPTIMALITY OF COLORING WITH A LATTICE*

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Abstract. For $z_1, z_2, z_3 \in \mathbb{Z}^2$, the *tristance* $d_3(z_1, z_2, z_3)$ is a generalization of the L_1-distance on $\mathbb{Z}^2$ to a quality that reflects the relative dispersion of three points rather than two. In this paper we prove that at least $3k^2$ colors are required to color the points of $\mathbb{Z}^2$, such that the tristance between any three distinct points, colored with the same color, is at least $4k$. We prove that $3k^2 + 3k + 1$ colors are required if the tristance is at least $4k + 2$. For the first case we show an infinite family of colorings with $3k^2$ colors and conjecture that these are the only colorings with $3k^2$ colors.

Key words. coloring, lattice, Lee sphere, tristance

AMS subject classifications. 05C15, 11H31, 52C15

DOI. 10.1137/S0895480104439589

1. Introduction.

Consider the *grid graph* $\mathcal{G} = (V, E)$ whose vertex set is $V = \mathbb{Z}^2$ and $\{(x_1, y_1), (x_2, y_2)\} \in E$ if $|x_1 - x_2| + |y_1 - y_2| = 1$. A coloring $\mathcal{F}$ is an onto function $\mathcal{F} : \mathbb{Z}^2 \to \{1, 2, \ldots, \chi\}$, where χ is the number of colors. We ask the following question: Given a positive integer t, what is the smallest number of colors required to color $\mathbb{Z}^2$, such that for any three points colored with the same color, the size of the minimum spanning tree which connects them is at least t?

This problem has an application in two-dimensional cluster error-correcting codes [1], [3]. For each color φ we assign a two-error-correcting code to the points colored with φ. We obtain an array which corrects any number of errors, if there exists a cluster of size t, which contains all the errors.

The problem has also combinatorial interest, as the coloring structure obtained is a generalization of perfect codes of $\mathbb{Z}^2$ in the L_1-metric and tiling of $\mathbb{Z}^2$ with Lee spheres (see [4]).

Lee spheres and lattices have an important role in our discussion.

The L_1-*distance* between two elements of $\mathbb{Z}^2$, $z_1 = (x_1, y_1)$, $z_2 = (x_2, y_2)$, is defined by

$$d_2(z_1, z_2) = |x_1 - x_2| + |y_1 - y_2|.$$

Clearly, the length of the shortest path which connects z_1 and z_2 in $\mathcal{G}$ is $d_2(z_1, z_2)$.

For a given element $\varsigma \in \mathbb{Z}^2$, the *Lee sphere* of radius k, $\mathcal{S}_k(\varsigma)$ (or $\mathcal{S}(\varsigma)$ if k is known), is defined by [4]

$$\mathcal{S}_k(\varsigma) = \{z : d_2(\varsigma, z) \le k\}.$$

Clearly, $|\mathcal{S}_k(\varsigma)| = 2k^2 + 2k + 1$.

*Received by the editors January 12, 2004; accepted for publication (in revised form) August 28, 2004; published electronically May 20, 2005. The results of this paper were presented in part at the IEEE International Symposium on Information Theory, Chicago, 2004.

http://www.siam.org/journals/sidma/18-4/43958.html

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A *lattice* of $\mathbb{Z}^2$ is a linear subspace of $\mathbb{Z}^2$. A lattice Λ with dimension two is defined by $\Lambda = \{a_1v_1 + a_2v_2 : a_1, a_2 \in \mathbb{Z}\}$, where $v_1 = (v_{11}, v_{12}),\ v_2 = (v_{21}, v_{22})$ are two linearly independent vectors in $\mathbb{Z}^2$, called the *basis* of Λ. The matrix

$$G = \begin{pmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{pmatrix}$$

having these vectors as rows is said to be a *generator matrix* of Λ. It is well known that $|\det G|$ is the number of cosets of Λ in $\mathbb{Z}^2$, i.e., $|\mathbb{Z}^2 / \Lambda| = |\det G|$. A lattice Λ with dimension two defines a coloring as follows: The points of each distinct coset of Λ are colored with the same color. Thus, there are $|\det G|$ distinct colors.

The solution for the following simpler question is known [1]:

Given a positive integer t, what is the smallest number of colors required to color $\mathbb{Z}^2$, such that for any two points colored with the same color, the length of the shortest path which connects them is at least t?

If $t = 2k+1$, then the number of colors is $2k^2 + 2k + 1$ [1]. A coloring is given by a lattice whose generator matrix is

$$\begin{pmatrix} 1 & 2k+1 \\ 0 & 2k^2+2k+1 \end{pmatrix}.$$

This lattice defines also a tiling of $\mathbb{Z}^2$ with Lee spheres of radius k [1], [4]. If $t = 2k$, then the number of colors is $2k^2$ [1]. A coloring is given by a lattice whose generator matrix is

$$\begin{pmatrix} k & k \\ 0 & 2k \end{pmatrix}.$$

Let $z_1, z_2, z_3 \in \mathbb{Z}^2$. The *tristance* $d_3(z_1, z_2, z_3)$ is a generalization of the L_1-distance. $d_3(z_1, z_2, z_3)$ is defined as the number of edges in a minimum spanning tree of z_1, z_2, z_3 in the grid graph $\mathcal{G}$. It is known [3] that if $z_1 = (x_1, y_1),\ z_2 = (x_2, y_2),\ z_3 = (x_3, y_3)$, then

$$d_3(z_1, z_2, z_3) = \left(\max_{1 \le i \le 3} x_i - \min_{1 \le i \le 3} x_i \right) + \left(\max_{1 \le i \le 3} y_i - \min_{1 \le i \le 3} y_i \right).$$

For a coloring $\mathcal{F} : \mathbb{Z}^2 \to \{1, 2, \ldots, \chi\}$, $d_3(\mathcal{F})$ is defined by

$$d_3(\mathcal{F}) = \min_{\substack{\mathcal{F}(z_1)=\mathcal{F}(z_2)=\mathcal{F}(z_3) \\ |\{z_1,z_2,z_3\}|=3}} d_3(z_1, z_2, z_3).$$

For a given t, a coloring $\mathcal{F}$ will be called a t-*coloring* if $d_3(\mathcal{F}) \ge t$.

Etzion and Vardy [3] proved that if $t = 4k$ ($t = 4k+2$), then any t-coloring defined by a lattice has at least $3k^2$ ($3k^2 + 3k + 1$) colors. Schwartz and Etzion [6] proved that if $t = 4k+1$ ($t = 4k+3$), then any t-coloring defined by a lattice has at least $3k^2 + 2k$ ($3k^2 + 5k + 2$) colors. For each t an optimal lattice coloring was given in [3].

In this paper we prove that if $t = 4k$ ($t = 4k+2$), then any t-coloring (and not just t-coloring defined by a lattice) has at least $3k^2$ ($3k^2 + 3k + 1$) colors. The result for $t = 4k$ is proved in section 2. In section 3 we show an infinite family of optimal colorings which we believe are the only optimal colorings. We conclude in sections 4 and 5 with extensions, some related questions, and problems for further research. There are also three appendices. In Appendix A we give a short description of the various types of geometric shapes used in our proofs. In Appendices B and C we give the detailed proofs of some of our results.

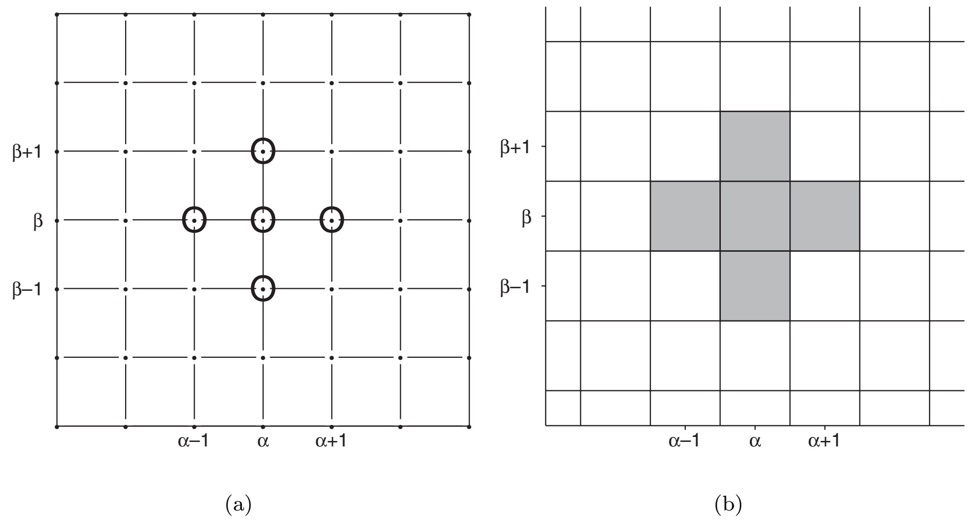


FIG. 1. *$\mathcal{S}_1((\alpha,\beta))$ in the grid graph and in the array.*

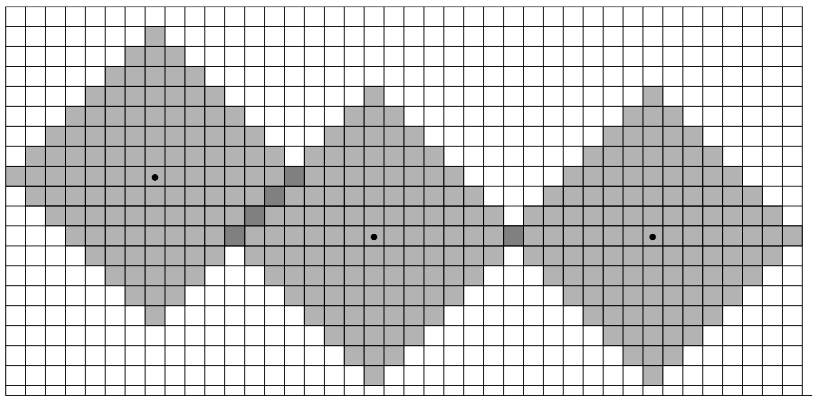
2. Optimality of the coloring. The main result of this paper is the following theorem.

THEOREM 1. *If $\mathcal{F}$ is a $4k$-coloring of $\mathbb{Z}^2$, then the number of colors in $\mathcal{F}$ is at least $3k^2$.*

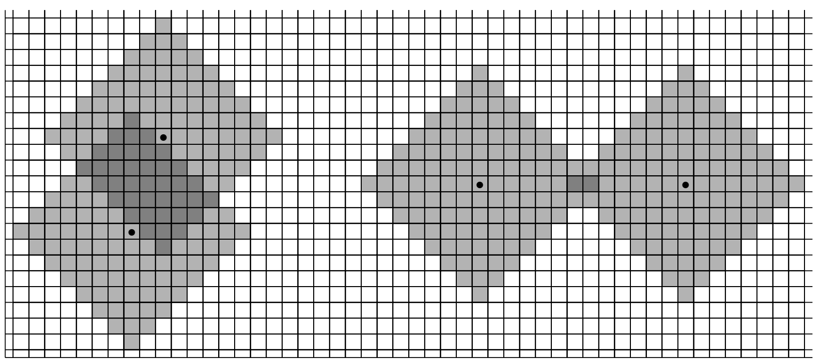
As the proof is very detailed, we first sketch the outline of the proof. For the simplicity of our presentation we will use an infinite array instead of the grid graph. Each element of $\mathbb{Z}^2$ is mapped into a corresponding cell of the array (see Figure 1). Let $\mathcal{C}$ be the set of cells in $\mathbb{Z}^2$ which are colored with the first color. These cells will be called *black cells*. For each black cell ς we define a neighborhood which contains ς. Each neighborhood contains either one black cell or two black cells, and two different neighborhoods are disjoint. We will prove that the size of a neighborhood with one black cell is at least $3k^2$ and the size of a neighborhood with two black cells is greater than $6k^2$. These properties will lead to an immediate proof of the theorem.

Hence the main part of the proof includes the definition of a neighborhood and the computation of its size. For this purpose, in what follows we consider all the Lee spheres with radius k whose centers are exactly all the black cells. Each one of these spheres, $\mathcal{S}(\varsigma)$, satisfies one of the following:

- If the sphere does not intersect another sphere, then the neighborhood of ς includes $\mathcal{S}(\varsigma)$ and additional cells above $\mathcal{S}(\varsigma)$.
- $\mathcal{S}(\varsigma)$ intersects other spheres, and any such intersection with another sphere contains cells which are on the same diagonal line, as depicted in Figure 2(a). This type of intersection will be called a *line-intersection*. The neighborhood of ς in this case includes $\mathcal{S}(\varsigma)$ (except maybe some of the intersection) and additional cells above $\mathcal{S}(\varsigma)$.
- $\mathcal{S}(\varsigma)$ intersects exactly one sphere, $\mathcal{S}(\varsigma_1)$, on more than one diagonal line, as depicted in Figure 2(b). This type of intersection will be called a *deep-intersection*. In this case we define a neighborhood which includes the union of $\mathcal{S}(\varsigma)$ and $\mathcal{S}(\varsigma_1)$ and additional cells above this union.



(a) Line-intersections



(b) Deep-intersections

Fig. 2. *Various types of intersections.* (a) *Line-intersections.* (b) *Deep-intersections.*

2.1. Intersections of spheres. For two black cells ς_1, ς_2, let $\mathcal{I}(\varsigma_1, \varsigma_2) = \mathcal{S}(\varsigma_1) \cap \mathcal{S}(\varsigma_2)$. By the definition of spheres with radius k we clearly have the following lemma.

Lemma 1. *Given two black cells* ς_1, ς_2,

(a) $\mathcal{I}(\varsigma_1, \varsigma_2) = \varnothing$ *if and only if* $d_2(\varsigma_1, \varsigma_2) > 2k$;
(b) $\mathcal{I}(\varsigma_1, \varsigma_2)$ *is a line-intersection if and only if* $d_2(\varsigma_1, \varsigma_2) = 2k$;
(c) $\mathcal{I}(\varsigma_1, \varsigma_2)$ *is a deep-intersection if and only if* $d_2(\varsigma_1, \varsigma_2) < 2k$.

The next lemma is an immediate result from the definition of the tristance.

Lemma 2. *Let t be a positive integer, and let* $z_0 = (0, 0)$, $z_1 = (\alpha, \beta)$, $z_2 = (x, y)$ *be three cells in* $\mathbb{Z}^2$ *such that* $\alpha, \beta \geq 0$ *and* $\alpha + \beta < t$. *The tristance* $d_3(z_0, z_1, z_2) < t$

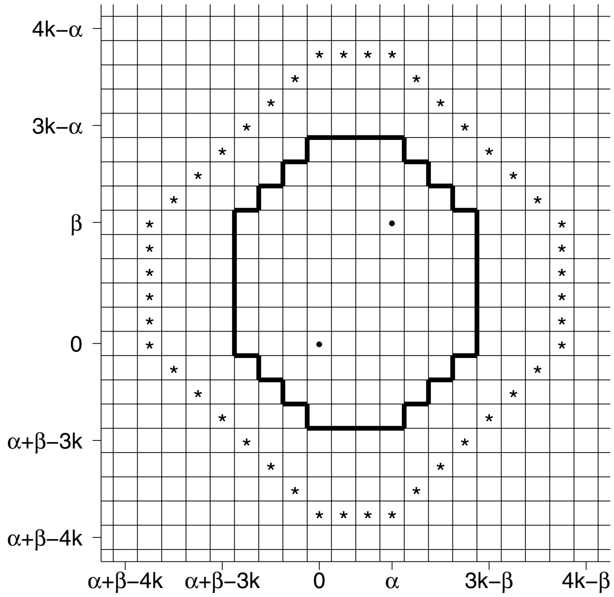


FIG. 3. *The polygon $\mathcal{P}(\varsigma_0,\varsigma_1)$ for $(\alpha,\beta)=(3,5)$ and $k=4$.*

iff

$$\begin{array}{rcccl} \alpha+\beta-t & < & x & < & t-\beta, \\ \alpha+\beta-t & < & y & < & t-\alpha, \\ \alpha+\beta-t & < & x+y & < & t, \\ \beta-t & < & y-x & < & t-\alpha. \end{array} \tag{1}$$

Note that if z_0 and z_1 are black cells, then there is no black cell inside the polygon defined by (1), i.e., inside the area defined by $*$ (inclusive) in Figure 3.

Let z_1, z_2 be two distinct black cells such that $d_2(z_1,z_2) < 3k$. We define $\mathcal{P}(z_1,z_2)$ to be the set of cells such that for each cell ς, if $\mathcal{P}(z_1,z_2)\cap\mathcal{S}(\varsigma)\neq\varnothing$, then $d_3(z_1,z_2,\varsigma) < 4k$.

LEMMA 3. *Let $\varsigma_0=(0,0)$, $\varsigma_1=(\alpha,\beta)$ be two black cells such that $|\alpha|\le 2k$, $0\le \beta\le 2k$, $d_2(\varsigma_0,\varsigma_1)=|\alpha|+\beta<3k$.*

If $\alpha\ge 0$, then

$$\mathcal{P}(\varsigma_0,\varsigma_1)=\left\{(x,y):\ \begin{array}{rcccl} \alpha+\beta-3k & < & x & < & 3k-\beta \\ \alpha+\beta-3k & < & y & < & 3k-\alpha \\ \alpha+\beta-3k & < & x+y & < & 3k \\ \beta-3k & < & y-x & < & 3k-\alpha \end{array}\right\}.$$

If $\alpha<0$, then

$$\mathcal{P}(\varsigma_0,\varsigma_1)=\left\{(x,y):\ \begin{array}{rcccl} \beta-3k & < & x & < & 3k-|\alpha|-\beta \\ |\alpha|+\beta-3k & < & y & < & 3k-|\alpha| \\ \beta-3k & < & x+y & < & 3k-|\alpha| \\ |\alpha|+\beta-3k & < & y-x & < & 3k \end{array}\right\}.$$

The cells of $\mathcal{P}(\varsigma_0, \varsigma_1)$ are inside the bold lines in Figure 3. If $|\alpha| + \beta < 3k - 1$, $|\alpha|, \beta > 0$, then $\mathcal{P}(\varsigma_0, \varsigma_1)$ is an octagon, as depicted in Figure 3. If $|\alpha| + \beta < 3k - 1$ and either $\alpha = 0$ or $\beta = 0$, then $\mathcal{P}(\varsigma_0, \varsigma_1)$ is a hexagon. If $|\alpha| + \beta = 3k - 1$, then $\mathcal{P}(\varsigma_0, \varsigma_1)$ is a rectangle whose opposite vertices are ς_0 and ς_1. Note that $\mathcal{P}(\varsigma_0, \varsigma_1)$ always contains the rectangle whose opposite vertices are ς_0 and ς_1. $\mathcal{P}(\varsigma_0, \varsigma_1)$ will be called the *polygon* of ς_0 and ς_1.

COROLLARY 1. *If $\varsigma_1, \varsigma_2, \varsigma_3$ are three distinct black cells, then $\mathcal{P}(\varsigma_1, \varsigma_2) \cap \mathcal{S}(\varsigma_3) = \varnothing$.*

COROLLARY 2. *Let $\varsigma_1, \varsigma_2, \varsigma_3$ be three distinct black cells. If $\mathcal{I}(\varsigma_1, \varsigma_2)$ is a deep-intersection, then $\mathcal{P}(\varsigma_1, \varsigma_2) \supset \mathcal{S}(\varsigma_1) \cup \mathcal{S}(\varsigma_2)$.*

COROLLARY 3. *Let $\varsigma_1, \varsigma_2, \varsigma_3$ be three distinct black cells. If $\mathcal{I}(\varsigma_1, \varsigma_2)$ is a deep-intersection, then $(\mathcal{S}(\varsigma_1) \cup \mathcal{S}(\varsigma_2)) \cap \mathcal{S}(\varsigma_3) = \varnothing$.*

COROLLARY 4. *Let $\varsigma_1, \varsigma_2, \varsigma_3$ be three distinct black cells. If $\mathcal{I}(\varsigma_1, \varsigma_2)$ and $\mathcal{I}(\varsigma_2, \varsigma_3)$ are line-intersections, then one of the following holds:*

1. $|\mathcal{I}(\varsigma_1, \varsigma_2)| = |\mathcal{I}(\varsigma_2, \varsigma_3)| = 1$.
2. *$\mathcal{I}(\varsigma_1, \varsigma_2)$ is on the line $x + y = l_1$ and $\mathcal{I}(\varsigma_2, \varsigma_3)$ is on the line $x + y = l_2$ for some l_1, l_2 such that $|l_1 - l_2| = 2k$.*
3. *$\mathcal{I}(\varsigma_1, \varsigma_2)$ is on the line $y - x = l_1$ and $\mathcal{I}(\varsigma_2, \varsigma_3)$ is on the line $y - x = l_2$ for some l_1, l_2 such that $|l_1 - l_2| = 2k$.*

2.2. Definition of neighborhood. In this subsection we define the *neighborhood* $\mathcal{N}(\varsigma)$ for any given black cell ς. First, we give some definitions concerning the sphere of a black cell. The right (left) *tip* of the sphere is the rightmost (leftmost) cell in the sphere. The *top* (*bottom*) of the sphere is the highest (lowest) cell in the sphere. A cell is called *free* if it is not contained in any sphere. For each black cell $\varsigma = (\alpha, \beta)$ we define a set

$$\mathcal{U}(\varsigma) = \{(x, y) : |x - \alpha| \le k - 1,\ \beta + 2 \le y \le \beta + k,\ (x, y) \notin \mathcal{S}(\varsigma)\}.$$

An example of $\mathcal{U}(\varsigma)$, $\varsigma = (0, 0)$, is depicted in Figure 4. We partition $\mathcal{U}(\varsigma)$ into two subsets $\mathcal{U}_l(\varsigma)$ and $\mathcal{U}_r(\varsigma)$, where

$$\mathcal{U}_r(\varsigma) = \{(x, y) : 0 < x - \alpha \le k - 1,\ \beta + 2 \le y \le \beta + k,\ (x, y) \notin \mathcal{S}(\varsigma)\}.$$

If $\mathcal{S}(\varsigma)$ has a deep-intersection with another sphere $\mathcal{S}(\varsigma_1)$, then $\mathcal{N}(\varsigma) = \mathcal{N}(\varsigma_1)$; i.e., ς and ς_1 have a joint neighborhood. In any other case each black cell has its own neighborhood. Note that by Corollary 3, if $\mathcal{I}(\varsigma, \varsigma_1)$ is a deep-intersection, then no other sphere intersects $\mathcal{S}(\varsigma) \cup \mathcal{S}(\varsigma_1)$.

The definition of $\mathcal{N}(\varsigma)$ will be done by assigning each cell in $\mathbb{Z}^2$ to at most one neighborhood. This assignment of a cell $z_0 = (a, b_0)$ is done as follows:

1. If z_0 is not a free cell
   - If z_0 belongs to exactly one sphere $\mathcal{S}(\varsigma)$, then $z_0 \in \mathcal{N}(\varsigma)$.
   - If $z_0 \in \mathcal{I}(\varsigma_1, \varsigma_2)$ and $\mathcal{I}(\varsigma_1, \varsigma_2)$ is a deep-intersection, then $z_0 \in \mathcal{N}(\varsigma_1) = \mathcal{N}(\varsigma_2)$.
   - If $z_0 \in \mathcal{I}(\varsigma_1, \varsigma_2)$, $\varsigma_1 = (\alpha, \beta)$, $\varsigma_2 = (\gamma, \delta)$, $\beta < \delta$, and $\mathcal{I}(\varsigma_1, \varsigma_2)$ is a line-intersection, then $z_0 \in \mathcal{N}(\varsigma_2)$.
   - If $z_0 \in \mathcal{I}(\varsigma_1, \varsigma_2)$, $\varsigma_1 = (\alpha, \beta)$, $\varsigma_2 = (\gamma, \beta)$, $\alpha < \gamma$, and $\mathcal{I}(\varsigma_1, \varsigma_2)$ is a line-intersection, then $z_0 \in \mathcal{N}(\varsigma_1)$ (note that z_0 is the right tip of $\mathcal{S}(\varsigma_1)$ and the left tip of $\mathcal{S}(\varsigma_2)$).
2. If z_0 is a free cell, then let $z_1 = (a, b_1)$, $b_1 < b_0$, be a cell in a sphere such that all the cells in the set $\{(a, d) : b_1 < d < b_0\}$ are free. If such a cell z_1 does not exist, then z_0 does not belong to any neighborhood.

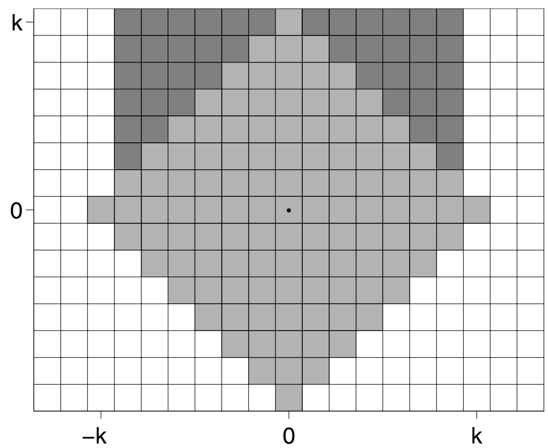


FIG. 4. *$\mathcal{U}(\varsigma)$, where ς is located at* $(0,0)$.

- If z_1 is not a tip of any sphere and $z_1 \in \mathcal{S}(\varsigma)$, then $z_0 \in \mathcal{N}(\varsigma)$.
- If z_1 is a tip of two spheres, $\mathcal{S}(\varsigma_1)$ and $\mathcal{S}(\varsigma_2)$, such that $\varsigma_1 = (\alpha, \beta)$ and $\varsigma_2 = (\alpha + 2k, \beta)$, then $z_0 \in \mathcal{N}(\varsigma_2)$.
- If z_1 is a tip of exactly one sphere, $\mathcal{S}(\varsigma)$, then let $z_2 = (a, b_2)$, $b_2 < b_1$, be a cell in a sphere such that all the cells in the set $\{(a, d) : b_2 < d < b_1\}$ are free. If such a cell z_2 does not exist, then $z_0 \in \mathcal{N}(\varsigma)$.
  - If z_2 is a cell of two distinct spheres, then $z_0 \in \mathcal{N}(\varsigma)$.
  - If z_2 is a cell of exactly one sphere $\mathcal{S}(\varsigma_1)$ and $\mathcal{U}(\varsigma_1) \cap \mathcal{S}(\varsigma) = \varnothing$, then $z_0 \in \mathcal{N}(\varsigma)$.
  - If z_2 is a cell of exactly one sphere $\mathcal{S}(\varsigma_1)$ and $\mathcal{U}(\varsigma_1) \cap \mathcal{S}(\varsigma) \neq \varnothing$, then $z_0 \in \mathcal{N}(\varsigma_1)$ if $z_0 \in \mathcal{P}(\varsigma_1, \varsigma)$ and $z_0 \in \mathcal{N}(\varsigma)$ if $z_0 \notin \mathcal{P}(\varsigma_1, \varsigma)$.

Note that in all cases, except one, z_0 is assigned to some neighborhood. The case analysis of the definition makes it clear that z_0 is assigned to at most one neighborhood. We would like to clarify that if z_0 is a free cell and z_1 is not a tip, then $z_1 \in \mathcal{I}(\varsigma_1, \varsigma_2)$, $\varsigma_1 \neq \varsigma_2$, only if $\mathcal{I}(\varsigma_1, \varsigma_2)$ is a deep-intersection and hence z_0 is assigned to exactly one neighborhood, $\mathcal{N}(\varsigma_1) = \mathcal{N}(\varsigma_2)$, in this case. Thus we have the following lemma.

LEMMA 4. *Each cell z_0 in $\mathbb{Z}^2$ belongs to at most one neighborhood.*

COROLLARY 5. *For two distinct black cells ς_1, ς_2, $\mathcal{N}(\varsigma_1) \cap \mathcal{N}(\varsigma_2) = \mathcal{N}(\varsigma_1) = \mathcal{N}(\varsigma_2)$ iff $\mathcal{I}(\varsigma_1, \varsigma_2)$ is a deep-intersection, and $\mathcal{N}(\varsigma_1) \cap \mathcal{N}(\varsigma_2) = \varnothing$ iff $\mathcal{I}(\varsigma_1, \varsigma_2)$ is not a deep-intersection.*

As a consequence of Corollary 5, we will denote by $\mathcal{N}(\varsigma_1, \varsigma_2)$ the common neighborhood of two black cells ς_1, ς_2 for which $\mathcal{I}(\varsigma_1, \varsigma_2)$ is a deep-intersection.

LEMMA 5. *If $\mathcal{S}(\varsigma_1) \cap \mathcal{S}(\varsigma_2)$ is a tip, where $\varsigma_1 = (\alpha, \beta)$, $\varsigma_2 = (\alpha + 2k, \beta)$, then $(\alpha + k, \beta) \in \mathcal{N}(\varsigma_1)$ and $(\alpha + k, \beta + 1) \in \mathcal{N}(\varsigma_2)$.*

Proof. By the definition of $\mathcal{N}(\varsigma_1)$ it is obvious that $(\alpha + k, \beta) \in \mathcal{N}(\varsigma_1)$. By Lemma 12 (see Appendix B), $(\alpha + k, \beta + 1) \in \mathcal{P}(\varsigma_1, \varsigma_2)$; therefore by Corollary 1 we have that $(\alpha + k, \beta + 1)$ is a free cell. Thus, by the definition of $\mathcal{N}(\varsigma_2)$ we have that $(\alpha + k, \beta + 1) \in \mathcal{N}(\varsigma_2)$. ☐

2.3. The size of a neighborhood. In this subsection we give a lower bound on the sizes of the neighborhoods defined in subsection 2.2. We first sketch the outline of the proof. There are two cases:

- If ς is a black cell for which $\mathcal{S}(\varsigma)$ does not have a deep-intersection with another sphere, we want to show a lower bound of $3k^2$ on $|\mathcal{N}(\varsigma)|$. If all the cells of $\mathcal{U}(\varsigma)$ are free, then $|\mathcal{N}(\varsigma) \cap \mathcal{S}(\varsigma)| + |\mathcal{U}(\varsigma)|$ is sufficient to obtain the bound. If $\mathcal{U}(\varsigma) \cap \mathcal{S}(\varsigma_1) \neq \varnothing$ (w.l.o.g. $\mathcal{U}_r(\varsigma) \cap \mathcal{S}(\varsigma_1) \neq \varnothing$) for some black cell ς_1, then $|\mathcal{N}(\varsigma) \cap \mathcal{P}(\varsigma, \varsigma_1)| + |\mathcal{U}_l(\varsigma) \setminus \mathcal{P}(\varsigma, \varsigma_1)|$ is sufficient to obtain the bound. This will complete the proof in most cases. If $\mathcal{U}_l(\varsigma) \cap \mathcal{S}(\varsigma_2) \neq \varnothing$ for some black cell ς_2, then in some cases we consider $|\mathcal{N}(\varsigma) \cap (\mathcal{P}(\varsigma, \varsigma_2) \setminus \mathcal{P}(\varsigma, \varsigma_1))|$.
- If ς_1, ς_2 are black cells for which $\mathcal{I}(\varsigma_1, \varsigma_2)$ is a deep-intersection, then we have to show a lower bound of $6k^2$ on $|\mathcal{N}(\varsigma_1, \varsigma_2)|$. In this case we consider $|\mathcal{N}(\varsigma_1, \varsigma_2) \cap \mathcal{P}(\varsigma_1, \varsigma_2)| + |(\mathcal{U}(\varsigma_1) \cup \mathcal{U}(\varsigma_2)) \setminus \mathcal{P}(\varsigma_1, \varsigma_2)|$. This will complete the proof in most cases. If $\mathcal{U}(\varsigma_1) \cap \mathcal{S}(\varsigma_3) \neq \varnothing$ (or $\mathcal{U}(\varsigma_2) \cap \mathcal{S}(\varsigma_4) \neq \varnothing$) for some black cell ς_3 (ς_4), then we consider $|\mathcal{N}(\varsigma_1, \varsigma_2) \cap (\mathcal{P}(\varsigma_1, \varsigma_3) \setminus \mathcal{P}(\varsigma_1, \varsigma_2))|$ (or $|\mathcal{N}(\varsigma_1, \varsigma_2) \cap (\mathcal{P}(\varsigma_2, \varsigma_4) \setminus \mathcal{P}(\varsigma_1, \varsigma_2))|$).

Each case will be proved in a separate lemma. For the proof of the first lemma, we also need the following result, which can be easily verified.

LEMMA 6. *Let $\varsigma_1 = (\alpha, \beta)$, $\varsigma_2 = (\gamma, \delta)$ be two black cells such that $\mathcal{I}(\varsigma_1, \varsigma_2)$ is not a deep-intersection. If $\mathcal{U}(\varsigma_1) \cap \mathcal{S}(\varsigma_2) \neq \varnothing$, then the following three conditions hold:*

$$\begin{array}{ccccc} 2k & \leq & |\gamma - \alpha| + |\delta - \beta| & < & 3k, \\ 0 & < & |\gamma - \alpha| & < & 2k, \\ 2 & \leq & \delta - \beta & \leq & 2k. \end{array}$$

Let $F(\varsigma)$ denote the set of free cells in $\mathcal{N}(\varsigma)$.

LEMMA 7. *If $k \geq 2$, and if ς is a black cell for which $\mathcal{S}(\varsigma)$ does not have a deep-intersection with another sphere, then $|\mathcal{N}(\varsigma)| \geq 3k^2$.*

Proof. Let ς be a black cell for which $\mathcal{S}(\varsigma)$ does not have a deep-intersection with another sphere. W.l.o.g. we can assume that ς is located at $(0, 0)$. We distinguish between two cases.

Case 1. $\mathcal{S}(\varsigma)$ does not intersect any other sphere $\mathcal{S}(\varsigma_1)$.

We have to compute the number of free cells in the neighborhood of ς. For this computation we first consider the set of cells $\mathcal{U}(\varsigma)$, defined earlier and depicted in Figure 4. We distinguish between the following subcases.

Case 1.1. All the cells of $\mathcal{U}(\varsigma)$ are free.

Therefore all the cells of $\mathcal{U}(\varsigma)$ belong to the neighborhood of ς. Hence $\mathcal{N}(\varsigma) \supseteq \mathcal{S}(\varsigma) \cup \mathcal{U}(\varsigma)$ and

$$|\mathcal{N}(\varsigma)| \geq (2k^2 + 2k + 1) + k(k-1) = 3k^2 + k + 1 > 3k^2$$

as required.

In the following two subcases there is a sphere $\mathcal{S}(\varsigma_1)$, $\varsigma_1 = (\alpha, \beta)$, which intersects the set $\mathcal{U}(\varsigma)$. W.l.o.g. we can assume that if $\mathcal{S}(\varsigma_2)$, $\varsigma_2 = (\gamma, \delta)$, also intersects $\mathcal{U}(\varsigma)$, then $|\gamma| \geq |\alpha|$. W.l.o.g. we can also assume that $\alpha > 0$.

Case 1.2. $\alpha \leq k$.

We claim that the number of cells in the union of the sphere of ς with the free cells of $\mathcal{N}(\varsigma)$ inside the polygon of ς and ς_1, $\mathcal{P}(\varsigma, \varsigma_1)$, is at least $3k^2$, i.e.,

$$|(F(\varsigma) \cap \mathcal{P}(\varsigma, \varsigma_1)) \cup \mathcal{S}(\varsigma)| \geq 3k^2.$$

The set $(F(\varsigma) \cap \mathcal{P}(\varsigma, \varsigma_1)) \cup \mathcal{S}(\varsigma)$ (an example is given in Figure 5), which is only a

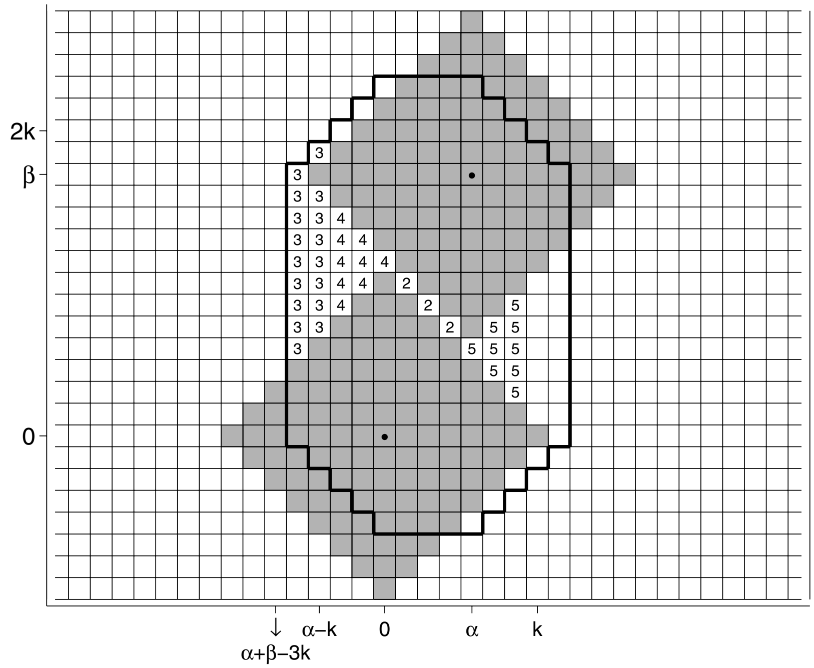


FIG. 5. *Case* 1.2 *with* $(\alpha, \beta) = (4, 12)$ *and* $k = 7$.

subset of $\mathcal{N}(\varsigma)$, contains the following disjoint subsets of cells:

(1) The sphere of ς, $\mathcal{S}(\varsigma)$, whose size is $2k^2 + 2k + 1$.

(2) The free cells of $\mathcal{N}(\varsigma)$ whose columns are between the top of $\mathcal{S}(\varsigma)$ and the bottom of $\mathcal{S}(\varsigma_1)$ (exclusive). This set of cells,

$$\{(x, y) : 0 < x < \alpha,\ k < x + y < \alpha + \beta - k\},$$

defines a parallelogram whose size is $(\alpha - 1)(\alpha + \beta - 2k - 1)$.

(3) The free cells of $\mathcal{N}(\varsigma)$ inside $\mathcal{P}(\varsigma, \varsigma_1)$ whose columns are between the left column of $\mathcal{P}(\varsigma, \varsigma_1)$ and the left tip of $\mathcal{S}(\varsigma_1)$ (inclusive). This set of cells,

$$\{(x, y) : \alpha + \beta - 3k + 1 \leq x \leq \alpha - k,\ k < y - x < 3k - \alpha\} \setminus \{(\alpha - k, \beta)\},$$

defines a parallelogram with a missing cell, $(\alpha - k, \beta)$, which is the left tip of $\mathcal{S}(\varsigma_1)$. The size of this set is $(2k - \beta)(2k - \alpha - 1) - 1$ if $\beta < 2k$ and 0 if $\beta = 2k$.

(4) The free cells of $\mathcal{N}(\varsigma)$ whose columns are between the left tip of $\mathcal{S}(\varsigma_1)$ (exclusive) and the top of $\mathcal{S}(\varsigma)$ (inclusive). This set is an arithmetic progression whose size is $(k - \alpha)(\beta - k - 2)$.

(5) The free cells of $\mathcal{N}(\varsigma)$ whose columns are between the bottom of $\mathcal{S}(\varsigma_1)$ (inclusive) and the right tip of $\mathcal{S}(\varsigma)$ (exclusive). This set is an arithmetic progression whose size is $(k - \alpha)(\beta - k - 2)$.

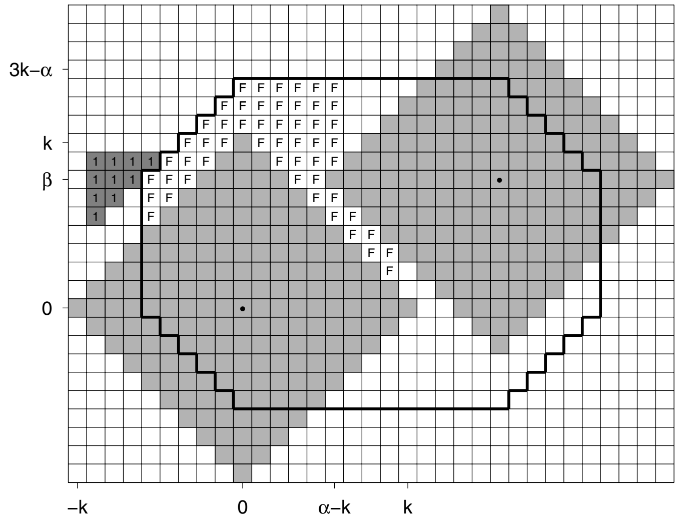


FIG. 6. *Case* 1.3 *with* $(\alpha,\beta) = (14,7)$ *and* $k = 9$.

Note that all the sets in (2) through (5) are indeed contained in $\mathcal{P}(\varsigma,\varsigma_1)$. Therefore

$$\begin{aligned}
&|(F(\varsigma)\cap\mathcal{P}(\varsigma,\varsigma_1))\cup\mathcal{S}(\varsigma)|\\
&\geq 2k^2+2k+1\\
&+(\alpha-1)(\alpha+\beta-2k-1)\\
&+(2k-\beta)(2k-\alpha-1)-1\\
&+2(k-\alpha)(\beta-k-2)\\
&=4k^2-2k(\alpha+1)+(\alpha+1)^2.
\end{aligned}$$

The minimum of $4k^2-2k(\alpha+1)+(\alpha+1)^2$ is when $\alpha+1=k$, and hence

$$|\mathcal{N}(\varsigma)|\geq|(F(\varsigma)\cap\mathcal{P}(\varsigma,\varsigma_1))\cup\mathcal{S}(\varsigma)|\geq 3k^2$$

as claimed.

Case 1.3. $\alpha > k$.

In a similar way to Case 1.2 we consider the set $(F(\varsigma)\cap\mathcal{P}(\varsigma,\varsigma_1))\cup\mathcal{S}(\varsigma)$ (an example is given in Figure 6) and compute its size. We obtain

$$|(F(\varsigma)\cap\mathcal{P}(\varsigma,\varsigma_1))\cup\mathcal{S}(\varsigma)|\geq\frac{5k^2}{2}+\frac{k}{2}(2\alpha-1)+\frac{1}{2}(-\alpha^2+\alpha+2).$$

Next, we consider $\mathcal{U}_l(\varsigma)\setminus\mathcal{P}(\varsigma,\varsigma_1)$, which contains the following isosceles right triangle (depicted in Figure 6):

$$TR_1=\{(x,y):-k+1\leq x,\ y\leq k-1,\ y-x\geq 3k-\alpha\}.$$

The size of TR_1 is $\frac{1}{2}(\alpha-k)(\alpha-k-1)$. If all the cells in $\mathcal{U}_l(\varsigma)\setminus\mathcal{P}(\varsigma,\varsigma_1)$ are free, then they are clearly free cells of $\mathcal{N}(\varsigma)$. Hence

$$|\mathcal{N}(\varsigma)|\geq|(F(\varsigma)\cap\mathcal{P}(\varsigma,\varsigma_1))\cup\mathcal{S}(\varsigma)|+|TR_1|\geq 3k^2+1.$$

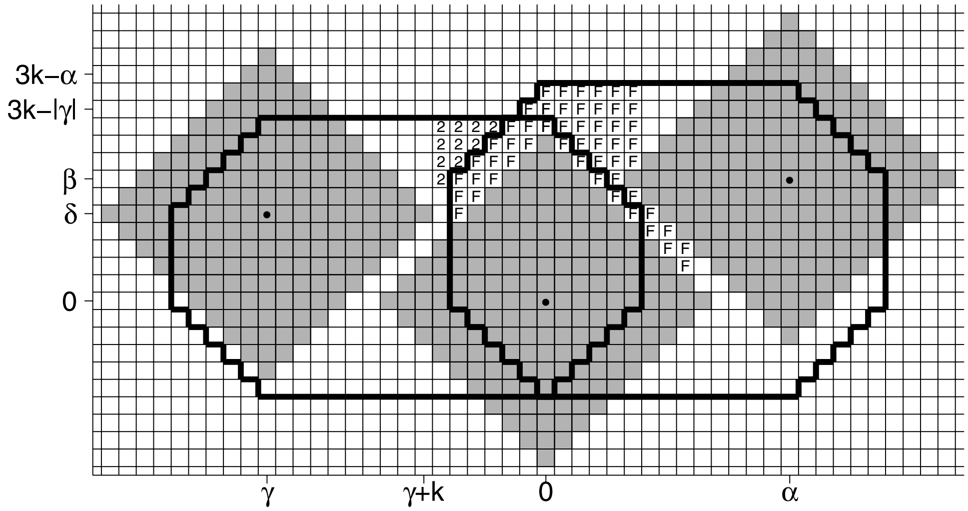


FIG. 7. *Case* 1.3 *with* $(\alpha, \beta) = (14, 7)$, $(\gamma, \delta) = (-16, 5)$, $k = 9$. *The cells marked by "2" belong to* TR_2.

To complete the proof we have to consider the case of a black cell $\varsigma_2 = (\gamma, \delta)$ such that $\mathcal{S}(\varsigma_2) \cap (\mathcal{U}_l(\varsigma) \setminus \mathcal{P}(\varsigma, \varsigma_1))$ is not empty. By the minimality of α we have $|\gamma| \geq \alpha$ and hence $|\gamma| > k$. We consider now another isosceles right triangle (depicted in Figure 7):

$$TR_2 = \{(x, y) : \gamma + k + 1 \leq x,\ y \leq 3k - |\gamma| - 1,\ y - x \geq 3k - \alpha\}.$$

Note that $TR_2 \subset \mathcal{P}(\varsigma, \varsigma_2)$ since $\alpha \leq |\gamma|$ and TR_2 does not intersect $\mathcal{P}(\varsigma, \varsigma_1) \cup \mathcal{S}(\varsigma_2)$. Hence, all the cells of TR_2 are free cells of $\mathcal{N}(\varsigma)$. The size of TR_2 is $\frac{1}{2}(\alpha - k)(\alpha - k - 1) = |TR_1|$. Therefore

$$|\mathcal{N}(\varsigma)| \geq |(F(\varsigma) \cap \mathcal{P}(\varsigma, \varsigma_1)) \cup \mathcal{S}(\varsigma)| + |TR_2| \geq 3k^2 + 1,$$

which completes the proof of this case.

Case 2. $\mathcal{S}(\varsigma)$ intersects other spheres and any such intersection is a line-intersection.

The proof is very similar to the proof of Case 1. If each cell (x, y), $|x| + y = k$ and $y > 0$, belongs only to $\mathcal{S}(\varsigma)$ (and not to other spheres), then by the definition of $\mathcal{N}(\varsigma)$ and Lemma 5 we have $\mathcal{S}(\varsigma) \subset \mathcal{N}(\varsigma)$ or $(\mathcal{S}(\varsigma) \cup \{(-k, 1)\}) \setminus \{(-k, 0)\} \subset \mathcal{N}(\varsigma)$, and hence the proof is identical to the one of Case 1.

Therefore, we assume that there exists $(x, y) \in \mathcal{I}(\varsigma, \varsigma_1)$, $\varsigma \neq \varsigma_1$, $\varsigma_1 = (\alpha, \beta)$, $y > 0$, and $|x| + y = k$. W.l.o.g. we assume that $\alpha \geq 0$. We distinguish between the following subcases.

Case 2.1. $\alpha = 0$.

By Lemma 1(b) we have $\beta = 2k$ and hence $\mathcal{U}(\varsigma) \subset \mathcal{P}(\varsigma, \varsigma_1)$ by Lemma 3. It follows by Corollary 4 and the definition of $\mathcal{N}(\varsigma)$ that $\mathcal{N}(\varsigma) \supset (\mathcal{S}(\varsigma) \cup \mathcal{U}(\varsigma)) \setminus \{(0, k), (-k, 0)\}$. Thus, as in Case 1.1 we have $|\mathcal{N}(\varsigma)| \geq 3k^2 + k - 1 > 3k^2$, as required.

Case 2.2. $0 < \alpha \leq k$.

The reasonings are identical to the ones in Case 1.2 with the following exceptions:

- The value in (2) is negative since it represents part of $\mathcal{I}(\varsigma, \varsigma_1)$ rather than free cells, which should be substracted from $\mathcal{S}(\varsigma)$ and, as a consequence, from $\mathcal{N}(\varsigma)$.

- Also in (4) one cell belongs to $\mathcal{I}(\varsigma, \varsigma_1)$, which causes the arithmetic progression to start in -1. The same is true for (5).
- By Corollary 4, $\mathcal{S}(\varsigma)$ can intersect another sphere $\mathcal{S}(\varsigma_2)$. If $\mathcal{I}(\varsigma, \varsigma_2)$ is a tip of both spheres, then we can always assume for the sake of the proof that $\mathcal{I}(\varsigma, \varsigma_2) \subset \mathcal{N}(\varsigma)$ by Lemma 5. By Corollary 4 and the definition of $\mathcal{N}(\varsigma)$, $\mathcal{I}(\varsigma, \varsigma_2) \subset \mathcal{N}(\varsigma)$ also if $\mathcal{I}(\varsigma, \varsigma_2)$ is not a tip. Therefore, we compute the size of $\mathcal{S}(\varsigma)$ as part of the size of $\mathcal{N}(\varsigma)$ and substract $\mathcal{I}(\varsigma, \varsigma_1)$ from $\mathcal{N}(\varsigma)$ in (2), (4), and (5).

$*$ The only case where the computation is different is when $\alpha = k$. In this case the values in (4) and (5) should be -1 and not 0. In (1) the left tip of $\mathcal{S}(\varsigma_1)$, $(0, k)$, does not belong to the set of cells, and hence the size of the set should be $k(k-1)$ and not $k(k-1) - 1$.

Therefore,

$$|\mathcal{N}(\varsigma)| \geq 4k^2 - 2k(\alpha + 1) + (\alpha + 1)^2 - 1 = 3k^2.$$

Case 2.3. $\alpha > k$.

In a similar way to Cases 1.3 and 2.2 we consider the set $(F(\varsigma) \cap \mathcal{P}(\varsigma, \varsigma_1)) \cup (\mathcal{S}(\varsigma) \setminus \mathcal{I}(\varsigma, \varsigma_1))$ and compute its size. We obtain

$$|(F(\varsigma) \cap \mathcal{P}(\varsigma, \varsigma_1)) \cup (\mathcal{S}(\varsigma) \setminus \mathcal{I}(\varsigma, \varsigma_1))| \geq \frac{5k^2}{2} + \frac{k}{2}(2\alpha - 1) + \frac{1}{2}(-\alpha^2 + \alpha).$$

By Lemmas 13, 14, and 15 (see Appendix B), we have that

$$|F(\varsigma) \setminus \mathcal{P}(\varsigma, \varsigma_1)| \geq \frac{1}{2}(\alpha - k)(\alpha - k - 1)$$

and hence

$$|\mathcal{N}(\varsigma)| \geq |(F(\varsigma) \cap \mathcal{P}(\varsigma, \varsigma_1)) \cup (\mathcal{S}(\varsigma) \setminus \mathcal{I}(\varsigma, \varsigma_1))| + |F(\varsigma) \setminus \mathcal{P}(\varsigma, \varsigma_1)| \geq 3k^2.$$

This completes the proof of the lemma. $\square$

Note that Case 1.3 can be solved similarly to Case 2.3, but the proof given in Case 1.3 is much simpler.

The proof of the following lemma has some similarity to the proof of Lemma 7. It is also very detailed, and hence it will be given in Appendix B.

LEMMA 8. *If $k \geq 2$, and if ς_1 and ς_2 are two black cells such that $\mathcal{I}(\varsigma_1, \varsigma_2)$ is a deep-intersection, then $|\mathcal{N}(\varsigma_1, \varsigma_2)| > 6k^2$.*

Proof of Theorem 1. If $k = 1$, the proof is trivial, and we leave it to the reader. Therefore, we assume that $k \geq 2$. Let $A(n)$ be any $n \times n$ subarray of $\mathbb{Z}^2$. For a color i let μ_i be the density of the cells in $\mathbb{Z}^2$ colored by i, i.e.,

$$\mu_i = \lim_{n \to \infty} \sup \frac{|\mathcal{F}^{-1}(i) \cap A(n)|}{n^2}.$$

Let χ be the number of colors in $\mathcal{F}$. Clearly, $\sum_{i=1}^{\chi} \mu_i = 1$, and by Lemmas 7 and 8 $\mu_i \leq \frac{1}{3k^2}$ for each i, $1 \leq i \leq \chi$. Hence, $1 = \sum_{i=1}^{\chi} \mu_i \leq \frac{\chi}{3k^2}$, and thus $\chi \geq 3k^2$. $\square$

3. An infinite family of optimal coloring. In this section we consider again only $4k$-colorings. We already know that any such coloring requires $3k^2$ colors. We would like to identify all the optimal $4k$-colorings, i.e., $4k$-colorings with $3k^2$ colors. First, note that the size of a neighborhood is at least $3k^2$.

CONJECTURE 1. *In an optimal $4k$-coloring each neighborhood has size $3k^2$.*

Remarks.

1. Note that a neighborhood can be defined for each cell of $\mathbb{Z}^2$ and not just for black cells.
2. The idea contained in Conjecture 1 is that in an optimal $4k$-coloring the "average" size of a neighborhood for each cell is $3k^2$. This idea is preserved if we rephrase the conjecture as "In an optimal $4k$-coloring each neighborhood with one black cell has size $3k^2$, and each neighborhood with two black cells has size $6k^2$." However, by Lemma 8, the size of a neighborhood with two black cells is greater than $6k^2$. Hence, we consider only the case where all the neighborhoods have size $3k^2$.

A coloring will be called *strongly optimal* if it satisfies the conjecture. One optimal coloring defined by a lattice Λ^R was given in [3]. The generator matrix of this lattice is

$$G^R = \begin{pmatrix} k & k \\ 0 & 3k \end{pmatrix}.$$

An isomorphic lattice Λ^L which also defines an optimal coloring has the generator matrix

$$G^L = \begin{pmatrix} -k & k \\ 0 & 3k \end{pmatrix}.$$

The cells of a given color in a strongly optimal coloring have a certain structure, as will be proved in what follows. Examples are depicted in Figure 8. We define the following sets:

$$\begin{aligned} \mathcal{D}_0^R &= \{(i,i) : i \in \mathbb{Z}\}, \\ \mathcal{D}_j^R &= (0,j) + \mathcal{D}_0^R, \qquad j \in \mathbb{Z}, \\ \mathcal{D}_0^L &= \{(-i,i) : i \in \mathbb{Z}\}, \\ \mathcal{D}_j^L &= (0,j) + \mathcal{D}_0^L, \qquad j \in \mathbb{Z}. \end{aligned}$$

A *shift vector* $\overrightarrow{s} = (\ldots, \overrightarrow{s}(-1), \overrightarrow{s}(0), \overrightarrow{s}(1), \ldots)$ is a function $\overrightarrow{s} : \mathbb{Z} \to \{0, 1, \ldots, k-1\}$. For each shift vector and integer h, $0 \le h \le 3k-1$, we define two one-to-one functions, $T^R_{\overrightarrow{s},h} : \Lambda^R \to \mathbb{Z}^2$, $T^L_{\overrightarrow{s},h} : \Lambda^L \to \mathbb{Z}^2$, as follows:

$$\begin{aligned} T^R_{\overrightarrow{s},h}((ik, 3jk+ik)) &= (ik + \overrightarrow{s}(j), h + 3jk + ik + \overrightarrow{s}(j)), \\ T^L_{\overrightarrow{s},h}((-ik, 3jk+ik)) &= (-ik - \overrightarrow{s}(j), h + 3jk + ik + \overrightarrow{s}(j)). \end{aligned}$$

The images $T^R_{\overrightarrow{s},h}(\Lambda^R)$ and $T^L_{\overrightarrow{s},h}(\Lambda^L)$ will be called *templates.*

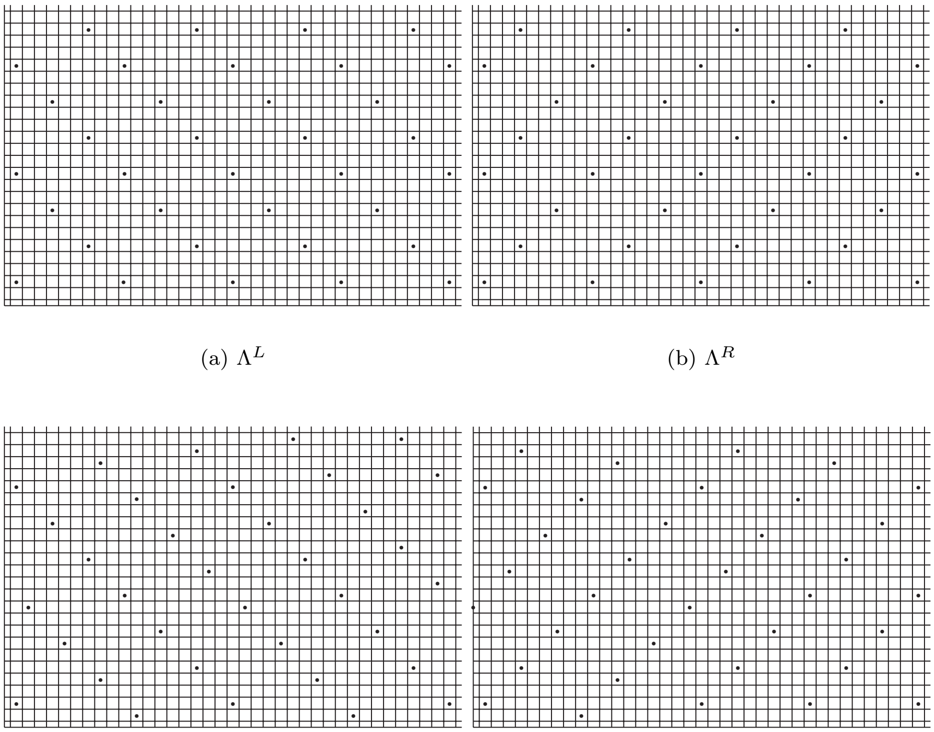
LEMMA 9. *If $\mathcal{F}$ is a strongly optimal coloring and φ is one of its colors, then the set of cells colored with φ, i.e., $\mathcal{F}^{-1}(\varphi)$, is a template.*

The proof of Lemma 9 is given in Appendix C. Examples of templates are given in Figure 8. Λ^R and Λ^L are templates with $h = 0$ and an allzero shift vector. Other templates are obtained from Λ^R in two steps:

- We lift the lattice by h; i.e., we obtain the set $(0,h) + \Lambda^R$.
- We shift the cells of $(0,h) + \Lambda^R$ in the diagonal $\mathcal{D}^R_{h+3jk}$ by $\overrightarrow{s}(j)$ to the right.

Similar templates are obtained from Λ^L.

Given a color φ, we say that φ is *R-oriented* (L-oriented) if the set of cells colored with φ is a template obtained from Λ^R (Λ^L).



(a) Λ^L (b) Λ^R

(c) $T^L(\Lambda^L)$ (d) $T^R(\Lambda^R)$

FIG. 8. *Lattices and templates.*

COROLLARY 6. *An R-oriented (L-oriented) color φ appears in the diagonal $\mathcal{D}_j^R$ ($\mathcal{D}_j^L$) iff it appears in the diagonal $\mathcal{D}_{j+3k}^R$ ($\mathcal{D}_{j+3k}^L$).*

LEMMA 10. *All the colors of the cells in the lattice $\Delta_{2k} = \{(ik, jk) : i + j \equiv 0 \pmod 2\}$ have the same orientation (R-oriented or L-oriented).*

Proof. Let $\varsigma_1 = (\alpha, \beta) \in \Delta_{2k}$; W.l.o.g. we assume that $\varphi_1 = \mathcal{F}(\varsigma_1)$ is R-oriented. By definition also $(\alpha + k, \beta + k)$ is colored with φ_1. Let $\varsigma_2 = (\alpha + k, \beta - k)$ and $\varphi_2 = \mathcal{F}(\varsigma_2)$. If φ_2 is L-oriented, then (α, β) is also colored with φ_2, a contradiction. Hence, φ_2 is R-oriented, and it can be easily verified that all the colors of the cells in Δ_{2k} have the same orientation. ∎

COROLLARY 7. *For $(\alpha, \beta) \in \mathbb{Z}^2$ all the colors of the cells in the set $(\alpha, \beta) + \Delta_{2k}$ have the same orientation.*

There are $2k^2$ disjoint cosets of Δ_{2k}. By Corollary 7, all the colors of the cells in a given coset have the same orientation. We say that the coset $(\alpha, \beta) + \Delta_{2k}$ is R-oriented (L-oriented) if the colors of the cells in the coset are R-oriented (L-oriented). We say that a cell $(\alpha, \beta) \in \mathbb{Z}^2$ is R-oriented (L-oriented) if it belongs to an R-oriented (L-oriented) coset. One can easily verify that a possible set of coset representatives is the set $\{(-j + i, j + i) : 0 \leq i, j < k\} \cup ((0, 1) + \{(-j + i, j + i) : 0 \leq i, j < k\})$. Note that the coset representative $(-j + i, j + i)$ lies in the intersection of the lines $y = x + 2j$ and $y = -x + 2i$ (an example is depicted in Figure 9). Clearly, we have the following lemma.

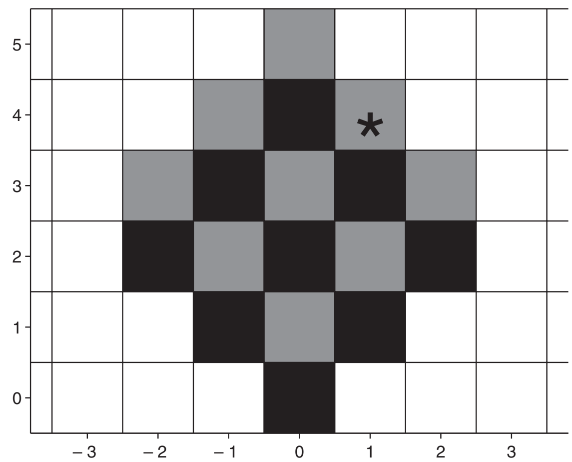


FIG. 9. *The set of coset representatives for $k = 3$. The coset representative marked by $*$ is $(1,4) = (-1+2, 1+2+1)$; it lies in the intersection of the lines $y = x+2\cdot 1+1$ and $y = -x+2\cdot 2+1$.*

LEMMA 11. *The number of R-oriented (L-oriented) colors on the diagonal $\mathcal{D}_j^R$ ($\mathcal{D}_j^L$) is equal to the number of R-oriented (L-oriented) cosets of Δ_{2k} which intersect $\mathcal{D}_j^R$ ($\mathcal{D}_j^L$).*

Since $(0, 2k) \in \Delta_{2k}$, it follows from Lemma 11 that the number of R-oriented (L-oriented) colors on the diagonal $\mathcal{D}_j^R$ ($\mathcal{D}_j^L$) is equal to the number of R-oriented (L-oriented) cosets of Δ_{2k} which intersect $\mathcal{D}_{j-2k}^R$ ($\mathcal{D}_{j-2k}^L$). Hence, by Corollary 6, we have the following corollary.

COROLLARY 8. *The number of R-oriented (L-oriented) cosets of Δ_{2k} which intersect $\mathcal{D}_j^R$ ($\mathcal{D}_j^L$) is equal to the number of R-oriented (L-oriented) cosets of Δ_{2k} which intersect $\mathcal{D}_{j+k}^R$ ($\mathcal{D}_{j+k}^L$).*

Since the number of cosets of Δ_{2k} which intersect $\mathcal{D}_j^R$ ($\mathcal{D}_j^L$) is exactly k, we have the following corollary.

COROLLARY 9. *The number of R-oriented (L-oriented) cosets of Δ_{2k} which intersect $\mathcal{D}_j^L$ ($\mathcal{D}_j^R$) is equal to the number of R-oriented (L-oriented) cosets of Δ_{2k} which intersect $\mathcal{D}_{j+k}^L$ ($\mathcal{D}_{j+k}^R$).*

An example for possible assignment of orientations to the cosets as forced by Corollaries 8 and 9 is given in Figure 10. Once the orientation of each coset is determined, we can color $\mathbb{Z}^2$. We first color all the R-oriented cells. Consider the parallelogram $M^R = \{(x, y) : 0 \le x < k,\ 0 \le y - x < 3k\}$. By Lemma 9, each R-oriented color appears exactly once in M^R. Let ψ be the number of colors which are R-oriented. We assign the ψ R-oriented colors arbitrarily to R-oriented cells in M^R. By Lemma 9, $\mathcal{F}((\alpha, \beta)) = \mathcal{F}((\alpha + k, \beta + k))$, and hence we have an assignment of colors to all the R-oriented cells in the strip $\{(x, y) : 0 \le y - x < 3k\}$. By Corollary 6, $\mathcal{D}_j^R$ and $\mathcal{D}_{j+3k}^R$ have the same R-oriented colors, but the assignment of the R-oriented colors in one diagonal is independent in the assignment in the other diagonals; i.e., in each diagonal $\mathcal{D}_j^R$ the order of the R-oriented colors along the diagonal can be different. In the same way we assign the L-oriented colors to the L-oriented cells, by using a parallelogram $M^L = \{(x, y) : 0 \le x < k,\ 0 \le x + y < 3k\}$.

We will now give a procedure which will enable us to determine all the possible assignments of orientations to the $2k^2$ cosets of Δ_{2k}. Recall that each coset $(-j +$

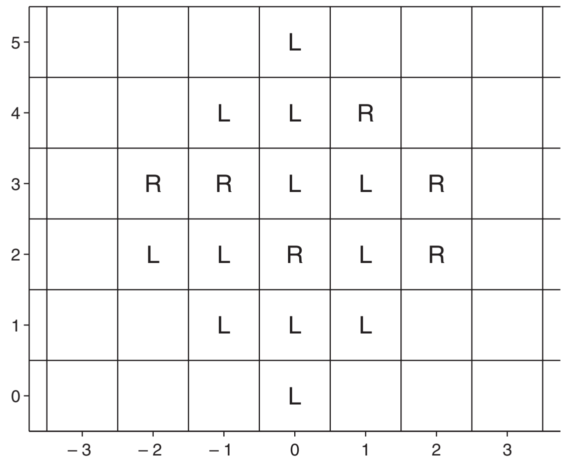


FIG. 10. *A possible assignment of orientations to the coset representatives for $k = 3$, which satisfies Corollaries* 8 *and* 9.

$i, j+i)$ (or $(-j+i, j+i+1)$) is associated with the lines $y = x+2j$ $(y = x+2j+1)$ and $y = -x+2i$ $(y = -x+2i+1)$.

We define a bipartite graph $G(V, E)$ (coset orientation graph) as follows: $V = V^L \cup V^R$, where

$$\begin{aligned} V^L &= \{(2i+b, L) : i \in \mathbb{Z}_k,\ b \in \mathbb{Z}_2\}, \\ V^R &= \{(2j+b, R) : j \in \mathbb{Z}_k,\ b \in \mathbb{Z}_2\}, \\ E &= \{\{(2i+b, L),\ (2j+b, R)\} : i, j \in \mathbb{Z}_k,\ b \in \mathbb{Z}^2\}. \end{aligned}$$

The edge $\{(2i+b, L),\ (2j+b, R)\}$ corresponds to the coset representative $(-j+i, j+i+b)$, which lies on the intersection of the lines $y = x+2j+b$ and $y = -x+2i+b$.

Each edge will get an assignment of either R or L, where the assignment will indicate the orientation of the corresponding coset. The only constraints on any assignment of orientations are Corollaries 8 and 9. For a vertex $v \in V$, let $\deg_L[v]$ $(\deg_R[v])$ denote the number of L-oriented edges incident to v. Note that $\deg_L[v] + \deg_R[v] = k$ for each $v \in V$. Corollary 8 implies that for $v = (\xi, L),\ 0 \le \xi \le k-1,\ \deg_L[(\xi, L)] = \deg_L[(\xi+k, L)]$, and Corollary 9 implies that for $v = (\xi, R),\ 0 \le \xi \le k-1,\ \deg_L[(\xi, R)] = \deg_L[(\xi+k, R)]$.

The procedure given below is not deterministic. It will produce all possible assignments. However, note that most of the assignments can be generated by different choices of the procedure.

THE ASSIGNMENT PROCEDURE.

- **Initialization:** $\deg_L[v] = 0$ for all $v \in V$.

(P.1) If $\deg_L[v] = k$ for all $v \in V$, then stop.
Either goto (P.2) or goto (P.5).

(P.2) Let $v_1 = (\xi, L)$ and $v_2 = (\xi+k, L)$ be two vertices such that $\deg_L[v_1] < k$.
Let $side := R$.

(P.3) Let $u_1 = (\xi_1, side)$ be a vertex such that $\deg_L[u_1] < k$.
Assign L to the edge $\{v_1, u_1\}$; increase $\deg_L[v_1]$ and $\deg_L[u_1]$.
Let $u_2 = (\xi_2, side)$ be a vertex such that $\deg_L[u_2] < k$.
Assign L to the edge $\{v_2, u_2\}$; increase $\deg_L[v_2]$ and $\deg_L[u_2]$.
If $u_1 \ne u_2$ and $\xi_1 \equiv \xi_2 \pmod{k}$, then goto (P.1).

(P.4) $v_1 := (\xi_1 + k \pmod{2k}, side)$.
$v_2 := (\xi_2 + k \pmod{2k}, side)$.
If $side = R$, then $side := L$, else $side := R$.
Goto (P.3).

(P.5) Assign R to all the edges which have not been assigned.

Note that the constraints of Corollaries 8 and 9 are satisfied since after (P.3) is performed either $\deg_L[(\xi, L)] = \deg_L[(\xi + k, L)]$ for all ξ, $0 \le \xi \le k-1$, or $\deg_L[(\xi, R)] = \deg_L[(\xi + k, R)]$ for all ξ, $0 \le \xi \le k-1$. Whenever (P.1) is reached both conditions hold.

From the discussion we have, it is clear that all the strongly optimal colorings are derived from "the assignment procedure." If Conjecture 1 is true, then these are all the optimal colorings. Hence we have the following conjecture.

CONJECTURE 2. *All the optimal $4k$-colorings are derived from "the assignment procedure."*

4. Some related results.

4.1. *t*-colorings with $t \neq 4k$.

THEOREM 2. *If $\mathcal{F}$ is a $(4k+2)$-coloring of $\mathbb{Z}^2$, then the number of colors in $\mathcal{F}$ is at least $3k^2 + 3k + 1$.*

Proof. Assume that for some k there exists a $(4k+2)$-coloring $\mathcal{F}$ of $\mathbb{Z}^2$ with $3k^2 + 3k$ colors. Let A be the set of $3k^2 + 3k$ colors of $\mathcal{F}$. We define the following coloring $\mathcal{F}' : \mathbb{Z}^2 \to A \times \mathbb{Z}_4$:

$$\mathcal{F}'((x,y)) = \begin{cases} \left(\mathcal{F}\left(\left\lfloor \frac{x}{2} \right\rfloor, \left\lfloor \frac{x}{2} \right\rfloor\right), 0\right), & x, y \text{ even}, \\ \left(\mathcal{F}\left(\left\lfloor \frac{x}{2} \right\rfloor, \left\lfloor \frac{x}{2} \right\rfloor\right), 1\right), & x \text{ even}, y \text{ odd}, \\ \left(\mathcal{F}\left(\left\lfloor \frac{x}{2} \right\rfloor, \left\lfloor \frac{x}{2} \right\rfloor\right), 2\right), & x \text{ odd}, y \text{ even}, \\ \left(\mathcal{F}\left(\left\lfloor \frac{x}{2} \right\rfloor, \left\lfloor \frac{x}{2} \right\rfloor\right), 3\right), & x, y \text{ odd}. \end{cases}$$

F' is an $(8k+4)$-coloring of $\mathbb{Z}^2$ with $12k^2 + 12k$ colors. However, by Theorem 1 an $(8k+4)$-coloring requires at least $3(2k+1)^2 = 12k^2 + 12k + 3$ colors, a contradiction. $\square$

CONJECTURE 3.

- *If $\mathcal{F}$ is a $(4k+1)$-coloring of $\mathbb{Z}^2$, then the number of colors in $\mathcal{F}$ is at least $3k^2 + 2k$.*
- *If $\mathcal{F}$ is a $(4k+3)$-coloring of $\mathbb{Z}^2$, then the number of colors in $\mathcal{F}$ is at least $3k^2 + 5k + 2$.*

Colorings defined by lattices which attain the bounds of Theorem 3 and Conjecture 3 were given in [3]. A proof for Conjecture 3 will imply the optimality of colorings in a graph similar to the grid graph (see [6]).

4.2. Colorings in finite arrays.

A t-coloring $\mathcal{F}$ can be defined similarly on finite arrays. Theorem 1 can be also proved on finite arrays.

THEOREM 3. *If $\mathcal{F}$ is a $4k$-coloring of a large enough $m \times n$ array, then the number of colors in $\mathcal{F}$ is at least $3k^2$.*

Proof. Let $L = 3k-1$, $R = 3k-1$, $D = 3k-1$, and $U = 4k-2$, and let m, n be integers such that

$$0 < m - U - D, \tag{2}$$

$$0 < n - R - L, \tag{3}$$

$$3k^2 - 1 < 3k^2 \frac{(m-U-D)(n-R-L)}{mn}. \tag{4}$$

Let A be an $m \times n$ array and $\mathcal{F}$ be a $4k$-coloring of A with χ colors. W.l.o.g. A is bounded by the cells $(-L,-D),(-L,-D+m),(-L+n,-D+m),(-L+n,-D)$. Let B be the $(m-U-D)\times(n-R-L)$ subarray of A, which is bounded by the cells $(0,0),(0,m-U-D),(n-R-L,m-U-D),(n-R-L,0)$. Note that by the detailed proofs of Lemmas 7 and 8, the neighborhood of any cell in B is contained in A.

For each color i, let a_i be the number of cells in B colored by i. By Lemmas 7 and 8, $3k^2 a_i \leq mn$ for each color i. Obviously,

$$\sum_{i=1}^{\chi} a_i = (m-U-D)\,(n-R-L)\,.$$

Hence

$$3k^2\,(m-U-D)\,(n-R-L) = 3k^2 \sum_{\varphi=1}^{\chi} a_{\varphi} \leq \chi mn,$$

$$3k^2 - 1 < 3k^2 \frac{(m-U-D)\,(n-R-L)}{mn} \leq \chi. \qquad \square$$

Clearly, Theorem 1 is an immediate consequence of Theorem 3, and hence we have an alternative proof of Theorem 1.

5. Conclusions. We have proved that the number of colors in a $4k$-coloring is at least $3k^2$. We have shown an infinite family of $4k$-colorings which attain this bound.

The original question can be asked more generally.

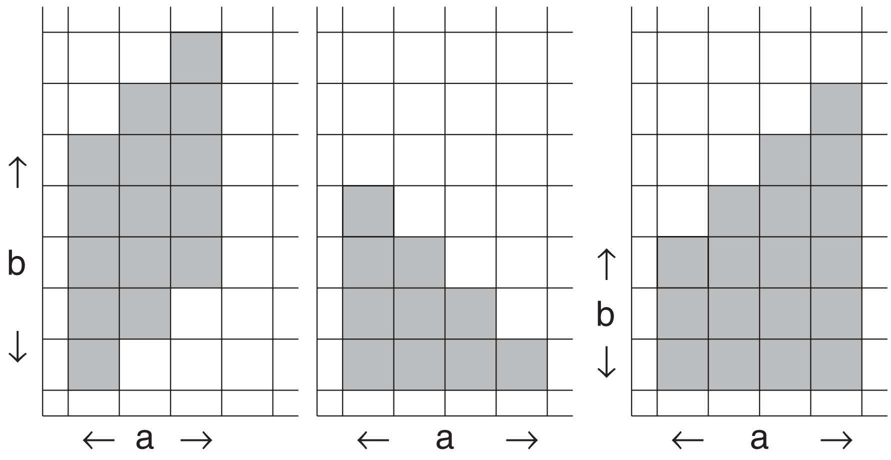
Given a positive integer r and a positive integer t, $t \geq r$, what is the smallest number of colors required to color $\mathbb{Z}^2$, such that for any r points colored with the same color, the size of the minimum spanning tree which connects them is at least t?

In this paper we discussed the case $r = 3$. Colorings with lattices for $r = 2$ are discussed in [1] and for $r \geq 3$ are discussed in [3]. For $r = 2$ all the optimal colorings can be identified with techniques which are similar to those used in section 3. Any generalization of the results in this paper for $r > 3$ would be very interesting. If a coloring defined by a lattice satisfies certain conditions, then the technique presented in section 3 can be used to obtain an infinite family of colorings with the same number of colors. Some of the colorings defined by lattices and conjectured to be optimal in [3] satisfy these conditions.

Appendix A. In the proofs of Lemmas 7 and 8 we use the areas of some geometric shapes. As these shapes are in fact polyominoes, their size is not necessarily equal to the size of the standard geometric shape.

- *Parallelogram.* The size of the parallelogram in Figure 11(a) is ab.
- *Isosceles right triangle.* The size of the isosceles right triangle in Figure 11(b), with legs of length a, is $\frac{a(a+1)}{2}$.
- *Right trapezoid.* In all the right trapezoids which are considered, the difference between any two consecutive columns is one. The size of the trapezoid in Figure 11(c) is $\frac{a(a+2b-1)}{2}$.

Note that the size of all these shapes can be computed as the sum of arithmetic progression. We also consider some arithmetic progressions with difference two between consecutive elements.



(a) A parallelogram. (b) An isosceles right triangle. (c) A right trapezoid.

Fig. 11. *Geometric shapes.*

Appendix B. In this appendix we will prove Lemma 8. First, we will give a few lemmas, some of which are also used in the proof of Lemma 7.

Lemma 12. *Let* $\varsigma_1 = (\alpha,\beta)$, $\varsigma_2 = (\gamma,\delta)$ *be two black cells such that* $|\gamma-\alpha| \leq 2k$, $0 \leq \delta-\beta \leq 2k$, $d_2(\varsigma_1,\varsigma_2) = |\gamma-\alpha| + \delta-\beta < 3k$.

If $\gamma-\alpha \geq 0$, *then*

$$\mathcal{P}(\varsigma_1,\varsigma_2) = \left\{ (x,y) : \begin{array}{ccccc} \gamma+\delta-\beta-3k & < & x & < & 3k+\alpha+\beta-\delta \\ \gamma+\delta-\alpha-3k & < & y & < & 3k+\alpha+\beta-\gamma \\ \gamma+\delta-3k & < & x+y & < & 3k+\alpha+\beta \\ \delta-\alpha-3k & < & y-x & < & 3k+\beta-\gamma \end{array} \right\}.$$

If $\gamma-\alpha < 0$, *then*

$$\mathcal{P}(\varsigma_1,\varsigma_2) = \left\{ (x,y) : \begin{array}{ccccc} \alpha-\beta+\delta-3k & < & x & < & 3k+\beta+\gamma-\delta \\ \alpha-\gamma+\delta-3k & < & y & < & 3k-\alpha+\beta+\gamma \\ \alpha+\delta-3k & < & x+y & < & 3k+\beta+\gamma \\ \delta-\gamma-3k & < & y-x & < & 3k-\alpha+\beta \end{array} \right\}.$$

Lemma 13. *Let* $\varsigma_0 = (0,0)$ *and* $\varsigma_1 = (\alpha,\beta)$, $\alpha > k$, $\beta \geq 0$, *be two black cells. If* $\mathcal{I}(\varsigma_0,\varsigma_1) \neq \varnothing$, *then* $|\mathcal{U}_l(\varsigma_0) \setminus \mathcal{P}(\varsigma_0,\varsigma_1)| \geq \frac{1}{2}(\alpha-k)(\alpha-k+1)$.

Proof.

$$\mathcal{U}_l(\varsigma_0) \setminus \mathcal{P}(\varsigma_0,\varsigma_1) \supseteq \{(x,y) : x \geq -k+1,\ y \leq k,\ y-x \geq 3k-\alpha\},$$

which is an isosceles right triangle with legs of length $\alpha-k$. ∎

An example for Lemma 13 is depicted in Figure 12.

Lemma 14. *Let* $\varsigma_0 = (0,0)$ *and* $\varsigma_1 = (\alpha,\beta)$, $\alpha > k$, $\beta \geq 0$, *be two black cells, such that* $\mathcal{I}(\varsigma_0,\varsigma_1) \neq \varnothing$, *and let* $\varsigma_2 = (\gamma,\delta)$ *be a black cell such that* $\mathcal{U}_l(\varsigma_0) \cap \mathcal{S}(\varsigma_2) \neq \varnothing$. *If* $|\gamma| \geq k$, *then* $|\mathcal{N}(\varsigma_0) \cap (\mathcal{P}(\varsigma_0,\varsigma_2) \setminus \mathcal{P}(\varsigma_0,\varsigma_1))| \geq \frac{1}{2}(\alpha-k+1)(\alpha-k)-1$.

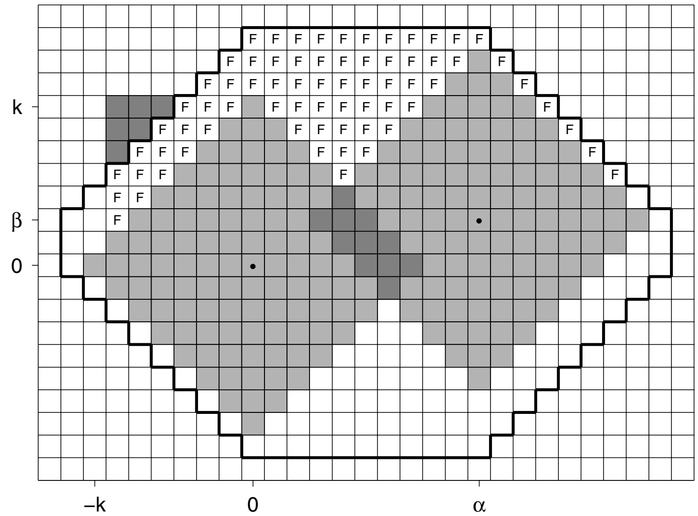


FIG. 12. *An example for Lemma* 13 *with* $(\alpha, \beta) = (10, 2)$ *and* $k = 7$.

Proof. We consider the set of free cells of $\mathcal{N}(\varsigma_0)$ inside $\mathcal{P}(\varsigma_0, \varsigma_2)$ and above $\mathcal{P}(\varsigma_0, \varsigma_1)$. This set contains the following two disjoint subsets V_1 and V_2:

$$V_1 = \{(x, y) : 1 \leq x,\ y \geq 3k - \alpha,\ x + y \leq 3k + \gamma - 1\},$$
$$V_2 = \{(x, y) : \gamma + k \leq x \leq 0,\ y \leq 3k - |\gamma| - 1,\ y - x \geq 3k - \alpha\} \setminus \{(\gamma + k, \delta)\}.$$

If $\alpha > |\gamma| + 1$, then V_1 is an isosceles right triangle and V_2 is a right trapezoid with a missing cell, as depicted in Figure 13(a). If $\alpha \leq |\gamma| + 1$, then V_1 is an empty set and V_2 is an isosceles right triangle with a missing cell, as depicted in Figure 13(b). In both cases

$$|\mathcal{N}(\varsigma_0) \cap (\mathcal{P}(\varsigma_0, \varsigma_2) \setminus \mathcal{P}(\varsigma_0, \varsigma_1))| \geq |V_1| + |V_2| = \frac{1}{2}(\alpha - k + 1)(\alpha - k) - 1. \qquad \square$$

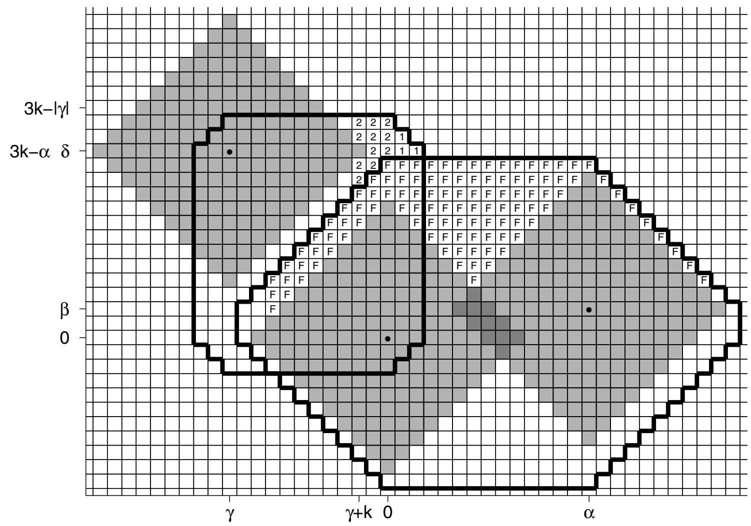
LEMMA 15. *Let* $\varsigma_0 = (0, 0)$ *and* $\varsigma_1 = (\alpha, \beta)$, $\alpha > k, \beta \geq 0$, *be two black cells, such that* $\mathcal{I}(\varsigma_0, \varsigma_1) \neq \varnothing$, *and let* $\varsigma_2 = (\gamma, \delta)$ *be a black cell such that* $\mathcal{U}_l(\varsigma_0) \cap \mathcal{S}(\varsigma_2) \neq \varnothing$. *If* $|\gamma| < k$, *then* $|\mathcal{N}(\varsigma_0) \cap (\mathcal{P}(\varsigma_0, \varsigma_2) \setminus \mathcal{P}(\varsigma_0, \varsigma_1))| \geq \frac{1}{2}(\alpha - k)(\alpha - k - 1)$.

Proof. Since $\mathcal{U}_l(\varsigma_0) \cap \mathcal{S}(\varsigma_2) \neq \varnothing$ it follows that $\delta \leq 2k$. If $\delta < 2k$, we consider the set of free cells of $\mathcal{N}(\varsigma_0)$ inside $\mathcal{P}(\varsigma_0, \varsigma_2)$ and above $\mathcal{P}(\varsigma_0, \varsigma_1)$. This set contains the following two disjoint subsets V_1 and V_2:

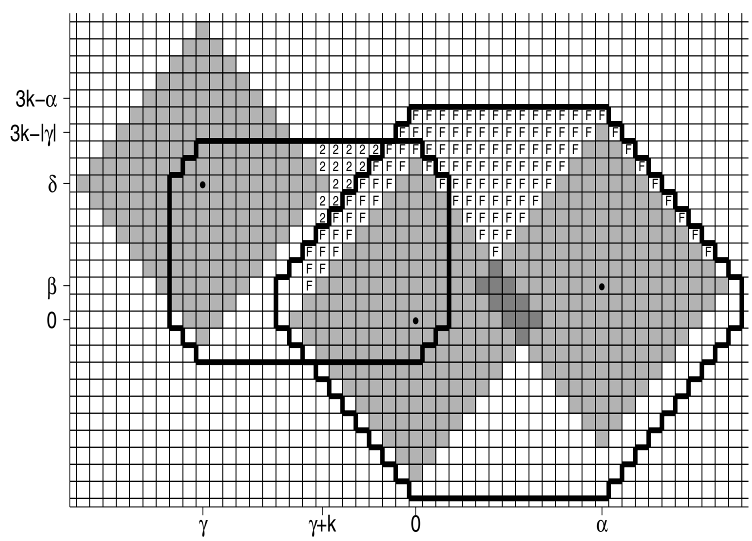
$$V_1 = \{(x, y) : x \leq \gamma + k,\ y \geq 3k - \alpha,\ y - x \leq \delta - \gamma - k - 1\},$$
$$V_2 = \{(x, y) : \gamma + k + 1 \leq x \leq 3k + \gamma - \delta - 1,\ y \geq 3k - \alpha,\ x + y \leq 3k + \gamma - 1\}.$$

V_1 is an isosceles right triangle and V_2 is a right trapezoid, as depicted in Figure 14, and hence

$$|\mathcal{N}(\varsigma_0) \cap (\mathcal{P}(\varsigma_0, \varsigma_2) \setminus \mathcal{P}(\varsigma_0, \varsigma_1))| \geq |V_1| + |V_2| = \frac{1}{2}(\alpha - k)(\alpha - k - 1).$$



(a) $(\gamma, \delta) = (-11, 13)$



(b) $(\gamma, \delta) = (-16, 8)$

FIG. 13. *Lemma* 14 *applied on* $(\alpha, \beta) = (14, 2)$ *and* $k = 9$.

If $\delta = 2k$, then $\mathcal{P}(\varsigma_0, \varsigma_2)$ does not include cells of the form $(\gamma + k, y)$, and hence the set V_1 as defined above is not contained in $\mathcal{P}(\varsigma_0, \varsigma_2)$. Moreover, $|\mathcal{U}_l(\varsigma_0) \cap \mathcal{S}(\varsigma_2)| = 1$ and $\mathcal{N}(\varsigma_0) \cap \mathcal{P}(\varsigma_0, \varsigma_2) \supset \mathcal{U}_l(\varsigma_0) \setminus \mathcal{S}(\varsigma_2)$. Hence, it follows by Lemma 13 that

$$|\mathcal{N}(\varsigma_0) \cap (\mathcal{P}(\varsigma_0, \varsigma_2) \setminus \mathcal{P}(\varsigma_0, \varsigma_1))| \geq |\mathcal{U}_l(\varsigma_0) \setminus \mathcal{P}(\varsigma_0, \varsigma_1)| - 1 = \frac{1}{2}(\alpha - k + 1)(\alpha - k) - 1. \qquad \square$$

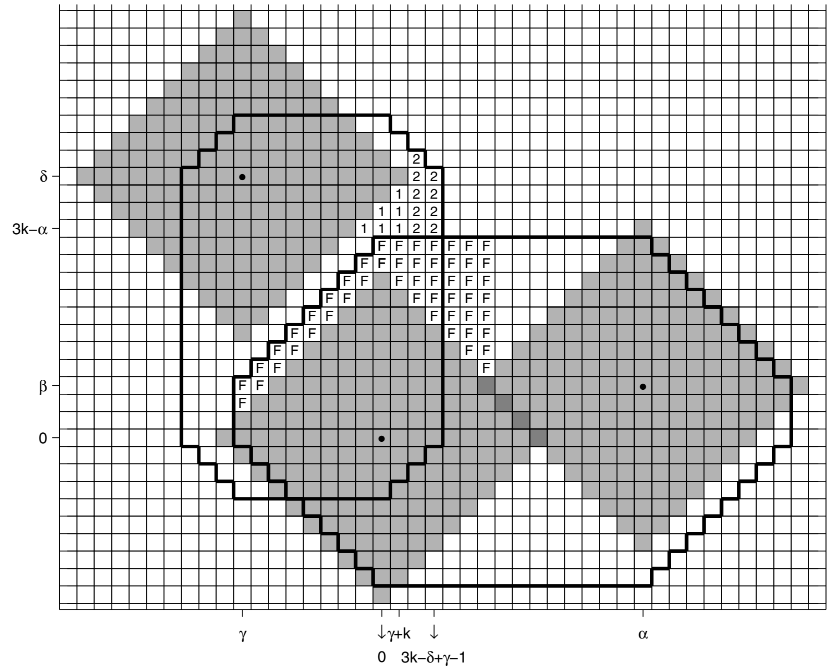


FIG. 14. *Case* 2.3 *with* $(\alpha,\beta) = (15,3)$, $(\gamma,\delta) = (-8,15)$, $k = 9$. *The cells of* V_1 *and* V_2 *are marked "*1*" and "*2*."*

Note that in Lemmas 13, 14, and 15, if $\mathcal{I}(\varsigma_0,\varsigma_1)$ is a deep-intersection, then $\mathcal{N}(\varsigma_0) = \mathcal{N}(\varsigma_0,\varsigma_1)$.

We now proceed to prove more results which will be useful in the proof of Lemma 8. In what follows we assume that $\varsigma_0 = (0,0)$ and $\varsigma_1 = (\alpha,\beta)$, $\alpha,\beta \geq 0$, are black cells for which $\mathcal{I}(\varsigma_0,\varsigma_1)$ is a deep-intersection, i.e., $\alpha+\beta < 2k$. First, we have to compute the size of $|\mathcal{I}(\varsigma_0,\varsigma_1)|$. We have found a few different methods to compute $|\mathcal{I}(\varsigma_0,\varsigma_1)|$; none of them is elegant. Therefore we leave the proof of the following lemma to the reader.

LEMMA 16.

$$|\mathcal{I}(\varsigma_0,\varsigma_1)| = \begin{cases} 2(k-\beta+1)k-\beta+\left\lfloor\frac{\beta^2-\alpha^2}{2}\right\rfloor+1 & \textit{if } \alpha<\beta, \\ 2(k-\alpha+1)k-\alpha+\left\lfloor\frac{\alpha^2-\beta^2}{2}\right\rfloor+1 & \textit{if } \beta\leq\alpha. \end{cases}$$

For a black cell $\varsigma = (\gamma,\delta)$, let

$$T_l(\varsigma) = \{(\gamma-k,y) : y>\delta,\ (\gamma-k,\eta) \text{ is a free cell for } \delta+1\leq\eta\leq y\}\,,$$
$$T_r(\varsigma) = \{(\gamma+k,y) : y>\delta,\ (\gamma+k,\eta) \text{ is a free cell for } \delta+1\leq\eta\leq y\}\,.$$

An example is depicted in Figure 15.

LEMMA 17. *If* $\alpha+\beta\leq k+1$, *then* $T_l(\varsigma_0)\subset\mathcal{N}(\varsigma_0,\varsigma_1)$ *and* $|\mathcal{P}(\varsigma_0,\varsigma_1)\cap T_l(\varsigma_0)| = 2k-\alpha-1$.

Proof. A cell $z\in T_l(\varsigma_0)$ does not belong to $\mathcal{N}(\varsigma_0,\varsigma_1)$ if there exists a black cell ς_2 such that $\mathcal{U}(\varsigma_2)\cap\mathcal{S}(\varsigma_0)\neq\varnothing$ and $z\in\mathcal{P}(\varsigma_2,\varsigma_0)$. Assume that such a cell $\varsigma_2=(\gamma,\delta)$ exists. Clearly, $\gamma,\delta<0$. $d_3(\varsigma_0,\varsigma_1,\varsigma_2) = \alpha-\gamma+\beta-\delta\geq 4k$, and hence

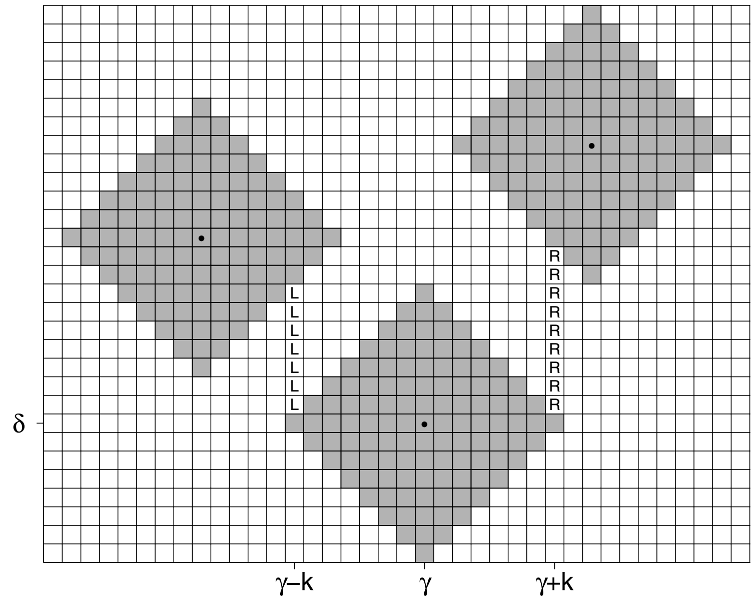


FIG. 15. $T_l(\varsigma)$ *and* $T_r(\varsigma)$.

$|\gamma| + |\delta| = -(\gamma + \delta) \geq 4k - (\alpha + \beta) \geq 3k - 1$. By Lemma 6, if $|\gamma| + |\delta| > 3k - 1$, then $\mathcal{U}(\varsigma_2) \cap \mathcal{S}(\varsigma_0) = \varnothing$. If $|\gamma| + |\delta| = 3k - 1$ and $\mathcal{U}(\varsigma_2) \cap \mathcal{S}(\varsigma_0) \neq \varnothing$, then $\mathcal{P}(\varsigma_2, \varsigma_0)$ is a rectangle and $z \notin \mathcal{P}(\varsigma_2, \varsigma_0)$, a contradiction.

Thus, such a black cell ς_2 does not exist, $T_l(\varsigma_0) \subset \mathcal{N}(\varsigma_0, \varsigma_1)$, and one can easily verify that $|\mathcal{P}(\varsigma_0, \varsigma_1) \cap T_l(\varsigma_0)| = 2k - \alpha - 1$. ◻

LEMMA 18. *If* $\alpha \leq k + 1$, $\alpha + \beta \geq k + 1$, *then* $|\mathcal{N}(\varsigma_0, \varsigma_1) \cap \mathcal{P}(\varsigma_0, \varsigma_1) \cap T_l(\varsigma_0)| \geq 3k - 2\alpha - \beta$.

Proof. A cell $z \in T_l(\varsigma_0)$ does not belong to $\mathcal{N}(\varsigma_0, \varsigma_1)$ if there exists a black cell ς_2 such that $\mathcal{U}(\varsigma_2) \cap \mathcal{S}(\varsigma_0) \neq \varnothing$ and $z \in \mathcal{P}(\varsigma_2, \varsigma_0)$. Assume that such a cell $\varsigma_2 = (\gamma, \delta)$ exists. Clearly, $\gamma, \delta < 0$.

By the definition of $\mathcal{P}(\varsigma_2, \varsigma_0)$, if $k < |\gamma| < 2k$, then

$$\mathcal{N}(\varsigma_2) \cap T_l(\varsigma_0) = \{(-k, y) : 1 \leq y \leq 3k - |\gamma| - |\delta| - 1\},$$

and if $|\gamma| \leq k$, then $\mathcal{N}(\varsigma_2) \cap T_l(\varsigma_0) = \{(-k, y) : 1 \leq y \leq 2k - |\delta| - 1\}$. Hence, $|\mathcal{N}(\varsigma_2) \cap T_l(\varsigma_0)| \leq 3k - |\gamma| - |\delta| - 1$. It is easy to verify that $|\mathcal{P}(\varsigma_0, \varsigma_1) \cap T_l(\varsigma_0)| = 2k - \alpha - 1$, and therefore

$$\begin{aligned} |\mathcal{N}(\varsigma_0, \varsigma_1) \cap \mathcal{P}(\varsigma_0, \varsigma_1) \cap T_l(\varsigma_0)| &\geq |\gamma| + |\delta| - \alpha - k \\ &= \alpha + \beta + |\gamma| + |\delta| - 2\alpha - \beta - k \\ &= d_3(\varsigma_0, \varsigma_1, \varsigma_2) - 2\alpha - \beta - k \\ &\geq 3k - 2\alpha - \beta. \end{aligned}$$

Note that $3k - 2\alpha - \beta \geq 0$ since $\alpha \leq k + 1$ and $\alpha + \beta < 2k$. ◻

LEMMA 19. *If $\alpha \le k+1$, then $T_r(\varsigma_1) \subset \mathcal{N}(\varsigma_0,\varsigma_1)$ and $|\mathcal{P}(\varsigma_0,\varsigma_1) \cap T_r(\varsigma_1)| = 2k-\alpha-\beta-1$.*

Proof. A cell $z \in T_r(\varsigma_1)$ does not belong to $\mathcal{N}(\varsigma_0,\varsigma_1)$ if there exists a black cell ς_2 such that $\mathcal{U}(\varsigma_2) \cap \mathcal{S}(\varsigma_1) \neq \varnothing$ and $z \in \mathcal{P}(\varsigma_2,\varsigma_1)$. Assume that such a cell $\varsigma_2 = (\gamma,\delta)$ exists. Clearly, $\alpha < \gamma$ and $\delta < \beta$.

If $\delta > 0$, then $\gamma - \alpha = d_3(\varsigma_0,\varsigma_1,\varsigma_2) - \alpha - \beta \ge 4k - (\alpha+\beta) > 2k$, and therefore by Lemma 6 we have $\mathcal{U}(\varsigma_2) \cap \mathcal{S}(\varsigma_1) = \varnothing$, a contradiction. If $\delta \le 0$, then $|\alpha-\gamma| + |\beta-\delta| = \gamma - \alpha + \beta - \delta = d_3(\varsigma_0,\varsigma_1,\varsigma_2) - \alpha \ge 4k - \alpha \ge 3k-1$. By Lemma 6, if $|\alpha-\gamma|+|\beta-\delta| > 3k-1$, then $\mathcal{U}(\varsigma_2) \cap \mathcal{S}(\varsigma_1) = \varnothing$. If $|\alpha-\gamma|+|\beta-\delta| = 3k-1$ and $\mathcal{U}(\varsigma_2) \cap \mathcal{S}(\varsigma_1) \neq \varnothing$, then $\mathcal{P}(\varsigma_2,\varsigma_1)$ is a rectangle and $z \notin \mathcal{P}(\varsigma_2,\varsigma_1)$, a contradiction.

Thus, such a black cell ς_2 does not exist, $T_r(\varsigma_1) \subset \mathcal{N}(\varsigma_0,\varsigma_1)$, and one can easily verify that $|\mathcal{P}(\varsigma_0,\varsigma_1) \cap T_r(\varsigma_1)| = 2k-\alpha-\beta-1$. ◻

LEMMA 20. *If $\alpha < \beta \le k$, then $T_r(\varsigma_0) \subset \mathcal{N}(\varsigma_0,\varsigma_1)$ and $|\mathcal{P}(\varsigma_0,\varsigma_1) \cap T_r(\varsigma_0)| = \beta-\alpha-1$.*

Proof. A cell $z \in T_r(\varsigma_0)$ does not belong to $\mathcal{N}(\varsigma_0,\varsigma_1)$ if there exists a black cell ς_2 such that $\mathcal{U}(\varsigma_2) \cap \mathcal{S}(\varsigma_0) \neq \varnothing$ and $z \in \mathcal{P}(\varsigma_2,\varsigma_0)$. Assume that such a cell $\varsigma_2 = (\gamma,\delta)$ exists. Clearly, $\gamma > 0$ and $\delta < 0$.

If $\gamma \le \alpha$, then $|\delta| = d_3(\varsigma_0,\varsigma_1,\varsigma_2) - \alpha - \beta \ge 4k - (\alpha+\beta) > 2k$, and by Lemma 6 we have $\mathcal{U}(\varsigma_2) \cap \mathcal{S}(\varsigma_0) = \varnothing$, a contradiction. If $\alpha < \gamma$, then $\gamma + |\delta| = d_3(\varsigma_0,\varsigma_1,\varsigma_2) - \beta \ge 4k - \beta \ge 3k$, and by Lemma 6 we have $\mathcal{U}(\varsigma_2) \cap \mathcal{S}(\varsigma_0) = \varnothing$, a contradiction.

Thus, such a black cell ς_2 does not exist, $T_r(\varsigma_0) \subset \mathcal{N}(\varsigma_0,\varsigma_1)$, and it is easy to verify that $|\mathcal{P}(\varsigma_0,\varsigma_1) \cap T_r(\varsigma_0)| = \beta-\alpha-1$. ◻

LEMMA 21. *If $\alpha < \beta$ and $k < \beta$, then $|\mathcal{N}(\varsigma_0,\varsigma_1) \cap \mathcal{P}(\varsigma_0,\varsigma_1) \cap T_r(\varsigma_0)| \ge k-\alpha$.*

Proof. A cell $z \in T_r(\varsigma_0)$ does not belong to $\mathcal{N}(\varsigma_0,\varsigma_1)$ if there exists a black cell ς_2 such that $\mathcal{U}(\varsigma_2) \cap \mathcal{S}(\varsigma_0) \neq \varnothing$ and $z \in \mathcal{P}(\varsigma_2,\varsigma_0)$. Assume that such a cell $\varsigma_2 = (\gamma,\delta)$ exists. Clearly, $\gamma > 0$ and $\delta < 0$.

By symmetry and similar arguments to those of Lemma 18 we have that $|\mathcal{N}(\varsigma_2) \cap \mathcal{P}(\varsigma_0,\varsigma_1) \cap T_r(\varsigma_0)| \le 2k-|\delta|-1$ if $\gamma \le k$ and $|\mathcal{N}(\varsigma_2) \cap \mathcal{P}(\varsigma_0,\varsigma_1) \cap T_r(\varsigma_0)| \le 3k-\gamma-|\delta|-1$ otherwise. $T_r(\varsigma_0) = \{(k,y) : 1 \le y \le \beta-\alpha-1\}$, and hence $|T_r(\varsigma_0)| = \beta-\alpha-1$. Therefore, we have the following:

- If $\gamma > \alpha$, then

$$\begin{aligned} |\mathcal{N}(\varsigma_0,\varsigma_1) \cap T_r(\varsigma_0)| &\ge (\beta-\alpha-1) - (3k-\gamma-|\delta|-1) \\ &= d_3(\varsigma_0,\varsigma_1,\varsigma_2) - \alpha - 3k \\ &\ge k-\alpha. \end{aligned}$$

- If $\gamma \le \alpha$, then $\gamma < k$;

$$\begin{aligned} |\mathcal{N}(\varsigma_0,\varsigma_1) \cap \mathcal{P}(\varsigma_0,\varsigma_1) \cap T_r(\varsigma_0)| &\ge (\beta-\alpha-1) - (2k-|\delta|-1) \\ &= d_3(\varsigma_0,\varsigma_1,\varsigma_2) - 2\alpha - 2k \\ &\ge 2k-2\alpha \\ &> k-\alpha. \end{aligned}$$

Note that $k-\alpha > 0$ since $\alpha < \beta$ and $\alpha+\beta < 2k$. ◻

LEMMA 22. *If $0 < \alpha < \beta$, then*

$$|\mathcal{N}(\varsigma_0,\varsigma_1) \cap \mathcal{P}(\varsigma_0,\varsigma_1) \setminus (T_l(\varsigma_0) \cup T_r(\varsigma_0) \cup T_r(\varsigma_1))| = 6k^2 + (\alpha-2)k - \frac{\alpha^2}{2} + \frac{3\alpha}{2} - \alpha\beta + 3. \tag{5}$$

Proof. Let $V = \mathcal{N}(\varsigma_0,\varsigma_1) \cap \mathcal{P}(\varsigma_0,\varsigma_1) \setminus (T_l(\varsigma_0) \cup T_r(\varsigma_0) \cup T_r(\varsigma_1))$. V is partitioned into seven subsets $\Pi_1,\Pi_2,\Pi_3,\Pi_4,\Pi_5,\Pi_6,\Pi_7$ as follows (see Figure 16):

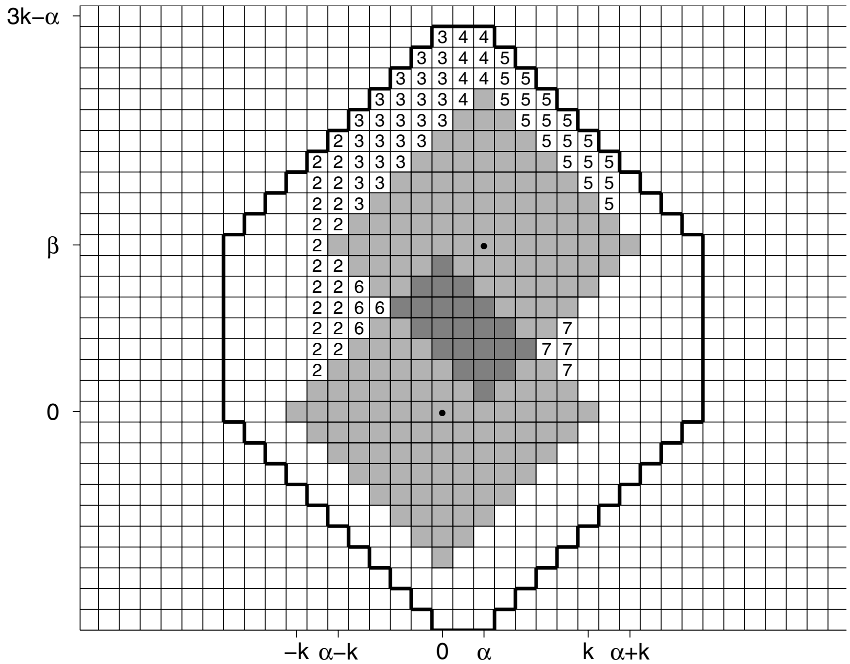


FIG. 16. *Lemma* 22 *applied on* $(\alpha, \beta) = (2, 8)$ *and* $k = 7$.

1. $\Pi_1 = \mathcal{S}(\varsigma_0) \cup \mathcal{S}(\varsigma_1)$, and by Lemma 16
$$|\Pi_1| = |\mathcal{S}(\varsigma_0)| + |\mathcal{S}(\varsigma_1)| - |\mathcal{I}(\varsigma_0, \varsigma_1)| = 2k^2 + 2(\beta+1)k + \beta - \left\lfloor \frac{\beta^2 - \alpha^2}{2} \right\rfloor + 1.$$
2. $\Pi_2 = \{(x, y) : -k+1 \le x \le \alpha - k,\ k+1 \le y - x \le 3k - \alpha - 1\} \setminus \{(\alpha - k, \beta)\}$. Π_2 is a parallelogram with a missing cell and $|\Pi_2| = (2k - \alpha - 1)\alpha - 1$.
3. $\Pi_3 = \{(x, y) : \alpha - k + 1 \le x \le 0,\ \beta - \alpha + k + 1 \le y - x \le 3k - \alpha - 1\}$. Π_3 is also a parallelogram and $|\Pi_3| = (2k - \beta - 1)(k - \alpha)$.
4. $\{(x, y) : 1 \le x \le \alpha,\ \beta - \alpha + k + 1 \le y - x,\ y \le 3k - \alpha - 1\}$. Π_4 is a right trapezoid and $|\Pi_4| = (2k - \beta - \frac{\alpha+3}{2})\alpha$.
5. $\Pi_5 = \{(x, y) : \alpha + 1 \le x \le \alpha + k - 1,\ \alpha + \beta + k + 1 \le x + y \le 3k - 1\}$. Π_5 is a parallelogram and $|\Pi_5| = (2k - \alpha - \beta - 1)(k - 1)$.
6. $\Pi_6 = \{(x, y) : \alpha - k + 1 \le x,\ k + 1 \le y - x,\ x + y \le \alpha + \beta - k - 1\}$. Π_6 is an arithmetic progression and $|\Pi_6| = \lfloor \frac{(\beta - \alpha - 2)^2}{4} \rfloor$.
7. $\Pi_7 = \{(x, y) : x \le k - 1,\ k + 1 \le x + y,\ y - x \le \beta - \alpha - k - 1\}$ and clearly $|\Pi_7| = |\Pi_6|$.

Therefore,
$$|V| = \sum_{i=1}^{7} |\Pi_i| = 6k^2 + (\alpha - 2)k - \frac{\alpha^2}{2} + \frac{3\alpha}{2} - \alpha\beta + 3. \qquad \square$$

LEMMA 23. *If* $0 < \alpha < \beta$ *and* $\alpha + \beta \le k + 1$, *then* $|\mathcal{N}(\varsigma_0, \varsigma_1)| > 6k^2$.

Proof. By Lemmas 17 and 19 we have that $T_l(\varsigma_0) \cup T_r(\varsigma_1) \subset \mathcal{N}(\varsigma_0, \varsigma_1)$, $|\mathcal{P}(\varsigma_0, \varsigma_1) \cap T_l(\varsigma_0)| = 2k - \alpha - 1$, $|\mathcal{P}(\varsigma_0, \varsigma_1) \cap T_r(\varsigma_1)| = 2k - \alpha - \beta - 1$, and hence by Lemma 22 we have

$$(6)\quad |\mathcal{N}(\varsigma_0,\varsigma_1)| \geq |(\mathcal{N}(\varsigma_0,\varsigma_1)\cap\mathcal{P}(\varsigma_0,\varsigma_1))\setminus T_r(\varsigma_0)| = 6k^2+(\alpha+2)k-\frac{\alpha^2}{2}-\frac{\alpha}{2}-\beta(\alpha+1)+1.$$

From (6) and since $\alpha+\beta \leq k+1$ it follows that

$$\begin{aligned}|\mathcal{N}(\varsigma_0,\varsigma_1)| &\geq 6k^2+(\alpha+2)k-\frac{\alpha^2}{2}-\frac{\alpha}{2}-(k+1-\alpha)(\alpha+1)+1\\ &= 6k^2+k+\frac{\alpha(\alpha-1)}{2} > 6k^2. \qquad \square\end{aligned}$$

LEMMA 24. *If* $0 \leq \beta \leq \alpha$, *then*

$$(7)\quad \begin{aligned}&|(\mathcal{N}(\varsigma_0,\varsigma_1)\cap\mathcal{P}(\varsigma_0,\varsigma_1))\setminus(T_l(\varsigma_0)\cup T_r(\varsigma_1))|\\ &\geq 6k^2+(2\alpha-\beta-2)k+\alpha+\tfrac{2\beta-5\alpha^2-2\alpha\beta+\beta^2}{4}+2.\end{aligned}$$

Proof. Let $V=(\mathcal{N}(\varsigma_0,\varsigma_1)\cap\mathcal{P}(\varsigma_0,\varsigma_1))\setminus(T_l(\varsigma_0)\cup T_r(\varsigma_1))$. V is partitioned into five subsets $\Pi_1,\Pi_2,\Pi_3,\Pi_4,\Pi_5$ as follows (see Figure 17):

1. $\Pi_1=\mathcal{S}(\varsigma_0)\cup\mathcal{S}(\varsigma_1)$, and by Lemma 16, $|\Pi_1|=2k^2+2(\alpha+1)k+\alpha-\lfloor\frac{\alpha^2-\beta^2}{2}\rfloor+1$.
2. $\Pi_2=\{(x,y): -k+1\leq x\leq 0\ , k+1\leq y-x\leq 3k-\alpha-1\}$. Π_2 is a parallelogram and $|\Pi_2|=(2k-\alpha-1)k$.
3. $\Pi_3=\{(x,y): 1\leq x\leq\lfloor\frac{\alpha-\beta}{2}\rfloor,\ k+1\leq x+y,\ y\leq 3k-\alpha-1\}$. Π_3 is a right trapezoid and $|\Pi_3|=\frac{1}{2}(4k-\lfloor\frac{3\alpha+\beta+3}{2}\rfloor)\lfloor\frac{\alpha-\beta}{2}\rfloor$.
4. $\Pi_4=\{(x,y): \lfloor\frac{\alpha-\beta}{2}\rfloor+1\leq x\leq\alpha-1,\ \beta-\alpha+k+1\leq y-x,\ y\leq 3k-\alpha-1\}$. Π_4 is a right trapezoid and $|\Pi_4|=\frac{1}{2}(4k-\lfloor\frac{3\alpha+3\beta+4}{2}\rfloor)\lfloor\frac{\alpha+\beta-1}{2}\rfloor$.
5. $\Pi_5=\{(x,y): \alpha\leq x\leq\alpha+k-1,\ \alpha+\beta+k+1\leq x+y\leq 3k-1\}$. Π_5 is a parallelogram and $|\Pi_5|=(2k-\alpha-\beta-1)k$.

Therefore,

$$\begin{aligned}|V|=\sum_{i=1}^{5}|\Pi_i| &= 6k^2+(2\alpha-\beta-2)k+\alpha+\left\lfloor\frac{2\beta-3\alpha^2-2\alpha\beta-\beta^2}{4}\right\rfloor-\left\lfloor\frac{\alpha^2-\beta^2}{2}\right\rfloor+2\\ &\geq 6k^2+(2\alpha-\beta-2)k+\alpha+\frac{2\beta-5\alpha^2-2\alpha\beta+\beta^2}{4}+2. \qquad \square\end{aligned}$$

LEMMA 25. *If* $0\leq\beta\leq\alpha$ *and* $\alpha+\beta\leq k+1$, *then* $|\mathcal{N}(\varsigma_0,\varsigma_1)|>6k^2$.

Proof. By Lemmas 17 and 19 we have that $T_l(\varsigma_0)\cup T_r(\varsigma_1)\subset\mathcal{N}(\varsigma_0,\varsigma_1)$, $|\mathcal{P}(\varsigma_0,\varsigma_1)\cap T_l(\varsigma_0)|=2k-\alpha-1$, $|\mathcal{P}(\varsigma_0,\varsigma_1)\cap T_r(\varsigma_1)|=2k-\alpha-\beta-1$, and hence by Lemma 24 we have

$$(8)\quad \begin{aligned}|\mathcal{N}(\varsigma_0,\varsigma_1)| &\geq |\mathcal{N}(\varsigma_0,\varsigma_1)\cap\mathcal{P}(\varsigma_0,\varsigma_1)|\\ &\geq 6k^2+(2\alpha-\beta+2)k-\alpha-\beta+\frac{2\beta-5\alpha^2-2\alpha\beta+\beta^2}{4}.\end{aligned}$$

A lower bound on $|\mathcal{N}(\varsigma_0,\varsigma_1)|$ for a fixed α and in the given range of β is obtained for the largest possible value of β. We distinguish between two cases:

- If $\alpha\geq\frac{k+1}{2}$, then the largest value of β is $k-\alpha+1$. Substituting $\beta=k-\alpha+1$ in (8) implies that

  $$(9)\quad |\mathcal{N}(\varsigma_0,\varsigma_1)|\geq 5\frac{1}{4}k^2+(2\alpha+1)k-\frac{\alpha^2}{2}-1\frac{1}{2}\alpha-\frac{1}{4}.$$

  A lower bound on $|\mathcal{N}(\varsigma_0,\varsigma_1)|$ in the given range of α is obtained for $\alpha=\frac{k+1}{2}$. Substituting $\alpha=\frac{k+1}{2}$ in (9) we obtain $|\mathcal{N}(\varsigma_0,\varsigma_1)|>6k^2$.

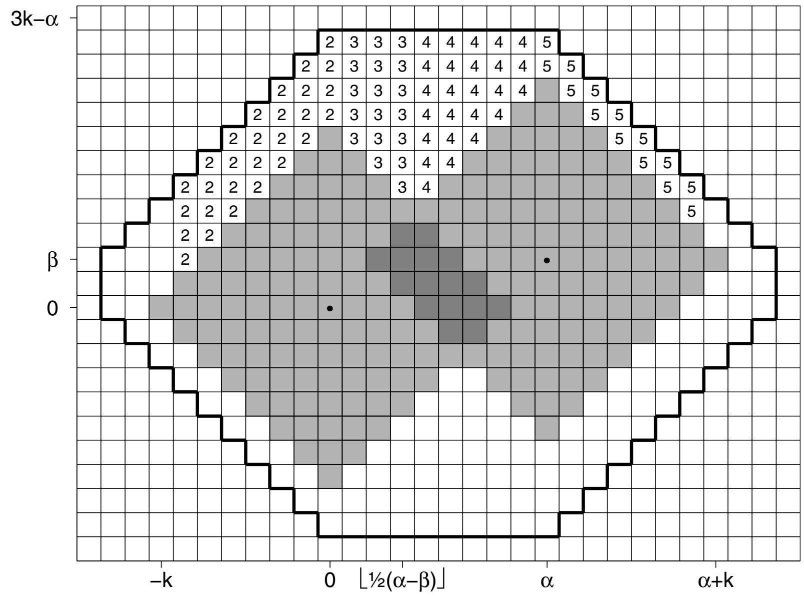


FIG. 17. *Lemma* 24 *applied on* $(\alpha,\beta)=(9,2)$ *and* $k=7$.

- If $1 \le \alpha \le \frac{k}{2}$, then the largest value of β is α, and

$$|\mathcal{N}(\varsigma_0,\varsigma_1)| \ge 6k^2 + (\alpha+2)k - \frac{3\alpha(\alpha+1)}{2}. \tag{10}$$

  A lower bound on $|\mathcal{N}(\varsigma_0,\varsigma_1)|$ in the given range of α is obtained for $\alpha=1$ if $k>10$ and for $\alpha=\left\lfloor\frac{k}{2}\right\rfloor$ if $2\le k\le 10$. Substituting $\alpha=1$ or $\alpha=\frac{k}{2}$, respectively, in (10) we obtain $|\mathcal{N}(\varsigma_0,\varsigma_1)|>6k^2$. ◻

LEMMA 26. *If* $\alpha+\beta>k$, *then* $|\mathcal{U}_r(\varsigma_1)\setminus\mathcal{P}(\varsigma_0,\varsigma_1)|=\frac{1}{2}(\alpha+\beta-k)(\alpha+\beta-k+1)$.

Proof.

$$\mathcal{U}_r(\varsigma_1)\setminus\mathcal{P}(\varsigma_0,\varsigma_1)=\{(x,y): x\le\alpha+k-1,\ y\le\beta+k,\ x+y\ge 3k\}$$

is an isosceles right triangle with legs of length $\alpha+\beta-k$. ◻

An example of $\mathcal{U}_r(\varsigma_1)\setminus\mathcal{P}(\varsigma_0,\varsigma_1)$ is depicted in Figure 18.

LEMMA 27. *If* $\varsigma_2=(\gamma,\delta)$ *is a black cell such that* $\mathcal{U}_r(\varsigma_1)\cap\mathcal{S}(\varsigma_2)\ne\varnothing$, *then* $\alpha+\beta>k$.

Proof. By Lemma 6, $\alpha+\beta>\gamma+\delta-3k=d_3(\varsigma_0,\varsigma_1,\varsigma_2)-3k\ge k$. ◻

LEMMA 28. *Let* $\varsigma_2=(\gamma,\delta)$ *be a black cell such that* $\mathcal{U}_r(\varsigma_1)\cap\mathcal{S}(\varsigma_2)\ne\varnothing$. *If* $\gamma\ge\alpha+k$, *then* $|\mathcal{N}(\varsigma_0,\varsigma_1)\cap(\mathcal{P}(\varsigma_1,\varsigma_2)\setminus\mathcal{P}(\varsigma_0,\varsigma_1))|\ge\frac{1}{2}(\alpha+\beta-k)(\alpha+\beta-k+1)-1$.

Proof. We consider the set of free cells of $\mathcal{N}(\varsigma_0,\varsigma_1)$ inside $\mathcal{P}(\varsigma_1,\varsigma_2)$ and above $\mathcal{P}(\varsigma_0,\varsigma_1)$. This set contains the following two disjoint subsets V_1 and V_2 (see Figure 19):

$$V_1=\{(x,y): x\le\alpha-1,\ y\ge 3k-\alpha,\ y-x\le 3k-\gamma-1+\beta\},$$
$$V_2=\{(x,y): \alpha\le x\le\gamma-k,\ y\le 3k-\gamma-1+\alpha+\beta,\ x+y\ge 3k\}\setminus\{(\gamma-k,\delta)\}.$$

If $\gamma<2\alpha+\beta-1$, then V_1 is an isosceles right triangle and V_2 is a right trapezoid with a missing cell, as depicted in Figure 19(a). If $\gamma\ge 2\alpha+\beta-1$, then V_1 is an empty set

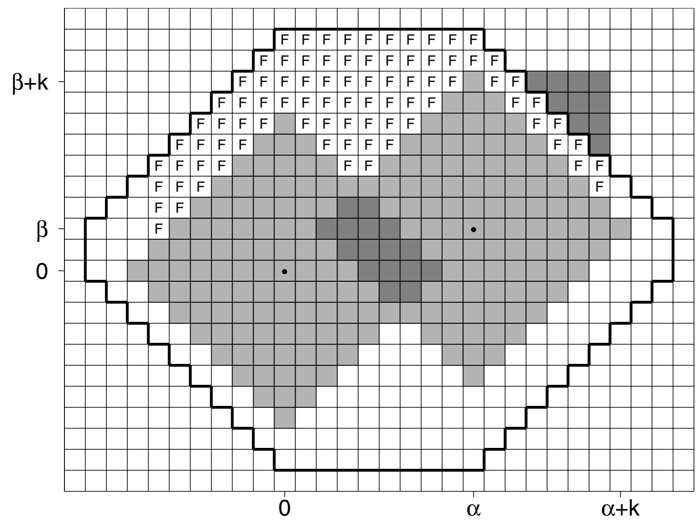


FIG. 18. *An example of* $\mathcal{U}_r(\varsigma_1) \setminus \mathcal{P}(\varsigma_0, \varsigma_1)$ *with* $(\alpha, \beta) = (9, 2)$ *and* $k = 7$.

and V_2 is an isosceles right triangle with a missing cell, as depicted in Figure 19(b). In both cases

$$|\mathcal{N}(\varsigma_0,\varsigma_1) \cap (\mathcal{P}(\varsigma_1,\varsigma_2) \setminus \mathcal{P}(\varsigma_0,\varsigma_2))| \geq |V_1| + |V_2| = \frac{1}{2}(\alpha+\beta-k)(\alpha+\beta-k+1) - 1. \qquad \square$$

LEMMA 29. *Let* $\varsigma_2 = (\gamma, \delta)$ *be a black cell such that* $\mathcal{U}_r(\varsigma_1) \cap \mathcal{S}(\varsigma_2) \neq \varnothing$. *If* $\gamma < \alpha + k$, *then* $|\mathcal{N}(\varsigma_0,\varsigma_1) \cap (\mathcal{P}(\varsigma_1,\varsigma_2) \setminus \mathcal{P}(\varsigma_0,\varsigma_1))| \geq \frac{1}{2}(\alpha+\beta-k)(\alpha+\beta-k+1) - 2$.

Proof. Since $\mathcal{U}_r(\varsigma_1) \cap \mathcal{S}(\varsigma_2) \neq \varnothing$, it follows that $\delta \leq \beta + 2k$. If $\delta < \beta + 2k$, we consider the set of free cells inside $\mathcal{P}(\varsigma_1, \varsigma_2)$ and above $\mathcal{P}(\varsigma_0, \varsigma_1)$. This set contains the following four disjoint subsets V_1, V_2, V_3, V_4 (see Figure 20):

$$\begin{aligned}
V_1 &= \{(x,y) : x \geq \gamma - k,\ y \geq 3k - \alpha,\ x + y \leq \gamma + \delta - k - 1\},\\
V_2 &= \{(x,y) : \gamma + \delta - \beta - 3k + 1 \leq x \leq \gamma - k - 1,\ y \geq 3k - \alpha,\ y - x \leq 3k - \gamma - 1 + \beta\},\\
V_3 &= T_l(\varsigma_2) \cap \mathcal{P}(\varsigma_1,\varsigma_2) = \{(\gamma - k, y) : \delta + 1 \leq y \leq 2k + \beta - 1\},\\
V_4 &= \{(\alpha + k - 1, y) : 2k - \alpha + 1 \leq y \leq \alpha + \delta - \gamma - 2\}.
\end{aligned}$$

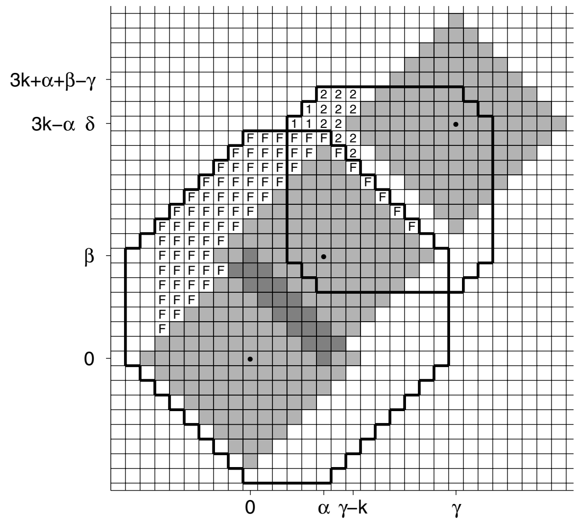
V_1 is a isosceles right triangle, V_2 is a right trapezoid, and

$$\begin{aligned}
|V_1| + |V_2| &= \frac{1}{2}(\alpha+\beta-k)(\alpha+\beta-k-1),\\
|V_3| + |V_4| &= 2\alpha + \beta - \gamma - 3 = \alpha + \beta - (\gamma - \alpha) - 3 \geq \alpha + \beta - k - 2.
\end{aligned}$$

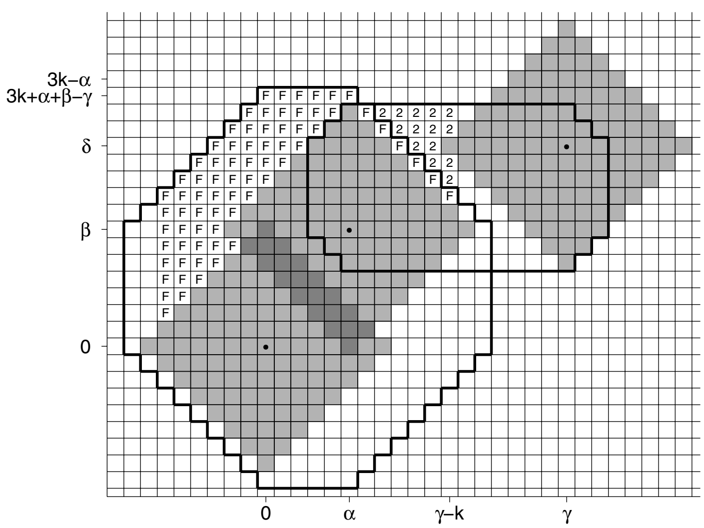
Hence

$$|\mathcal{N}(\varsigma_0,\varsigma_1) \cap (\mathcal{P}(\varsigma_1,\varsigma_2) \setminus \mathcal{P}(\varsigma_0,\varsigma_1))| \geq |V_1| + |V_2| + |V_3| + |V_4| \geq \frac{1}{2}(\alpha+\beta-k)(\alpha+\beta-k+1) - 2.$$

If $\delta = \beta + 2k$, then $\mathcal{P}(\varsigma_1, \varsigma_2)$ does not include cells of the form $(\gamma - k, y)$, and hence the set V_1 as defined above is not contained in $\mathcal{P}(\varsigma_1, \varsigma_2)$. Moreover, $|\mathcal{U}_r(\varsigma_1) \cap \mathcal{S}(\varsigma_2)| = 1$



(a) $(\gamma, \delta) = (14, 16)$



(b) $(\gamma, \delta) = (18, 12)$

FIG. 19. *Lemma* 28 *applied on* $(\alpha, \beta) = (5, 7)$ *and* $k = 7$. *The cells of* V_1 *and* V_2 *are marked "*1*" and "*2*."*

and $\mathcal{N}(\varsigma_0, \varsigma_1) \supset \mathcal{U}_r(\varsigma_1) \setminus \mathcal{S}(\varsigma_2)$. Hence, it follows by Lemmas 26 and 27 that

$$\begin{aligned} |\mathcal{N}(\varsigma_0, \varsigma_1) \cap (\mathcal{P}(\varsigma_1, \varsigma_2) \setminus \mathcal{P}(\varsigma_0, \varsigma_1))| &\geq |\mathcal{U}_r(\varsigma_1) \setminus \mathcal{P}(\varsigma_0, \varsigma_1)| - 1 \\ &= \frac{1}{2}(\alpha + \beta - k)(\alpha + \beta - k + 1) - 1. \qquad \square \end{aligned}$$

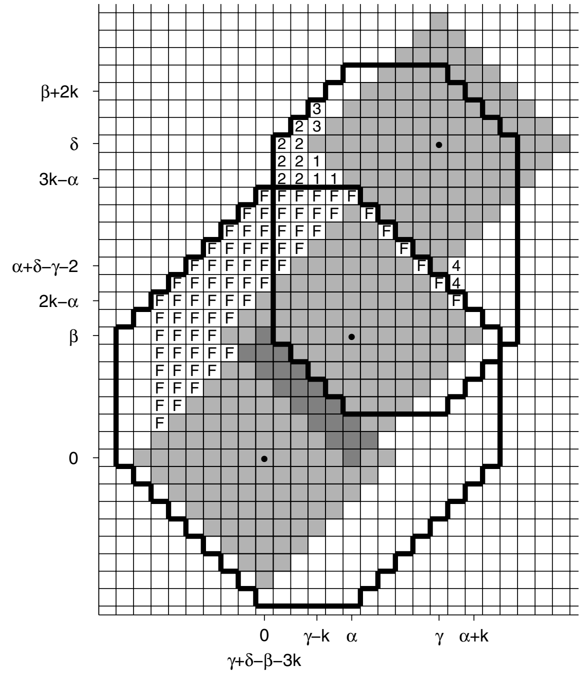


FIG. 20. *Lemma* 29 *applied on* $(\alpha,\beta) = (5, 7)$, $(\gamma,\delta) = (10, 18)$, $k = 7$.

LEMMA 30. *If* $0 < \alpha < \beta \le k$ *and* $\alpha + \beta > k + 1$, *then* $|\mathcal{N}(\varsigma_0, \varsigma_1)| > 6k^2$.

Proof. By Lemmas 18, 19, and 20 we have that

$$|\mathcal{N}(\varsigma_0, \varsigma_1) \cap \mathcal{P}(\varsigma_0, \varsigma_1) \cap (T_l(\varsigma_0) \cup T_r(\varsigma_0) \cup T_r(\varsigma_1))| \ge 5k - 4\alpha - \beta - 2. \tag{11}$$

By Lemmas 26, 28, and 29 we have that

$$|\mathcal{N}(\varsigma_0, \varsigma_1) \setminus \mathcal{P}(\varsigma_0, \varsigma_1)| \ge \frac{1}{2}(\alpha + \beta - k)(\alpha + \beta - k + 1) - 2. \tag{12}$$

Combining (5) with (11) and (12) we obtain

$$|\mathcal{N}(\varsigma_0, \varsigma_1)| \ge 6\frac{1}{2}k^2 + \left(2\frac{1}{2} - \beta\right)k - 2\alpha + \frac{\beta^2}{2} - \frac{\beta}{2} - 1.$$

Since $\alpha < \beta$ it follows that

$$|\mathcal{N}(\varsigma_0, \varsigma_1)| \ge 6\frac{1}{2}k^2 + \left(2\frac{1}{2} - \beta\right)k + \frac{\beta^2}{2} - \frac{5\beta}{2} + 1. \tag{13}$$

A lower bound on $|\mathcal{N}(\varsigma_0, \varsigma_1)|$ in the given range of β is obtained for $\beta = k$. Substituting $\beta = k$ in (13) we obtain $|\mathcal{N}(\varsigma_0, \varsigma_1)| \ge 6k^2 + 1$. ∎

LEMMA 31. *If* $0 < \alpha < \beta$ *and* $k < \beta$, *then* $|\mathcal{N}(\varsigma_0, \varsigma_1)| > 6k^2$.

Proof. By Lemmas 18, 19, and 21 we have that

$$|\mathcal{N}(\varsigma_0, \varsigma_1) \cap \mathcal{P}(\varsigma_0, \varsigma_1) \cap (T_l(\varsigma_0) \cup T_r(\varsigma_0) \cup T_r(\varsigma_1))| \ge 6k - 4\alpha - 2\beta - 1. \tag{14}$$

By Lemmas 26, 28, and 29 we have that

$$|\mathcal{N}(\varsigma_0,\varsigma_1)\setminus \mathcal{P}(\varsigma_0,\varsigma_1)| \geq \frac{1}{2}(\alpha+\beta-k+1)(\alpha+\beta-k)-2. \tag{15}$$

Combining (5) with (14) and (15) we obtain

$$|\mathcal{N}(\varsigma_0,\varsigma_1)| \geq 6\frac{1}{2}k^2+\left(3\frac{1}{2}-\beta\right)k-2\alpha+\frac{\beta^2}{2}-\frac{3\beta}{2}.$$

Since $\alpha < 2k-\beta$ it follows that

$$|\mathcal{N}(\varsigma_0,\varsigma_1)| \geq 6\frac{1}{2}k^2-\left(\frac{1}{2}+\beta\right)k+\frac{\beta^2}{2}+\frac{\beta}{2}+2. \tag{16}$$

A lower bound on $|\mathcal{N}(\varsigma_0,\varsigma_1)|$, in the given range of β, is obtained for $\beta = k+1$. Substituting $\beta = k+1$ in (16) we obtain $|\mathcal{N}(\varsigma_0,\varsigma_1)| \geq 6k^2+3$. $\square$

LEMMA 32. *If $\alpha = 0$, then $|\mathcal{N}(\varsigma_0,\varsigma_1)| > 6k^2$.*

Proof. This case is the only one in which some of the cells in $\mathcal{P}(\varsigma_0,\varsigma_1)$ above the left tip of ς_0 belong to $T_l(\varsigma_1)$. Thus we have the following variant of Lemma 22:

$$\begin{aligned}
&|\mathcal{N}(\varsigma_0,\varsigma_1)\cap\mathcal{P}(\varsigma_0,\varsigma_1)\setminus(T_l(\varsigma_0)\cup T_l(\varsigma_1)\cup T_r(\varsigma_0)\cup T_r(\varsigma_1))|\\
&= 6k^2+(\alpha-2)k-\frac{\alpha^2}{2}+\frac{3\alpha}{2}-\alpha\beta+4\\
&= 6k^2-2k+4
\end{aligned}$$

since in the proof $|\Pi_2| = 0$. On the other hand, Lemma 17 takes the following form:

If $\beta \leq k+1$, then $T_l(\varsigma_0)\cup T_l(\varsigma_1)\subset\mathcal{N}(\varsigma_0,\varsigma_1)$ and $|T_l(\varsigma_0)\cup T_l(\varsigma_1)| = 2k-2$.

Lemma 18 takes the following form:

If $\beta > k+1$, then $|\mathcal{N}(\varsigma_0,\varsigma_1)\cap\mathcal{P}(\varsigma_0,\varsigma_1)\cap(T_l(\varsigma_0)\cup T_l(\varsigma_1))| \geq 3k-\beta-1$.

Therefore, $|\mathcal{N}(\varsigma_0,\varsigma_1)| > 6k^2$ as computed in Lemmas 23 and 31. $\square$

LEMMA 33. *If $\beta \leq \alpha \leq k+1$ and $\alpha+\beta > k+1$, then $|\mathcal{N}(\varsigma_0,\varsigma_1)| > 6k^2$.*

Proof. By Lemmas 18 and 19 we have that

$$|\mathcal{N}(\varsigma_0,\varsigma_1)\cap\mathcal{P}(\varsigma_0,\varsigma_1)\cap(T_l(\varsigma_0)\cup T_r(\varsigma_1))| \geq 5k-3\alpha-2\beta-1. \tag{17}$$

By Lemma 26, 28, and 29 we have that

$$|\mathcal{N}(\varsigma_0,\varsigma_1)\setminus \mathcal{P}(\varsigma_0,\varsigma_1)| \geq \frac{1}{2}(\alpha+\beta-k+1)(\alpha+\beta-k)-2. \tag{18}$$

Combining (7) with (17) and (18) we obtain

$$|\mathcal{N}(\varsigma_0,\varsigma_1)| \geq 6\frac{1}{2}k^2+\left(\alpha-2\beta+2\frac{1}{2}\right)k+\frac{3\beta^2-3\alpha^2+2\alpha\beta-4\beta-6\alpha}{4}-1. \tag{19}$$

A lower bound on $|\mathcal{N}(\varsigma_0,\varsigma_1)|$ for a fixed α and in the given range of β is obtained for the largest possible value of β. We distinguish between three cases:

- If $\beta \le \alpha - 3$, then the largest value of β is $\alpha - 3$. Substituting $\beta = \alpha - 3$ in (19) implies that

$$|\mathcal{N}(\varsigma_0, \varsigma_1)| \ge 6\frac{1}{2}k^2 + \left(8\frac{1}{2} - \alpha\right)k + \frac{\alpha^2 - 17\alpha}{2} + 8\frac{3}{4}. \tag{20}$$

  A lower bound on $|\mathcal{N}(\varsigma_0, \varsigma_1)|$ in the given range of α is obtained for $\alpha = k+1$. Substituting $\alpha = k + 1$ in (20) we obtain $|\mathcal{N}(\varsigma_0, \varsigma_1)| \ge 6k^2 + \frac{3}{4}$.
- If $\alpha - 2 \le \beta \le \alpha - 1$, then the largest value of β is $\alpha - 1$. Substituting $\beta = \alpha - 1$ in (19) implies that

$$|\mathcal{N}(\varsigma_0, \varsigma_1)| \ge 6\frac{1}{2}k^2 + \left(4\frac{1}{2} - \alpha\right)k + \frac{\alpha^2 - 9\alpha}{2} + \frac{3}{4}. \tag{21}$$

  A lower bound on $|\mathcal{N}(\varsigma_0, \varsigma_1)|$ in the given range of α is obtained for $\alpha = k$ (note that $\alpha+\beta < 2k$). Substituting $\alpha = k$ in (21) we obtain $|\mathcal{N}(\varsigma_0, \varsigma_1)| \ge 6k^2 + \frac{3}{4}$.
- If $\alpha = \beta$, then

$$|\mathcal{N}(\varsigma_0, \varsigma_1)| \ge 6\frac{1}{2}k^2 + \left(2\frac{1}{2} - \alpha\right)k + \frac{\alpha^2 - 5\alpha}{2} - 1. \tag{22}$$

  A lower bound on $|\mathcal{N}(\varsigma_0, \varsigma_1)|$ in the given range of α is obtained for $\alpha = k-1$ (note that $\alpha+\beta < 2k$). Substituting $\alpha = k-1$ in (22) we obtain $|\mathcal{N}(\varsigma_0, \varsigma_1)| \ge 6k^2 + 2$.

Thus, $|\mathcal{N}(\varsigma_0, \varsigma_1)| > 6k^2$. ∎

LEMMA 34. *If $\varsigma_2 = (\gamma_1, \delta_1)$ and $\varsigma_3 = (\gamma_2, \delta_2)$ are two black cells such that $\mathcal{U}_l(\varsigma_0) \cap \mathcal{S}(\varsigma_2) \ne \varnothing$ and $\mathcal{U}_r(\varsigma_1) \cap \mathcal{S}(\varsigma_3) \ne \varnothing$, then $\mathcal{P}(\varsigma_0, \varsigma_2) \cap \mathcal{P}(\varsigma_1, \varsigma_3) = \varnothing$.*

Proof. $\mathcal{P}(\varsigma_0, \varsigma_1) \cap \mathcal{S}(\varsigma_2) = \varnothing$ by Corollary 1, and hence $\delta_1 - \gamma_1 - k \ge 3k - \alpha$, i.e., $3k - |\gamma_1| - \delta_1 - 1 \le \alpha - k - 1$.

$\mathcal{P}(\varsigma_0, \varsigma_1) \cap \mathcal{S}(\varsigma_3) = \varnothing$ by Corollary 1, and hence $\gamma_2 + \delta_2 - k \ge 3k$, i.e., $\gamma_2 + \delta_2 - \beta - 3k + 1 \ge k - \beta + 1$.

Since $\alpha + \beta < 2k$, it follows that $\alpha - k - 1 < k - \beta + 1$.

Therefore, $3k - |\gamma_1| - \delta_1 - 1 < \gamma_2 + \delta_2 - \beta - 3k + 1$, and it can be readily verified that $\mathcal{P}(\varsigma_0, \varsigma_2) \cap \mathcal{P}(\varsigma_1, \varsigma_3) = \varnothing$ by Lemmas 3 and 12. ∎

LEMMA 35. *If $\beta \le \alpha$ and $\alpha \ge k + 2$, then $|\mathcal{N}(\varsigma_0, \varsigma_1)| > 6k^2$.*

Proof. By Lemmas 13, 14, 15, 26, 28, 29, and 34 we have that

$$|\mathcal{N}(\varsigma_0, \varsigma_1)| \setminus \mathcal{P}(\varsigma_0, \varsigma_1)| \ge \frac{1}{2}(\alpha - k)(\alpha - k - 1) + \frac{1}{2}(\alpha + \beta - k + 1)(\alpha + \beta - k) - 2.$$

Combining with (7) we obtain

$$|\mathcal{N}(\varsigma_0, \varsigma_1)| \ge 7k^2 - 2(\beta + 1)k + \alpha + \beta + \frac{3\beta^2 - \alpha^2 + 2\alpha\beta}{4}. \tag{23}$$

Since $\alpha+\beta < 2k$ and $\alpha \ge k+2$ it follows that $\beta \le k-3$. A lower bound on $|\mathcal{N}(\varsigma_0, \varsigma_1)|$ for a fixed β and in the given range of α is obtained for the largest possible value of α, i.e., $2k - 1 - \beta$. Substituting $\alpha = 2k - 1 - \beta$ in (23) implies that

$$|\mathcal{N}(\varsigma_0, \varsigma_1)| \ge 6k^2 + k - \beta - 1\frac{1}{4} \ge 6k^2 + 1\frac{3}{4}. \qquad \square$$

Proof of Lemma 8. W.l.o.g. we assume that $\varsigma_1 = (0,0)$ and $\varsigma_2 = (\alpha, \beta),\ \alpha, \beta \geq 0$.

- If $\alpha + \beta \leq k+1$, then by Lemmas 23, 25, and 32, $|\mathcal{N}(\varsigma_1, \varsigma_2)| > 6k^2$.
- If $\alpha + \beta > k+1$ and $\alpha < \beta$, then by Lemmas 30, 31, and 32, $|\mathcal{N}(\varsigma_1, \varsigma_2)| > 6k^2$.
- If $\alpha + \beta > k+1$ and $\alpha \geq \beta$, then by Lemmas 33 and 35 $|\mathcal{N}(\varsigma_1, \varsigma_2)| > 6k^2$.

Thus, $|\mathcal{N}(\varsigma_1, \varsigma_2)| > 6k^2$. ◻

Appendix C. In this appendix we will prove Lemma 9. By considering the various cases in the proof of Lemma 7, one can readily verify that in most cases the size of the neighborhood is greater than $3k^2$. In some cases, where the formulas indicate that the size of a neighborhood can be $3k^2$, we can find some cells in the neighborhood which were not counted, e.g., above the tips. There are two cases in which the size of the neighborhood of a given black cell $\varsigma = (\alpha, \beta)$ can be $3k^2$, depending on the position of the black cells surrounding it:

- Either $(\alpha + k, \beta + k)$ is a black cell or $(\alpha - k, \beta + k)$ is a black cell (Case 2.2 item marked with $*$ in Lemma 7).
- Either $(\alpha + k + 1, \beta + k - 1)$ is a black cell or $(\alpha - k - 1, \beta + k - 1)$ is a black cell (a specific value in Case 2.3 of Lemma 7).

In the latter case one can verify that the coloring cannot be strongly optimal. Therefore, we have the following lemma.

LEMMA 36. *If $\mathcal{F}$ is a coloring for which all the neighborhoods have size $3k^2$, then for each black cell (α, β), either $(\alpha + k, \beta + k)$ is a black cell or $(\alpha - k, \beta + k)$ is a black cell.*

W.l.o.g. we assume that (α, β) and $(\alpha + k, \beta + k)$ are black cells for some $(\alpha, \beta) \in \mathbb{Z}^2$.

LEMMA 37. *If (α, β) and $(\alpha + k, \beta + k)$ are black cells, then $(\alpha + ik, \beta + ik)$ is a black cell for each $i \geq 0$.*

Proof. By Lemma 36, if $(\alpha + k, \beta + k)$ is a black cell, then either $(\alpha + 2k, \beta + 2k)$ is a black cell or $(\alpha, \beta + 2k)$ is a black cell. But $(\alpha, \beta + 2k)$ cannot be a black cell since $d_3((\alpha, \beta), (\alpha + k, \beta + k), (\alpha, \beta + 2k)) = 3k < 4k$. ◻

For each $i \geq 0$ let $\varsigma_i = (\alpha + ik, \beta + ik)$; by Lemma 12 we have the following lemma.

LEMMA 38. *All the cells on the line $y - x = 2k + \beta - \alpha - 1$, where $x > \alpha - k$, are inside $\bigcup_{i=0}^{\infty} \mathcal{P}(\varsigma_i, \varsigma_{i+1})$. None of the cells on the line $y - x = 2k + \beta - \alpha$ belongs to $\bigcup_{i=0}^{\infty} \mathcal{P}(\varsigma_i, \varsigma_{i+1})$.*

LEMMA 39. *There exists $j,\ -2k + 1 \leq j \leq -k + 1$, such that $(\alpha + j + ik, \beta + 3k + j + ik)$ are black cells for all $i \geq 0$.*

Proof. Recall that all the black cells satisfy Case 2.2 in the proof of Lemma 7. Note that in this case if $|\mathcal{N}(\varsigma_i)| = 3k^2$, then $\mathcal{N}(\varsigma_i) \subset \mathcal{P}(\varsigma_i, \varsigma_{i+1}) \cup \mathcal{S}(\varsigma_i)$, and hence all the cells on the line $y - x = 2k + \beta - \alpha$ for $x > \alpha - k$ belong to some spheres whose black cells are on the line $y - x = 3k + \beta - \alpha$. In particular, the cell $(\alpha - k + 1, \beta + k + 1)$ belongs to some sphere. Therefore, there is a black cell $(\alpha + j, \beta + 3k + j)$ for some $j,\ -2k + 1 \leq j \leq -k + 1$. The next black cell on the line $y - x = 3k + \beta - \alpha$ must be either $(\alpha + j + k, \beta + 3k + j + k)$ or $(\alpha + j + k + 1, \beta + 3k + j + k + 1)$. Since the size of the neighborhood of $(\alpha + j, \beta + 3k + j)$ is $3k^2$, it follows that $(\alpha + j + k, \beta + 3k + j + k)$ is a black cell, and thus by Lemma 37, for each $i \geq 0$, $(\alpha + j + ik, \beta + 3k + j + ik)$ is a black cell. ◻

COROLLARY 10. *For each $l \geq 0$ there exists $j_l, -2k + 1 \leq j_l \leq -k + 1$, such that $(\alpha + \Sigma_{m=1}^{l} j_m, \beta + 3kl + \Sigma_{m=1}^{l} j_m)$ is a black cell.*

Let $\mathcal{R}_{\alpha,\beta} = \{(x, y) : y \geq \frac{2k+1}{k-1}(x - \alpha) + \beta,\ y \geq \frac{2k+1}{-k+1}(x - \alpha) + \beta\}$.

COROLLARY 11. *There exist a shift vector $\overrightarrow{s}$ and an integer $h,\ 0 \leq h \leq 3k - 1$, such that $T^R_{\overrightarrow{s}, h}(\Lambda^R) \cap \mathcal{R}_{\alpha,\beta}$ consists of all the black cells inside $\mathcal{R}_{\alpha,\beta}$.*

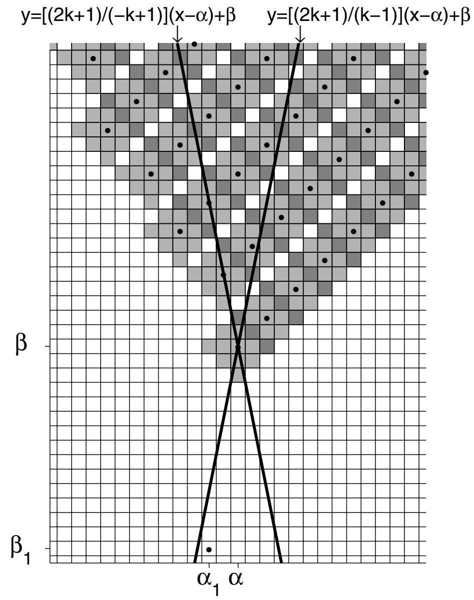


FIG. 21. *Corollary* 11 *with* $k = 2$.

By Theorem 3, there exists a black cell (α_1, β_1) in the region

$$\left\{(x,y) : y < \frac{2k+1}{k-1}(x-\alpha)+\beta,\ y < \frac{2k+1}{-k+1}(x-\alpha)+\beta\right\}.$$

We can start our discussion in this appendix with (α_1, β_1) instead of (α, β). The scenario is depicted in Figure 21.

COROLLARY 12. *There exist a shift vector* $\overrightarrow{s}$ *and an integer* h, $0 \leq h \leq 3k-1$, *such that either* $T^{R}_{\overrightarrow{s},h}(\Lambda^R) \cap \mathcal{R}_{\alpha_1,\beta_1}$ *or* $T^{L}_{\overrightarrow{s},h}(\Lambda^L) \cap \mathcal{R}_{\alpha_1,\beta_1}$ *consists of all the black cells inside* $\mathcal{R}_{\alpha_1,\beta_1}$.

Note that $\mathcal{R}_{\alpha_1,\beta_1} \supset \mathcal{R}_{\alpha,\beta}$; we can define an infinite sequence of cells (α_i, β_i), $i \geq 0$, such that $\mathcal{R}_{\alpha_{i+1},\beta_{i+1}} \supset \mathcal{R}_{\alpha_i,\beta_i}$ and $\mathcal{R}_{\alpha_i,\beta_i} \rightarrow_{i\rightarrow\infty} \mathbb{Z}^2$, and hence Lemma 9 is proved by Corollary 12.

Acknowledgment. We would like to thank an anonymous referee for his valuable comments.

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