Two-Dimensional Interleaving Schemes With Repetitions: Constructions and Bounds

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***Abstract*—Two-dimensional interleaving schemes with repetitions are considered. These schemes are required for the correction of two-dimensional bursts (or clusters) of errors in applications such as optical recording and holographic storage. We assume that a cluster of errors may have an arbitrary shape, and is characterized solely by its area t. Thus, an *interleaving scheme* $A(t, r)$ of strength t with r repetitions is an (infinite) array of integers defined by the property that every integer appears no more than r times in any connected component of area t. The problem is to minimize, for a given t and r, the *interleaving degree* $\deg A(t, r)$, which is the total number of distinct integers contained in the array. Optimal interleaving schemes for $r = 1$ (no repetitions) have been devised in earlier work. Here, we consider interleaving schemes for $r \geqslant 2$. Such schemes reduce the overall redundancy, yet are considerably more difficult to construct and analyze. To this end, we generalize the concept of L_1-distance and introduce the notions of tristance, quadristance, and more generally r-dispersion. We focus on the special class of interleaving schemes, called *lattice interleavers*, that is akin to the class of linear codes in coding theory. We construct efficient lattice interleavers for $r = 2, 3, 4$ and some higher values of r. For $r = 2, 3$, we show that these lattice interleavers are either optimal for all t or asymptotically optimal for $t \to \infty$. We present the results of an extensive computer search that yields the optimal lattice interleavers for $r = 2, 3, 4, 5, 6$, and t up to about 1000. Finally, we consider an alternative connectivity model for clusters, where two elements in an array are connected if they are adjacent horizontally, vertically, or *diagonally*. We establish relations between interleavers for this model and interleavers for the standard horizontal/vertical connectivity model, and show that these models become equivalent for $t \to \infty$. We conclude with some conjectures and open problems.**

***Index Terms*—Bursts, clusters, lattices, L_1-distance, multidimensional interleaving, r-dispersion.**

I. Introduction

A ONE-DIMENSIONAL error burst of length t is a set of t or fewer errors confined to t consecutive locations [21]. The most common approach for dealing with one-dimensional error bursts is interleaving. The idea is to assign consecutive symbols in a data sequence to a number of separate codewords (or codes). For example, to implement the correction of bursts of length 4, one can use four different codewords drawn from a code that corrects τ errors, while encoding, or *interleaving*, the one-dimensional data sequence as follows: $1234123412341234\cdots$. Here, the symbols $1, 2, 3,$ and 4 correspond to the first, second, third, and fourth codewords, respectively. It is obvious that any τ bursts of length up to 4 can be corrected in this fashion. In general, this straightforward interleaving scheme, which requires t different codewords to correct bursts of length t, is optimal in the sense that there is no other interleaving scheme that can accomplish the same error correction with less than t different codewords drawn from a τ-error-correcting code. An alternative way to correct any τ bursts of length up to 4 is to use two different codewords from a code that corrects 2τ errors, while interleaving the one-dimensional data sequence as follows: $1122112211221122\cdots$. This is an interleaving scheme with two *repetitions*, in that the same integer appears (at most) twice within a burst of length 4. This scheme is also optimal, in the same sense as the $1234123412341234\cdots$ interleaving scheme. Furthermore, if the error-correcting codes used are maximum-distance separable (MDS), both interleaving schemes have the same redundancy of 8τ, regardless of the length of the data sequence.

The optimal one-dimensional interleaving schemes, both with and without repetitions, are straightforward. However, in two dimensions it is not at all obvious how to interleave a minimal number of codewords so that any burst of size up to t can be corrected. To begin with, one needs to define two-dimensional bursts, or clusters, of errors. Most two-dimensional burst-correcting codes that have been studied in the literature so far [1], [4], [9], [13], [16], [17] correct error bursts of a given rectangular shape, say $t_1 \times t_2$ rectangular arrays. In [6], [12], and [22] the authors consider metrics given by the rank of an array: a particular case is the correction of "criss-cross" errors. Metrics for different channels are presented in [11]. In this work, we assume that a *cluster of errors* can have an arbitrary shape, as long as it maintains horizontal/vertical connectivity. Thus, a two-dimensional cluster is characterized solely by its total area or size t. This is the natural generalization to two dimensions of the concept of a one-dimensional burst of length t. Important applications where the correction of such two-dimensional error clusters is required are optical recording (e.g., page-oriented optical memories [20]) and holographic storage (e.g., volume holographic memory [14], [15]).

Given the foregoing notion of a cluster, one may define a two-dimensional *interleaving scheme* $A(t, r)$ of strength t with r repetitions as an infinite array of integers characterized by the property that every integer appears at most r times in any cluster of size t. The *interleaving degree* of $A(t, r)$, denoted

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$\deg A(t, r)$, is the total number of distinct integers contained in the array. An interleaving scheme $A(t, r)$ is said to be *optimal* if $\deg A(t, r)$ is the minimum possible for the given t and r.

Blaum, Bruck, and Vardy [3] have constructed optimal two-dimensional interleaving schemes without repetitions for all t. Thus, the case $r = 1$ is solved, and the minimum possible interleaving degree is $\lceil t^2/2 \rceil$. Although interleaving schemes (without repetitions) in three and higher dimensions were also considered in [3], we will focus exclusively on two dimensions in this work. Blaum, Bruck, and Farrell [2] have generalized the two-dimensional interleaving schemes of [3] in such a way that each integer appears at most twice in any cluster of size t. However, the methods developed in [2] are limited in their scope and applicability. On the other hand, it is obvious from the work of [2], [3] that the problem of constructing $A(t, r)$ to minimize $\deg A(t, r)$ becomes much more challenging for $r \geqslant 2$. At this time, preciously little is known about two-dimensional interleaving schemes with repetitions.

It is natural to ask why interleaving schemes with repetitions might be preferable to the ones without repetitions. The following example provides some motivation. Suppose we need to record a 1024×1024 array of symbols over GF(256), and would like to correct $\tau = 4$ clusters of size up to $t = 256$ using MDS codes, say shortened Reed–Solomon (RS) codes over GF(256). An optimal interleaving scheme without repetitions has degree $\lceil t^2/2 \rceil = 2^{15}$. Thus, we need $2^{15} = 32\,768$ codewords drawn from a $(32, 24, 9)$ shortened RS code. It follows that $2^{15}(32-24) = 262\,144$ of the recorded symbols must be allocated to the redundancy due to error-correction coding, an overall redundancy rate of 25%. However, later in this paper, we will construct an interleaving scheme *with two repetitions* whose degree is $3t^2/16 = 3 \cdot 2^{12}$. With this interleaving scheme, to accomplish the same error correction, we need only $3 \cdot 2^{12} = 12288$ codewords drawn from an $(85, 69, 17)$ or an $(86, 70, 17)$ shortened RS code. This results in only $3 \cdot 2^{16} = 196\,608$ redundant symbols, and overall redundancy rate of 18.75%. We can do even better with three repetitions. In particular, we will construct in Section IV an interleaving scheme $A(256, 3)$ with $\deg A(256, 3) = 6498$. In this case, we need 6498 codewords drawn from a $(161, 137, 25)$ or a $(162, 138, 25)$ shortened RS code. This results in only $6498 \cdot 24 = 155\,952$ redundant symbols, or an overall redundancy rate of about 14.87%. The conclusion from this example is that, in contrast to the one-dimensional case, in two dimensions, judiciously constructed interleaving schemes *with repetitions* can lead to significant savings in the overall redundancy due to error-correction coding.

The rest of this paper is organized as follows. We start in the next section with some basic concepts and auxiliary results. First, we introduce the notion of r-*dispersion* that turns out to be crucial in the design of two-dimensional interleaving schemes with repetitions. The r-dispersion may be thought of as a generalization of the L_1-distance [10], [24] to a quantity that reflects a property of r points for $r \geqslant 2$. Thus 2-dispersion is just the L_1-distance. For $r = 3, 4, 5$, we refer to the corresponding r-dispersion as *tristance*, *quadristance*, and *quintistance*, respectively. Efficient methods for computing these dispersions are also presented in the next section. In Section II-B, we introduce a special class of interleaving schemes based on two-dimensional lattices, which we call *lattice interleavers*. Lattice interleavers are akin to linear codes in coding theory: both classes are distinguished by the fact that a certain linearity property is imposed on their structure. So far, *all* the best known interleaving schemes, with or without repetitions, belong to the class of lattice interleavers, and we will focus almost exclusively on lattice interleavers in the remainder of this work.

In Section III, we consider interleaving schemes for two repetitions. First, we construct lattice interleavers $A(t, 2)$ for all t, and compute the corresponding tristance. We then present lower bounds on the degree of lattice interleavers with two repetitions, which show that our constructions are optimal for even t and asymptotically optimal for odd t. Finally, we develop the methodology for an elaborate computer search that produces optimal lattice interleavers with two repetitions for all $t \leqslant 1003$. These results support our conjecture that the lattice interleavers constructed in this section are, in fact, optimal for all values of t, both even and odd.

In Section IV, we present analogous constructions, bounds, and computer search for lattice interleavers with three repetitions. In particular, we prove a lower bound which shows that our constructions are optimal for $t \equiv 0 \bmod 9$, and asymptotically optimal for other t. The computer search yields optimal lattice interleavers $A(t, 3)$ for $t \leqslant 999$. We conjecture that for all higher values of t, optimal lattice interleavers may be obtained from our construction.

In Section V, we construct lattice interleavers with four repetitions for all strengths t and compute their quintistance. Although we do not have lower bounds on the interleaving degree in this case, we conjecture that these lattice interleavers are, in fact, optimal, except for $t = 4, 5, 16, 53, 70$. The results of an exhaustive computer search confirm this conjecture up to $t = 1008$.

Section VI deals with multiple repetitions. In particular, we construct the lattice interleavers $A(18k, 5)$, $A(14k, 6)$, and $A(33k, 7)$ for all $k \in \mathbb{Z}^+$. We believe that these lattice interleavers are optimal for $r = 5, 6$, and 7, respectively. For higher values of r, we exhibit certain infinite families of lattice interleavers, such as $A(rk, r)$ and $A(r^2k, r)$ for all $k \in \mathbb{Z}^+$, and compute the interleaving degree in each case. These families make it possible to establish general asymptotic results for large r.

All the results discussed so far pertain to the conventional [2]–[4] connectivity model where elements in a cluster are connected either vertically or horizontally. In Section VII, we consider interleaving schemes for an alternative cluster connectivity model. Namely, we assume that two elements in an array are connected if they are adjacent horizontally, vertically, or *diagonally*. First, we construct lattice interleavers with two repetitions for this model and present the results of a computer search, which confirms that these interleavers are optimal for all $t \leqslant 143$, except $t = 2, 3, 5$. More importantly, for general t and r, we show that there is a relation between interleaving schemes for this connectivity model and interleaving schemes for the standard horizontal/vertical connectivity model. We use this relation to prove that, under a certain assumption, the two connectivity models become equivalent for large t. We conclude the paper in Section VIII with a summary of redundancy savings obtained through the interleaving schemes constructed in Sec-

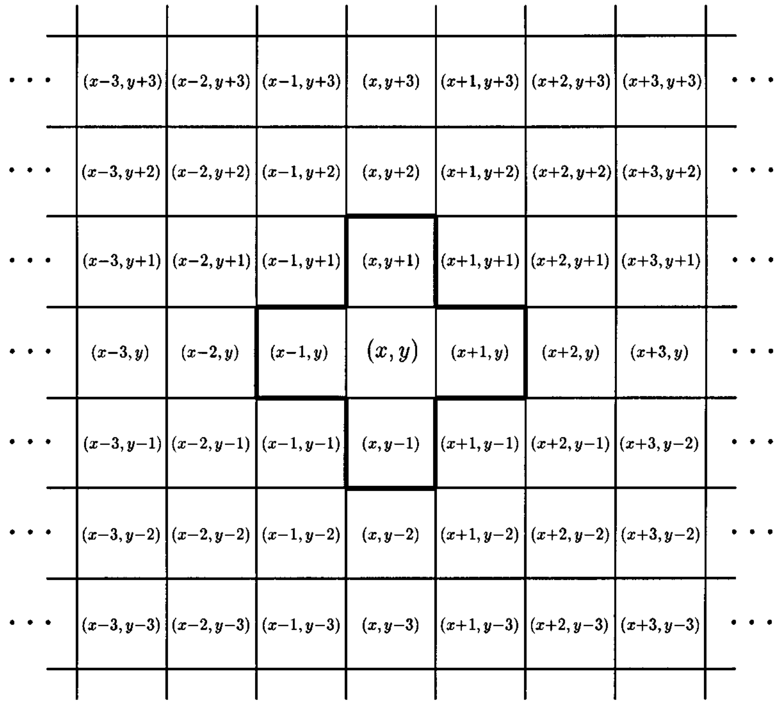

Fig. 1. The connectivity model for clusters.

tions III–VI. We also discuss some conjectures and open problems for future research.

II. Basic Concepts and Auxiliary Results

We start by elaborating upon some of the definitions made in the previous section. Consider an infinite two-dimensional array A with integer entries. We refer to the locations, or positions, of the array as *elements* of A. We assume that the array has the topology of $\mathbb{Z} \times \mathbb{Z} = \mathbb{Z}^2$; that is, there is a one-to-one correspondence between points $(x, y) \in \mathbb{Z}^2$ and elements $[x, y]$ of the array. We say that an element $[x, y]$ is *connected* to the elements $[x{+}1, y]$, $[x{-}1, y]$, $[x, y{+}1]$, and $[x, y{-}1]$. This connectivity structure is depicted in Fig. 1. An alternative connectivity model will be introduced in Section VII. A *path* of length n from an element a_0 to an element a_n in A is a sequence of $n+1$ distinct elements $a_0, a_1, \ldots, a_n$ such that a_i is connected to a_{i-1} for all $i = 1, 2, \ldots, n$.

Definition 2.1: A set $\mathcal{S}$ of t elements of A is said to be a *cluster* of size t if $t = 1$ or if $t \geqslant 2$ and any two elements of $\mathcal{S}$ belong to a path that is contained in $\mathcal{S}$.

The concept of a cluster of size t generalizes the concept of a one-dimensional burst of length t. This generalizes further to multiple dimensions [3], although we will not pursue such generalizations.

Definition 2.2: The array A is an *interleaving scheme* of strength t with r repetitions, denoted $A(t, r)$, if no integer appears more than r times in any cluster of size t in A. The total number of distinct integers that appear as entries of A is the *interleaving degree* of A, denoted $\deg A$ or $\deg A(t, r)$.

The main problem studied in this paper can be now stated as follows: for given t and r, construct $A(t, r)$ so that $\deg A(t, r)$ is minimized. We focus mainly on low values of r, in particular $r = 2, 3, 4$.

It would be often convenient to formulate the same problem in different terms. Consider a graph $\mathcal{G} = (V, E)$ whose vertex set is $V = \mathbb{Z}^2$ and whose edge set is defined as follows: there is an edge $\{z_1, z_2\} \in E$ if and only if $d(z_1, z_2) = 1$, where $d(\cdot, \cdot)$ is the L_1-distance. The graph $\mathcal{G}$ is often called the *grid graph* on $\mathbb{Z}^2$, and the corresponding connectivity structure is illustrated in Fig. 2.

It is obvious that this graph is planar, and we will sometimes identify the edges of $\mathcal{G}$ with the corresponding line segments in $\mathbb{R}^2$, as depicted in Fig. 2. In terms of the grid graph, a cluster of size t is just a connected subgraph with t vertices. An interleaving scheme $A(t, r)$ is then a coloring of the vertices of $\mathcal{G}$ with the property that no color appears more than r times in a cluster of size t, and the interleaving degree $\deg A(t, r)$ is the total number of colors. We will use the two equivalent formulations (that of Fig. 1 and that of Fig. 2) interchangeably throughout this paper.

A. The r-Dispersion and its Computation

For any two points $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$ in $\mathbb{Z}^2$, the L_1-*distance* between z_1 and z_2 is defined as

$$d(z_1, z_2) = |x_1 - x_2| + |y_1 - y_2|.$$

Clearly, $d(z_1, z_2) = \delta$ if and only if the size of the smallest cluster that contains both points is $\delta + 1$. This observation provides the basis for a generalization of the notion of L_1-distance to a quantity that reflects a property of three or more points. In particular, for any three points $z_1, z_2, z_3 \in \mathbb{Z}^2$, we define the *tristance* $d_3(z_1, z_2, z_3)$ as one less than the size of the smallest cluster that contains the three points. More generally, we have the following definition.

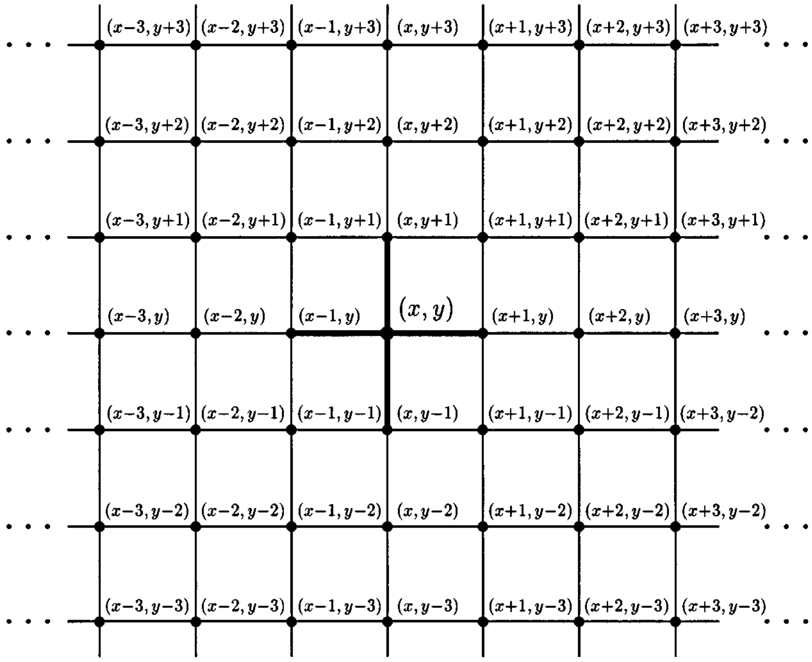

Fig. 2. The connectivity model viewed as a grid graph.

Definition 2.3: Given any r points $z_1, z_2, \ldots, z_r \in \mathbb{Z}^2$, the *$r$-dispersion* $d_r(z_1, z_2, \ldots, z_r)$ is defined as one less than the size of the smallest cluster that contains all of these points.

Notice that $d_1(z) = 0$ for all $z \in \mathbb{Z}^2$, while $d_2(z_1, z_2)$ is just the L_1-distance between z_1 and z_2. Also notice that the r points $z_1, z_2, \ldots, z_r$ in Definition 2.3 do not have to be distinct. However, if the multiset $\{z_1, z_2, \ldots, z_r\}$ contains only $\rho < r$ distinct points, say $z_{i_1}, z_{i_2}, \ldots, z_{i_\rho}$, then obviously

$$d_r(z_1, z_2, \ldots, z_r) = d_\rho(z_{i_1}, z_{i_2}, \ldots, z_{i_\rho}). \tag{1}$$

Finally, we observe that the definition of r-dispersion has a nice interpretation in terms of the grid graph $\mathcal{G}$. Given r vertices $z_1, z_2, \ldots, z_r$ in $V = \mathbb{Z}^2$, the r-dispersion $d_r(z_1, z_2, \ldots, z_r)$ is just the number of edges in a minimum spanning tree for $z_1, z_2, \ldots, z_r$. The following simple properties of r-dispersion are obvious from this viewpoint.

Lemma 2.1: Let $\mathcal{S} = \{z_1, z_2, \ldots, z_r\}$ be a set of r distinct points in $\mathbb{Z}^2$ and let $\mathcal{S}' = \{z'_1, z'_2, \ldots, z'_\rho\}$ be any subset of $\mathcal{S}$. Then $d_r(z_1, z_2, \ldots, z_r) \geqslant d_\rho(z'_1, z'_2, \ldots, z'_\rho)$.

Lemma 2.2: Let

$$z_1 = (x_1, y_1),\ z_2 = (x_2, y_2),\ \ldots,\ z_r = (x_r, y_r)$$

be r points in $\mathbb{Z}^2$. Then

$$d_r(z_1, z_2, \ldots, z_r) \geqslant \left(\max_{1 \leqslant i \leqslant r} x_i - \min_{1 \leqslant i \leqslant r} x_i\right) + \left(\max_{1 \leqslant i \leqslant r} y_i - \min_{1 \leqslant i \leqslant r} y_i\right).$$

Proof: Think of the cluster that contains $z_1, z_2, \ldots, z_r$ as a subgraph T of the grid graph $\mathcal{G}$. It is easy to see that the number of horizontal edges in T is at least $\max_{1 \leqslant i \leqslant r} x_i - \min_{1 \leqslant i \leqslant r} x_i$ while the number of vertical edges in T is at least $\max_{1 \leqslant i \leqslant r} y_i - \min_{1 \leqslant i \leqslant r} y_i$. □

Given a set $\mathcal{S} \subseteq \mathbb{Z}^2$ with $|\mathcal{S}| \geqslant r$, we define the *minimum r-dispersion* $d_r(\mathcal{S})$ of $\mathcal{S}$ as follows. If $|\mathcal{S}| = r$, then $d_r(\mathcal{S})$ is given directly by Definition 2.3, while if $|\mathcal{S}| > r$ then

$$d_r(\mathcal{S}) \stackrel{\text{def}}{=} \min_{\substack{\mathcal{S}' \subset \mathcal{S} \\ |\mathcal{S}'| = r}} d_r(\mathcal{S}').$$

Let A be an infinite two-dimensional array of integers, and suppose that $\deg A = m$. Without loss of generality (w.l.o.g.), we shall henceforth regard A as a labeling of $\mathbb{Z}^2$ by the integers $1, 2, \ldots, m$. For each $i = 1, 2, \ldots, m$, let $A_i \subset \mathbb{Z}^2$ denote the set of points in A that are labeled by the integer i. The significance of r-dispersion in our context follows from the following fundamental theorem.

Theorem 2.3: The array A is an interleaving scheme of strength t with r repetitions if and only if the minimum $(r+1)$-dispersion of each of the sets $A_1, A_2, \ldots, A_{\deg A}$ is at least t.

Proof: ($\Rightarrow$) Suppose that $d_{r+1}(A_i) \geqslant t$. Then the smallest cluster that contains $r+1$ or more points labeled by the integer i must have size at least $t+1$. If this is true for all i, then any cluster of size t or less cannot contain more than r points labeled by the same integer, which is the defining property of the interleaving scheme $A(t, r)$. ($\Leftarrow$) Suppose that A is an interleaving scheme of strength t with r repetitions, and consider $r+1$ distinct points $z_1, z_2, \ldots, z_{r+1} \in A_i$ for some i. Since the integer i cannot appear more than r times in a cluster of size t in A, it follows that the size of the smallest cluster that contains the points $z_1, z_2, \ldots, z_{r+1} \in A_i$ is at least $t+1$. Thus, $d_{r+1}(z_1, z_2, \ldots, z_{r+1}) \geqslant t$. Since this is true for any $z_1, z_2, \ldots, z_{r+1} \in A_i$, we have $d_{r+1}(A_i) \geqslant t$. □

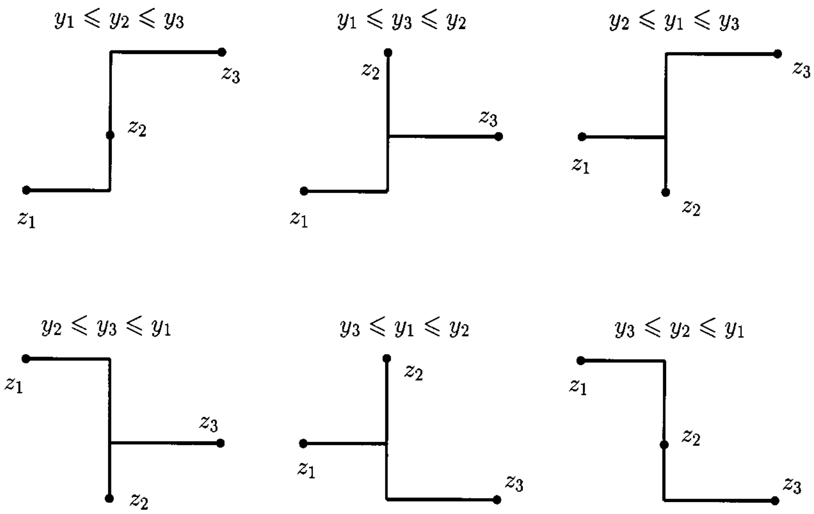

Fig. 3. Minimal spanning trees for three points (see Theorem 2.4).

As pointed out by a referee, the strength of A could, in fact, be strictly greater than t unless the minimum $(r+1)$-dispersion of at least one of the sets $A_1, A_2, \ldots, A_{\deg A}$ is exactly t.

The remainder of this subsection deals with methods for the computation of r-dispersion. We start with a simple explicit formula for the tristance. Notice that the L_1-distance between (x_1, y_1) and (x_2, y_2) can be written as

$$\max\{x_1, x_2\} - \min\{x_1, x_2\} + \max\{y_1, y_2\} - \min\{y_1, y_2\}.$$

Thus, the bound of Lemma 2.2 holds with equality for $r = 2$. The next theorem shows that this is also true for $r = 3$.

Theorem 2.4: Let $z_1 = (x_1, y_1)$, $z_2 = (x_2, y_2)$, and $z_3 = (x_3, y_3)$ be three distinct points in $\mathbb{Z}^2$. Then

$$d_3(z_1, z_2, z_3) = \left(\max_{1\leqslant i\leqslant 3} x_i - \min_{1\leqslant i\leqslant 3} x_i\right) + \left(\max_{1\leqslant i\leqslant 3} y_i - \min_{1\leqslant i\leqslant 3} y_i\right). \quad (2)$$

Proof: In view of Lemma 2.2, it would suffice to exhibit a spanning tree that contains exactly

$$\max_{1\leqslant i\leqslant 3} x_i - \min_{1\leqslant i\leqslant 3} x_i$$

horizontal edges and

$$\max_{1\leqslant i\leqslant 3} y_i - \min_{1\leqslant i\leqslant 3} y_i$$

vertical edges. W.l.o.g., we assume that $x_1 \leqslant x_2 \leqslant x_3$. The vertices of the spanning tree are then given by

$$\{(i, y_1): x_1 \leqslant i \leqslant x_2\} \cup \{(i, y_3): x_2 \leqslant i \leqslant x_3\} \cup \left\{(x_2, j): \min_{1\leqslant i\leqslant 3} y_i \leqslant j \leqslant \max_{1\leqslant i\leqslant 3} y_i\right\}$$

as shown in Fig. 3 for the six possible orders of y_1, y_2, y_3. It is easy to see that this tree contains exactly

$$\max_{1\leqslant i\leqslant 3} x_i - \min_{1\leqslant i\leqslant 3} x_i$$

horizontal edges and

$$\max_{1\leqslant i\leqslant 3} y_i - \min_{1\leqslant i\leqslant 3} y_i$$

vertical edges. □

Let $\mathcal{S} = \{z_1, z_2, \ldots, z_r\} \subset \mathbb{Z}^2$. We say that a point $z_i \in \mathcal{S}$ is *extremal* if it is strictly to the right, or strictly to the left, or strictly above, or strictly below all the other points in $\mathcal{S}$. The next lemma shows how to (iteratively) compute the r-dispersion of $\mathcal{S}$ as long as $\mathcal{S}$ has at least one extremal point.

Lemma 2.5: Let

$$z_1 = (x_1, y_1),\ z_2 = (x_2, y_2),\ \ldots,\ z_r = (x_r, y_r)$$

be r distinct points in $\mathbb{Z}^2$, and suppose that the point $z_1 = (x_1, y_1)$ is extremal. Specifically, w.l.o.g., assume that z_1 is strictly to the left of all the other points, so that $x_1 < x_i$ for $i = 2, 3, \ldots, r$. Define $z_1' = (x_1 + 1, y_1)$. Then

$$d_r(z_1, z_2, \ldots, z_r) = 1 + d_r(z_1', z_2, \ldots, z_r).$$

Proof: We claim that there exists a minimum spanning tree T for the points $z_1, z_2, \ldots, z_r$ that includes the edge $\{z_1, z_1'\}$. Otherwise let T' be an arbitrary minimum spanning tree for these r points. We modify T' to produce T as follows. Let H be the subgraph of T' consisting of all the edges that lie to the left of the line $x = x_1 + 1$, namely, those edges $\{(x, y), (x', y')\}$ for which $\max\{x, x'\} \leqslant x_1 + 1$. Further, let H_0 be the connected component of H (if H is a forest, otherwise $H_0 = H$) that contains the point $z_1 = (x_1, y_1)$. We define the integers $y_{\max}$ and $y_{\min}$ as follows:

$$y_{\max} \stackrel{\text{def}}{=} \max_{(x,y)\in V(H_0)} y \qquad y_{\min} \stackrel{\text{def}}{=} \min_{(x,y)\in V(H_0)} y$$

where $V(H_0)$ is the vertex set of H_0. It follows from Lemma 2.2 that H_0 contains at least $y_{\max} - y_{\min}$ vertical edges and at least one horizontal edge. We create T from T' by replacing H_0 with the tree T^* consisting of the $y_{\max} - y_{\min}$ vertical edges

$$\{(x_1 + 1, y), (x_1 + 1, y + 1)\}, \quad \text{for } y = y_{\min}, y_{\min}{+}1, \ldots, y_{\max}{-}1$$

and the single horizontal edge $\{z_1, z_1'\}$, as illustrated in Fig. 4. Clearly, the number of edges in $T = (T' \backslash H_0) \cup T^*$ does not exceed the number of edges in T'. It remains to show that T is connected and contains the points $z_1, z_2, \ldots, z_r$. Since z_1 is extremal, this follows directly from the fact that any vertex of H_0 of the type $(x_1{+}1, y)$ is also a vertex of T^*. Thus,

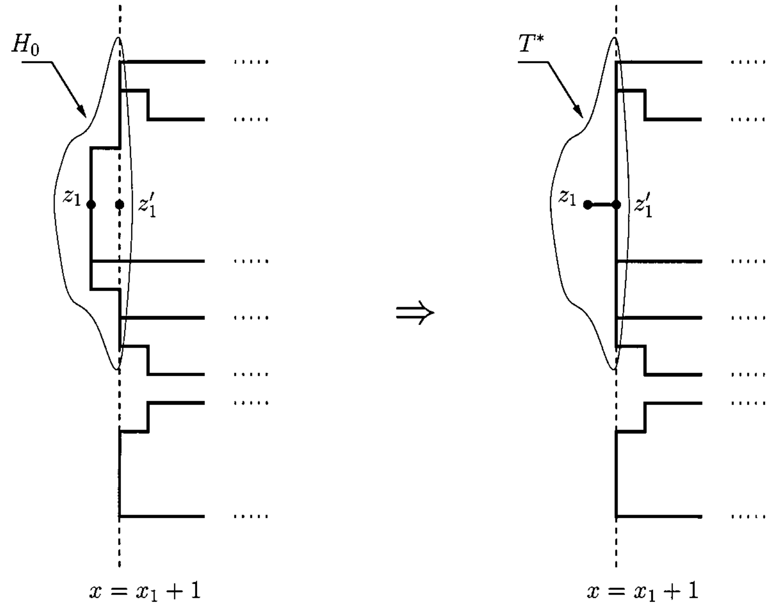

Fig. 4. Reduction of a minimal spanning tree (see Lemma 2.5).

T is a minimum spanning tree for the points $z_1, z_2, \ldots, z_r$. It is now obvious that removing the edge $\{z_1, z_1'\}$ from T produces a minimum spanning tree for the points $z_1', z_2, \ldots, z_r$, and the lemma follows. □

Let $z_1 = (x_1, y_1), z_2 = (x_2, y_2), \ldots, z_n = (x_n, y_n)$ be n distinct points in $\mathbb{Z}^2$. The *bounding rectangle* of $z_1, z_2, \ldots, z_n$ is defined as the smallest rectangle $\mathcal{R}(z_1, \ldots, z_n)$ with edges parallel to the axes that contains all the n points. Explicitly, let

$$\begin{aligned} x_{\max} &= \max\{x_1, x_2, \ldots, x_n\} \\ x_{\min} &= \min\{x_1, x_2, \ldots, x_n\} \\ y_{\max} &= \max\{y_1, y_2, \ldots, y_n\} \\ y_{\min} &= \min\{y_1, y_2, \ldots, y_n\}. \end{aligned}$$

Then, the bounding rectangle is

$$\mathcal{R}(z_1, \ldots, z_n) \stackrel{\text{def}}{=} \{(x, y) \in \mathbb{Z}^2: x_{\min} \leqslant x \leqslant x_{\max} \text{ and } y_{\min} \leqslant y \leqslant y_{\max}\}.$$

We also define the four *edges* of the bounding rectangle: the top edge $\overline{\mathcal{R}(z_1, \ldots, z_n)}$, the bottom edge $\underline{\mathcal{R}(z_1, \ldots, z_n)}$, the left edge $|\mathcal{R}(z_1, \ldots, z_n)$, and the right edge $\mathcal{R}(z_1, \ldots, z_n)|$, as follows:

$$\begin{aligned} &\overline{\mathcal{R}(z_1, \ldots, z_n)} \\ &\quad \stackrel{\text{def}}{=} \{(x, y) \in \mathbb{Z}^2: x_{\min} \leqslant x \leqslant x_{\max} \text{ and } y = y_{\max}\} \\ &\underline{\mathcal{R}(z_1, \ldots, z_n)} \\ &\quad \stackrel{\text{def}}{=} \{(x, y) \in \mathbb{Z}^2: x_{\min} \leqslant x \leqslant x_{\max} \text{ and } y = y_{\min}\} \\ &|\mathcal{R}(z_1, \ldots, z_n) \\ &\quad \stackrel{\text{def}}{=} \{(x, y) \in \mathbb{Z}^2: x = x_{\min} \text{ and } y_{\min} \leqslant y \leqslant y_{\max}\} \\ &\mathcal{R}(z_1, \ldots, z_n)| \\ &\quad \stackrel{\text{def}}{=} \{(x, y) \in \mathbb{Z}^2: x = x_{\max} \text{ and } y_{\min} \leqslant y \leqslant y_{\max}\}. \end{aligned}$$

The *corners* of the rectangle are intersections of perpendicular edges. Notice that each of the four edges must contain at least one of the points $z_1, z_2, \ldots, z_n$, and this point is extremal if and only if the corresponding edge contains no other points of $z_1, z_2, \ldots, z_n$. This proves the following lemma.

Lemma 2.6: A set $\mathcal{S} = \{z_1, z_2, \ldots, z_n\} \subset \mathbb{Z}^2$ has no extremal points if and only if each edge of the bounding rectangle $\mathcal{R}(z_1, \ldots, z_n)$ contains at least two points of $\mathcal{S}$.

Using Lemmas 2.5 and 2.6, we now derive an algorithm for the computation of the 4-dispersion $d_4(z_1, z_2, z_3, z_4)$, which we call *quadristance*. It is obvious that Lemma 2.5 can be applied iteratively as long as the four points z_1, z_2, z_3, z_4 are distinct and at least one of them is extremal. The iterations will terminate if and only if either

a) two of the points coincide, say $z_3 = z_4$. In this case,

$$d_4(z_1, z_2, z_3, z_4) = d_3(z_1, z_2, z_3)$$

by (1), and $d_3(z_1, z_2, z_3)$ is given by (2); or

b) the four points z_1, z_2, z_3, z_4 are distinct, but none of them is extremal. In this case, in view of Lemma 2.6, the four points must be at the corners of the bounding rectangle $\mathcal{R}(z_1, \ldots, z_4)$. W.l.o.g., assume that $z_1 = (x_1, y_1)$ is the lower left corner of the rectangle while $z_2 = (x_2, y_2)$ is the upper right corner. Then

$$d_4(z_1, z_2, z_3, z_4) = (x_2 - x_1) + (y_2 - y_1) + \min\{x_2 - x_1, y_2 - y_1\}.$$

This case is illustrated in Fig. 5.

Similar reasoning produces an algorithm for the computation of $d_5(z_1, z_2, z_3, z_4, z_5)$, which we call *quintistance*. Once again, we apply Lemma 2.5 iteratively as long as the five points z_1, z_2, z_3, z_4, z_5 are distinct and at least one of them is extremal. The iterations terminate if and only if either

a) two of the points coincide, say $z_4 = z_5$. In this case,

$$d_5(z_1, z_2, z_3, z_4, z_5) = d_4(z_1, z_2, z_3, z_4)$$

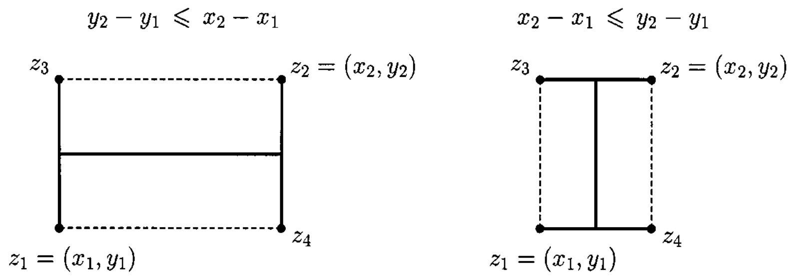

Fig. 5. Computation of quadristance.

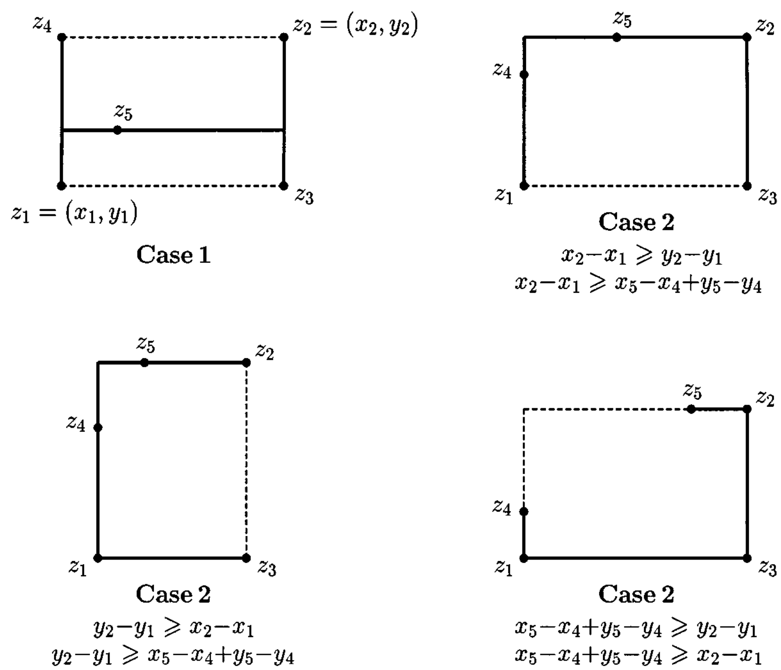

Fig. 6. Computation of quintistance.

and $d_4(z_1, z_2, z_3, z_4)$ is computed using the algorithm described in the foregoing paragraph; or

b) the five points z_1, z_2, z_3, z_4, z_5 are distinct, but none of them is extremal. Then, by Lemma 2.6, at least three of the five points, say z_1, z_2, and z_3, must be at the corners of the bounding rectangle $\mathcal{R}(z_1, \ldots, z_5)$. Once again, we may always assume w.l.o.g. that $z_1 = (x_1, y_1)$ is the lower left corner of the rectangle while $z_2 = (x_2, y_2)$ is the upper right corner.

Case 1) One of the points z_4, z_5 is at the fourth corner of $\mathcal{R}(z_1, \ldots, z_5)$. Then, as shown in Fig. 6, we have

$$\begin{aligned} d_5(z_1, z_2, z_3, z_4, z_5) &= (x_2 - x_1) + (y_2 - y_1) \\ &\quad + \min\{x_2 - x_1,\, y_2 - y_1\}. \end{aligned}$$

Case 2) The points $z_4 = (x_4, y_4)$ and $z_5 = (x_5, y_5)$ lie on two adjacent edges of $\mathcal{R}(z_1, \ldots, z_5)$. W.l.o.g., we can assume that $x_4 = x_1$ and $y_5 = y_2$, as illustrated in Fig. 6. In this case

$$\begin{aligned} &d_5(z_1, z_2, z_3, z_4, z_5) \\ &\quad = 2(x_2 - x_1) + 2(y_2 - y_1) \\ &\quad\quad - \max\{x_2 - x_1,\, y_2 - y_1,\, x_5 - x_4 + y_5 - y_4\}. \end{aligned}$$

For higher values of r, the r-dispersion can be computed in a similar manner, although the number of different terminating cases to consider grows exponentially with r. We were able to implement the computation of r-dispersion effectively up to $r = 8$. Our algorithms are based on Lemmas 2.5, 2.6, and Lemma 2.7 presented below. For each point $z = (x_0, y_0) \in \mathbb{Z}^2$, consider the vertical line $\mathfrak{X}_z$ and the horizontal line $\mathfrak{Y}_z$ given by

$$\mathfrak{X}_z = \{(x, y) \in \mathbb{Z}^2 \colon x = x_0\}$$
$$\mathfrak{Y}_z = \{(x, y) \in \mathbb{Z}^2 \colon y = y_0\}.$$

Given a set $\mathcal{S} = \{z_1, z_2, \ldots, z_n\} \subset \mathbb{Z}^2$, we define the linear grid

$$\mathcal{L}(z_1, \ldots, z_n) = \bigcup_{z \in \mathcal{S}} (\mathfrak{X}_z \cup \mathfrak{Y}_z).$$

Lemma 2.7: Let $z_1, z_2, \ldots, z_n$ be n points in $\mathbb{Z}^2$. Then there exists a minimum spanning tree T for $z_1, z_2, \ldots, z_n$ such that the set of vertices of T is a subset of

$$\mathcal{L}(z_1, \ldots, z_n) \cap \mathcal{R}(z_1, \ldots, z_n).$$

For the proof of Lemma 2.7, we refer the reader to [7]. The technical report [7] also contains a more detailed exposition of our algorithms for the computation of r-dispersion for $r \leqslant 8$.

For large r, we note that the problem of computing the r-dispersion of an arbitrary set of r points in $\mathbb{Z}^2$ is equivalent to the Steiner tree problem in a grid, under the L_1 metric. The latter problem is known to be NP-complete [10]. A polynomial-time approximation algorithm for this problem is given in [18], with the approximation ratio of 1.267. The best known exact algorithms may be found in [23], [24]. These algorithms make it possible to compute the r-dispersion of up to 2000 points.

B. Lattice Interleavers

Lattice interleavers, which are interleaving schemes based on lattices, will play an important role in our discussion. So far, *all* the best known interleaving schemes, with or without repetitions, belong to the class of lattice interleavers, and we will focus almost exclusively on lattice interleavers in the remainder of this paper. A *lattice* Λ is a discrete, nowhere dense, additive subgroup of the real n-space $\mathbb{R}^n$. W.l.o.g., we can assume that

$$\Lambda = \{u_1 v_1 + u_2 v_2 + \cdots + u_n v_n \colon u_1, u_2, \ldots, u_n \in \mathbb{Z}\} \quad (3)$$

where $\{v_1, v_2, \ldots, v_n\}$ is a set of linearly independent vectors in $\mathbb{R}^n$. A lattice Λ defined by (3) is a sublattice of $\mathbb{Z}^n$ if and only if $\{v_1, v_2, \ldots, v_n\} \subset \mathbb{Z}^n$. We will be interested solely in sublattices of $\mathbb{Z}^2$ unless stated otherwise. The vectors v_1, v_2 are called a *basis* for $\Lambda \subseteq \mathbb{Z}^2$, and the 2×2 matrix

$$\boldsymbol{G} = \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix}$$

having these vectors as its rows is said to be a *generator matrix* for Λ. The *volume* of a lattice Λ, denoted $V(\Lambda)$, is inversely proportional to the number of lattice points per unit volume. More precisely, $V(\Lambda)$ may be defined as the volume of the *fundamental parallelogram* $\Pi(\Lambda)$, which is given by

$$\Pi(\Lambda) \stackrel{\text{def}}{=} \{\theta_1 v_1 + \theta_2 v_2 \colon 0 \leqslant \theta_1, \theta_2 < 1\}.$$

There is a simple expression for the volume of Λ, namely, $V(\Lambda) = |\det \boldsymbol{G}|$. The significance of volume in our context derives from the following well-known [5] observation: the index of a sublattice Λ of $\mathbb{Z}^2$ in $\mathbb{Z}^2$ is equal to its volume, that is, the order of the group $\mathbb{Z}^2/\Lambda$ is precisely $V(\Lambda)$. Thus, given a lattice $\Lambda \subseteq \mathbb{Z}^2$ of volume $V(\Lambda) = m$, the integer lattice $\mathbb{Z}^2$ can be partitioned into m additive cosets $\Lambda, x_1 + \Lambda, \ldots, x_{m-1} + \Lambda$ for some $x_1, x_2, \ldots, x_{m-1} \in \mathbb{Z}^2$.

Suppose $d_r(\Lambda) \geqslant t$. Then it follows from Theorem 2.3 that the partition of $\mathbb{Z}^2$ into additive cosets of Λ is an interleaving scheme $A_\Lambda(t, r-1)$ of strength t with $r-1$ repetitions. The degree of this scheme is $\deg A_\Lambda(t, r-1) = V(\Lambda)$. We call such an interleaving scheme $A_\Lambda(t, r-1)$ a *lattice interleaver*.

We now present several lemmas and theorems pertaining to lattices and to lattice interleavers. Given any two linearly independent vectors $z_1, z_2 \in \Lambda$, define the parallelogram

$$\mathcal{P}(z_1, z_2) \stackrel{\text{def}}{=} \{\theta_1 z_1 + \theta_2 z_2 \colon 0 \leqslant \theta_1, \theta_2 < 1\}.$$

Theorem 2.8: If $\mathcal{P}(z_1, z_2)$ contains no other points of Λ except the origin $(0,0)$ then the vectors z_1, z_2 form a basis for Λ and $\mathcal{P}(z_1, z_2)$ is a fundamental parallelogram.

Proof: Any two linearly independent vectors $z_1, z_2 \in \Lambda$ generate a sublattice Λ' of Λ. Clearly, it would suffice to show that the set $\Lambda \backslash \Lambda'$ is empty. Assume to the contrary that $\alpha \in \Lambda \backslash \Lambda'$, and notice that $\mathcal{P}(z_1, z_2)$ is certainly a fundamental parallelogram of Λ'. Thus, every point of $\mathbb{R}^2$, including α, can be represented as $\alpha = \alpha' + v$, where $\alpha' \in \Lambda'$ and $v \in \mathcal{P}(z_1, z_2)$. But $v = \alpha - \alpha'$ must be a nonzero point of Λ by linearity, which contradicts the initial assumption that the only point of Λ contained in $\mathcal{P}(z_1, z_2)$ is the origin $(0,0)$. □

Theorem 2.9: Let Λ be a sublattice of $\mathbb{Z}^2$ and let $\Lambda' = k\Lambda$ for a positive integer k. Then $d_r(\Lambda') = k d_r(\Lambda)$ for all $r \geqslant 1$, and $V(\Lambda') = k^2 V(\Lambda)$.

Proof: A generator matrix $\boldsymbol{G}'$ for Λ' is obtained by multiplying each of the four entries in a generator matrix $\boldsymbol{G}$ for Λ by the integer k. Thus, $V(\Lambda') = |\det \boldsymbol{G}'| = k^2 |\det \boldsymbol{G}| = k^2 V(\Lambda)$. It is obvious that $d_r(\Lambda') \leqslant k d_r(\Lambda)$ since a cluster of size $t+1$ that contains r points of Λ scales to a cluster of size $kt+1$ that contains the corresponding r points of Λ'. The converse inequality $d_r(\Lambda) \leqslant d_r(\Lambda')/k$ follows from Lemma 2.7. We omit the details. □

Lemma 2.10: Let Λ be a sublattice of $\mathbb{Z}^2$. Then for all $r \geqslant 2$, we have $d_r(\Lambda) \leqslant (r-1) d_2(\Lambda)$.

Proof: Suppose that $d_2(\Lambda) = t$, and w.l.o.g. let $z = (x, y)$ be a point in Λ such that $|x| + |y| = t$. Then it is easy to verify that $d_r(z, 2z, \ldots, rz) = (r-1)t$ and hence $d_r(\Lambda) \leqslant (r-1)t$. □

Given a point $z = (x, y) \in \mathbb{R}^2$, we define

$$\mathcal{X}(z) \stackrel{\text{def}}{=} x \quad \text{and} \quad \mathcal{Y}(z) \stackrel{\text{def}}{=} y.$$

We say that r lattice points $z_1, z_2, \ldots, z_r \in \Lambda$ are *minimal* if

$$d_r(z_1, z_2, \ldots, z_r) = d_r(\Lambda).$$

If the lattice points $z_1, z_2, \ldots, z_r \in \Lambda$ are minimal, we refer to $d_r(z_1, z_2, \ldots, z_r) = d_r(\Lambda)$ as the *r-dispersion of* Λ.

Lemma 2.11: Let Λ be a sublattice of $\mathbb{Z}^2$. Then there exist r minimal points $z_1, z_2, \ldots, z_r \in \Lambda$ such that $z_1 = (0,0)$ and $\mathcal{X}(z_i) \geqslant 0$ for $i = 1, 2, \ldots, r$. Similarly, there also exist r minimal points $z_1, z_2', \ldots, z_r' \in \Lambda$ such that $z_1 = (0,0)$ and $\mathcal{Y}(z_i') \geqslant 0$ for $i = 2, 3, \ldots, r$.

Proof: Let $(a_1, b_1), \ldots, (a_r, b_r)$ be some r minimal points in Λ, and assume w.l.o.g. that $a_1 = \min_{1 \leqslant i \leqslant r} a_i$. We then define $z_i = (a_i - a_1, b_i - b_1)$ for $i = 1, 2, \ldots, r$. The second statement of the lemma follows by a similar argument □

We say that a lattice Λ' is *isomorphic* to a lattice $\Lambda \subseteq \mathbb{Z}^2$ if Λ' may be obtained from Λ by reflection and/or rotation about the origin by a multiple of 90°. Notice that if Λ and Λ' are isomorphic then $V(\Lambda') = V(\Lambda)$ and $d_r(\Lambda') = d_r(\Lambda)$ for all r.

III. Lattice Interleavers for two Repetitions

For two repetitions, the strength of a lattice interleaver is equal to the tristance of the underlying lattice Λ, while its interleaving degree is equal to the volume of Λ. Thus, our goal in this section is to construct, for each given t, a lattice Λ such that $d_3(\Lambda)=t$ and $V(\Lambda)$ is the smallest possible among all lattices with tristance $\geqslant t$. We say that such a lattice Λ is *optimal*.

The constructions are presented in Section III-A. In Section III-B, we show that these constructions are optimal for even t. In Section III-C, we describe a computer search for optimal lattices.

A. Constructions

We distinguish between four cases, depending on the value of the strength t modulo 4. Specifically, for each $k=1,2,\ldots,$ we define the lattices Λ_{4k}, Λ_{4k+1}, Λ_{4k+2}, and Λ_{4k+3} by means of the corresponding generator matrices

$$\boldsymbol{G}_{4k}=\begin{bmatrix} k & k \\ 0 & 3k \end{bmatrix} \qquad \boldsymbol{G}_{4k+1}=\begin{bmatrix} k & k+1 \\ 0 & 3k+2 \end{bmatrix}$$
$$\boldsymbol{G}_{4k+2}=\begin{bmatrix} k+1 & k \\ 1 & 3k+1 \end{bmatrix} \quad \boldsymbol{G}_{4k+3}=\begin{bmatrix} k+1 & k+1 \\ 0 & 3k+2 \end{bmatrix}.$$

The following theorem establishes the tristance and the volume of Λ_{4k}, Λ_{4k+1}, Λ_{4k+2}, Λ_{4k+3}, and hence the strength and the interleaving degree of the corresponding lattice interleavers.

Theorem 3.1:

$$d_3(\Lambda_{4k})=4k \qquad V(\Lambda_{4k})=3k^2=\frac{3t^2}{16} \tag{4}$$
$$d_3(\Lambda_{4k+1})=4k+1 \quad V(\Lambda_{4k+1})=3k^2+2k=\frac{3t^2+2t-5}{16} \tag{5}$$
$$d_3(\Lambda_{4k+2})=4k+2 \quad V(\Lambda_{4k+2})=3k^2+3k+1=\frac{3t^2+4}{16} \tag{6}$$
$$d_3(\Lambda_{4k+3})=4k+3 \quad V(\Lambda_{4k+3})=3k^2+5k+2=\frac{3t^2+2t-1}{16}. \tag{7}$$

Proof: The expressions for the volumes of Λ_{4k}, Λ_{4k+1}, Λ_{4k+2}, Λ_{4k+3} in (4)–(7) follow straightforwardly by evaluating the determinants of the corresponding generator matrices. Note that the integer t in (4)–(7) denotes the tristance of the corresponding lattice in each case.

Observe that $(k,k)\in\Lambda_{4k}$ and thus $d_2(\Lambda_{4k})\leqslant k+k=2k$. Therefore, it follows from Lemma 2.10 that $d_3(\Lambda_{4k})\leqslant 4k$. Now let z_0, z_1, z_2 be three minimal points of Λ_{4k}. By Lemma 2.1, we can assume that $z_0=(0,0)$, $\mathcal{X}(z_1)\geqslant 0$, $\mathcal{X}(z_2)\geqslant 0$. To establish (4), it remains to prove that $d_3(z_0,z_1,z_2)\geqslant 4k$. By Lemma 2.11, this holds trivially if either $d_2(z_0,z_1)\geqslant 4k$ or $d_2(z_0,z_2)\geqslant 4k$. Thus, as potential candidates for z_1 and z_2, we need to examine only those points of Λ_{4k} whose first coordinate is nonnegative and whose L_1-distance from the origin is at most $4k-1$. Writing a generic point of Λ_{4k} as $u_1(k,k)+u_2(0,3k)$, we see that these two conditions leave only seven possible choices for (u_1,u_2), namely, $(0,-1)$, $(0,1)$, $(1,-1)$, $(1,0)$, $(2,-1)$, $(2,0)$, and $(3,0)$. These correspond to the following seven points

$$(0,-3k),\ (0,3k),\ (k,-2k),\ (k,k),\ (2k,-k),\ (2k,2k),\ (3k,0) \tag{8}$$

of Λ_{4k}. Given the above, there are only $\binom{7}{2}=21$ possible choices for z_1, z_2. It can be verified that $d_3(z_0,z_1,z_2)\geqslant 4k$ in each case. The expressions for the tristances of Λ_{4k+1}, Λ_{4k+2}, Λ_{4k+3} in (5)–(7) can be proved by a similar argument. In a manner analogous to (8), we find that there are only six relevant candidates for z_1, z_2 in each of Λ_{4k+1}, Λ_{4k+2}, Λ_{4k+3} that are given, respectively, by

$$(0,-3k-2),\quad (0,3k+2),\quad (k,-2k-1),\quad (k,k+1),\quad (2k,-k),\quad (3k,1) \tag{9}$$
$$(1,3k+1),\quad (k,-2k-1),\quad (k+1,k),\quad (2k+1,-k-1),\quad (2k+2,2k),\quad (3k+2,-1) \tag{10}$$
$$(0,-3k-2),\quad (0,3k+2),\quad (k+1,-2k-1),\quad (k+1,k+1),\quad (2k+2,-k). \tag{11}$$

It is easy to verify that $d_3(z_0,z_1,z_2)$ is at least $4k+1$, $4k+2$, and $4k+3$ in (9), (10), and (11), respectively. Finally, to establish the converse inequalities, consider the points

$$(k,k+1),(3k,1)\in\Lambda_{4k+1}$$
$$(1,3k+1),(k+1,k)\in\Lambda_{4k+2}$$
$$(k+1,k+1),(3k+3,1)\in\Lambda_{4k+3}.$$

□

B. Lower Bounds

Our main objective in this subsection is to prove that the family of lattices Λ_{4k} constructed in Section III-A is optimal for all k. As a corollary to this result, we will be able to show that the lattices Λ_{4k+2} are also optimal for all k, and establish lower bounds on the interleaving degree in other cases. We start with the following simple lemma.

Lemma 3.2: Let Λ be a sublattice of $\mathbb{Z}^2$ such that

$$|\mathcal{X}(z)|+|\mathcal{Y}(z)|\neq d_3(\Lambda)/2, \qquad \text{for all } z\in\Lambda.$$

Then there exists a lattice Λ' isomorphic to Λ and three minimal points $z'_0, z'_1, z'_2\in\Lambda'$ such that $z'_0=(0,0)$, $\mathcal{X}(z'_2)\geqslant\mathcal{X}(z'_1)\geqslant 0$, and $\mathcal{Y}(z'_1)\geqslant\mathcal{Y}(z'_2)\geqslant 0$.

Proof: Let $z_0=(x_0,x_1)$, $z_1=(x_1,x_2)$, and $z_2=(x_2,y_2)$ be three minimal points in Λ, and define

$$x_{\min}=\min\{x_0,x_1,x_2\}$$
$$x_{\max}=\max\{x_0,x_1,x_2\}$$
$$y_{\min}=\min\{y_0,y_1,y_2\}$$
$$y_{\max}=\max\{y_0,y_1,y_2\}.$$

We first show that the bounding rectangle $\mathcal{R}(z_0,z_1,z_2)$ is nondegenerate, that is, $x_{\min}\neq x_{\max}$ and $y_{\min}\neq y_{\max}$. Assume to the contrary that $y_{\min}=y_{\max}$, and let $x_0\leqslant x_1\leqslant x_2$. Using an appropriate shift (as in Lemma 2.11), we may further assume that $(x_0,y_0)=(0,0)$. Then

$$d_3(\Lambda)=d_3(z_0,z_1,z_2)=x_2=x_1+(x_2-x_1).$$

But we have $d_3(\Lambda)\leqslant 2x_1$ and $d_3(\Lambda)\leqslant 2(x_2-x_1)$ by Lemma 2.10. It follows that $x_1=d_3(\Lambda)/2$, contradicting the assumption that $|\mathcal{X}(z_1)|+|\mathcal{Y}(z_1)|\neq d_3(\Lambda)/2$. By the same argument, $x_{\min}\neq x_{\max}$, and hence the rectangle $\mathcal{R}(z_0,z_1,z_2)$ is nondegenerate.

A very similar argument shows that none of the three points z_0, z_1, z_2 can be properly inside the rectangle $\mathcal{R}(z_0, z_1, z_2)$. Indeed, assume to the contrary that z_1 does not belong to any one of the four edges of $\mathcal{R}(z_0, z_1, z_2)$. This implies that the other two points are at the opposite corners of $\mathcal{R}(z_0, z_1, z_2)$, and we assume w.l.o.g. that $z_0 = (0, 0)$ is in the lower left corner while z_2 is in the upper right corner. Then

$$\begin{aligned} d_3(\Lambda) &= d_3(z_0, z_1, z_2) = x_2 + y_2 \\ &= (x_1 + y_1) + (x_2 - x_1) + (y_2 - y_1). \end{aligned}$$

But by Lemma 2.10, we have

$$d_3(\Lambda) \leqslant 2(x_1+y_1) \quad \text{and} \quad d_3(\Lambda) \leqslant 2(x_2-x_1) + 2(y_2-y_1).$$

It follows that $x_1+y_1 = d_3(\Lambda)/2$, which again contradicts the assumption that $|\mathcal{X}(z_1)| + |\mathcal{Y}(z_1)| \neq d_3(\Lambda)/2$.

Thus, it follows that each of the points z_0, z_1, z_2 belongs to an edge of $\mathcal{R}(z_0, z_1, z_2)$. At least one of these points, say z_0, must be in a corner of $\mathcal{R}(z_0, z_1, z_2)$. Furthermore, we can assume w.l.o.g. that one of the two opposite edges (that is, the edges that do not contain z_0) contains the point z_1 while the other one contains the point z_2. By an appropriate shift (as in Lemma 2.11), we bring the point z_0 to the origin, so that $z_0 = (0, 0)$. We then rotate the lattice about the origin by a multiple of 90°, and possibly relabel the points z_1, z_2, to satisfy the conditions of the lemma. □

Taking into account the result of Lemma 3.2, we will distinguish between two cases, depending upon whether or not a lattice Λ contains a point whose L_1-distance from the origin is exactly half the tristance of Λ. These two cases are considered separately in the following two propositions.

Proposition 3.3: Let Λ be a sublattice of $\mathbb{Z}^2$ such that $d_3(\Lambda) = 4k$. Suppose that there exists a point $(\alpha, \beta) \in \Lambda$ such that $|\alpha| + |\beta| = 2k$. Then $V(\Lambda) \geqslant 3k^2$.

Proof: First, we assume w.l.o.g. that $\alpha \geqslant 0$ and $\beta \geqslant 0$, otherwise, rotate Λ by an appropriate multiple of 90° and notice that such rotation preserves both the volume and the tristance. We will further assume w.l.o.g. that $\alpha \geqslant \beta$. Otherwise, one could repeat the proof of this proposition, while interchanging the roles of the X and Y axes. The lattice points $(0, 0)$ and (α, β) lie on the line $\mathcal{L}$: $y = (\beta/\alpha)x$. Consider lattice points that lie above this line. Specifically, let Δ be the smallest positive real number such that the line

$$\mathcal{L}_\Delta: \quad y = (\beta/\alpha)x + \Delta \tag{12}$$

contains a point of Λ. Denote this point by (x', y'). By linearity, the lattice Λ must also contain points of the form $(x' + u\alpha, y' + u\beta)$ for all $u \in \mathbb{Z}$. Of these, we consider the unique point (x_0, y_0) such that $0 < x_0 \leqslant \alpha$. Notice that all these points lie on the line $\mathcal{L}_\Delta$, and, therefore, $y_0 = (\beta/\alpha)x_0 + \Delta$. We now establish a lower bound on y_0. Suppose that $y_0 \leqslant 2\beta$, and consider the three lattice points $z_1 = (x_0, y_0)$, $z_2 = (\alpha, \beta)$, and $z_3 = (2\alpha, 2\beta)$. Then, by Theorem 2.4, we have

$$\begin{aligned} d_3(z_1, z_2, z_3) &= (2\alpha - x_0) + (2\beta - \min\{\beta, y_0\}) \\ &= 4k - x_0 - \min\{\beta, y_0\} < 4k \end{aligned}$$

which contradicts the assumption that $d_3(\Lambda) = 4k$. Therefore, $y_0 > 2\beta$. Next, we establish a lower bound on Δ. To do so, we will distinguish between two cases: $0 < x_0 \leqslant \alpha - \beta$ and $\alpha - \beta < x_0 \leqslant \alpha$. In the latter case, we have

$$\begin{aligned} &d_3(z_1, z_2, z_3) \\ &\quad = (2\alpha - x_0) + (y_0 - \beta) \\ &\quad = 2\alpha - x_0 \frac{\alpha - \beta}{\alpha} + \Delta - \beta \quad [\text{since } y_0 = (\beta/\alpha)x_0 + \Delta] \\ &\quad \leqslant 2\alpha - \frac{(\alpha - \beta)^2}{\alpha} + \Delta - \beta \quad [\text{since } x_0 > \alpha - \beta \geqslant 0] \\ &\quad = (\alpha + \beta) - \frac{\beta^2}{\alpha} + \Delta = 2k - \frac{\beta^2}{\alpha} + \Delta. \end{aligned} \tag{13}$$

In the former case $0 < x_0 \leqslant \alpha - \beta$, we consider the lattice points $z_1 = (x_0, y_0)$ and $z_2 = (\alpha, \beta)$ as above, along with the origin $z_0 = (0, 0)$. Again, by Theorem 2.4, we have

$$\begin{aligned} d_3(z_0, z_1, z_2) &= \alpha + y_0 \\ &= \alpha + x_0 \frac{\beta}{\alpha} + \Delta \\ &\leqslant \alpha + \frac{(\alpha - \beta)\beta}{\alpha} + \Delta \\ &= 2k - \frac{\beta^2}{\alpha} + \Delta. \end{aligned} \tag{14}$$

It follows from (13), (14) in conjunction with the fact that $d_3(\Lambda) = 4k$ that $\Delta \geqslant 2k + (\beta^2/\alpha)$ in both cases. Since the two cases are exhaustive, this bound on Δ holds in general.

Now consider the parallelogram $\mathcal{P}(z_1, z_2)$ defined by the lattice points $z_1 = (x_0, y_0)$ and $z_2 = (\alpha, \beta)$ as illustrated in Fig. 7. It follows from the definition of Δ in (12) that this parallelogram contains no other points of Λ except the origin. Hence, by Theorem 2.8, it is a fundamental parallelogram of Λ, and the points z_1, z_2 form a basis for Λ. Thus,

$$\begin{aligned} V(\Lambda) &= \begin{vmatrix} \alpha & \beta \\ x_0 & y_0 \end{vmatrix} = \alpha\Delta \geqslant 2k\alpha + \beta^2 \\ &= 4k^2 - \beta(2k - \beta) \geqslant 3k^2 \end{aligned}$$

where the first inequality follows from $\Delta \geqslant 2k + (\beta^2/\alpha)$ while the second inequality follows by maximizing the expression $\beta(2k - \beta)$ over the range $0 \leqslant \beta \leqslant 2k$. □

Proposition 3.4: Let Λ be a sublattice of $\mathbb{Z}^2$ such that $d_3(\Lambda) = 4k$. Suppose that for all $z \in \Lambda$ we have

$$|\mathcal{X}(z)| + |\mathcal{Y}(z)| \neq 2k.$$

Then $V(\Lambda) > 3k^2$.

Proof: Let z_0, z_1, z_2 be three minimal points of Λ. Using Lemma 3.2, we can assume w.l.o.g. that $z_0 = (0, 0)$, $z_1 = (x_1, y_1)$, $z_2 = (x_2, y_2)$, where $0 \leqslant x_1 \leqslant x_2$ and $0 \leqslant y_2 \leqslant y_1$. Thus, we have

$$d_3(z_0, z_1, z_2) = x_2 + y_1 = 4k. \tag{15}$$

We start the proof by refining the inequalities $0 \leqslant x_1 \leqslant x_2$ and $0 \leqslant y_2 \leqslant y_1$. Suppose that $x_1 = x_2$, which means that the point z_1 is at the upper right corner of $\mathcal{R}(z_0, z_1, z_2)$. Then we have

$$d_2(z_0, z_2) + d_2(z_2, z_1) = (x_2 + y_2) + (y_1 - y_2) = x_2 + y_1 = 4k.$$

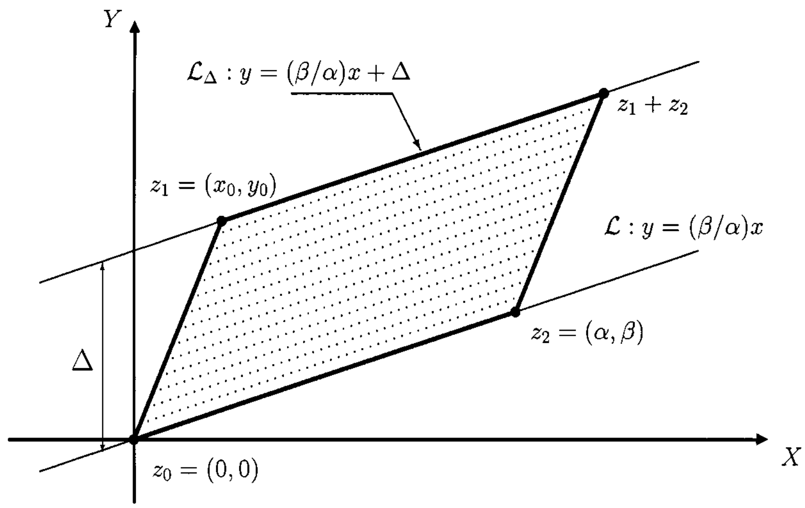

Fig. 7. Fundamental parallelogram for $z_1 = (x_0, y_0)$ and $z_2 = (\alpha, \beta)$.

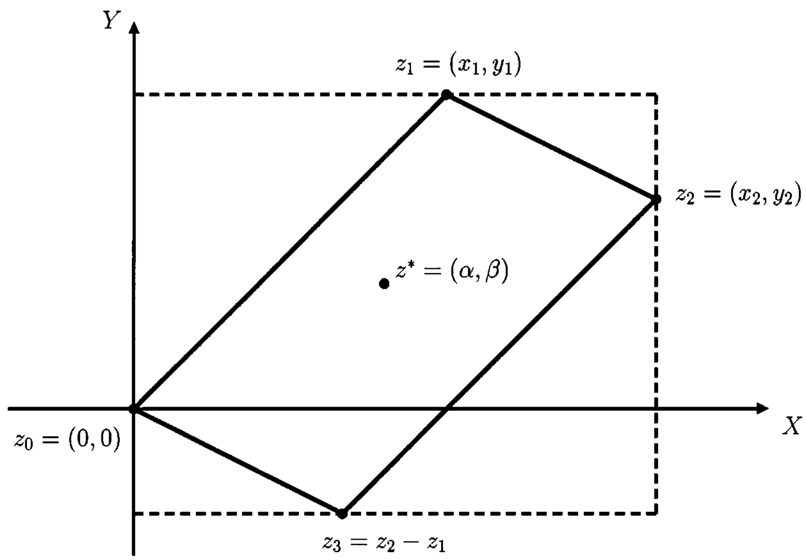

Fig. 8. Nonexistence of an interior point (see Proposition 3.3 and Lemma 3.5).

It follows that either $d_2(z_0, z_2)$ or $d_2(z_2, z_1)$ is at most $2k$. This is a contradiction, since it cannot be equal to $2k$ by the assumption that $|\mathcal{X}(z)| + |\mathcal{Y}(z)| \neq 2k$ for all $z \in \Lambda$, while if it were strictly less than $2k$ then $d_3(\Lambda) < 4k$ by Lemma 2.10. Hence $x_1 \neq x_2$. By a similar argument $y_1 \neq y_2$.

It follows from (15) that either $x_2 \leqslant 2k$ or $y_1 \leqslant 2k$. We assume w.l.o.g. that $y_1 \leqslant 2k$, otherwise repeat the proof of this proposition, while interchanging the roles of the X and Y axes. Thus, we write $y_1 = 2k - \delta$ and $x_2 = 2k + \delta$, where $0 \leqslant \delta < 2k$. As we already observed, the L_1-distance between any two points of Λ is strictly greater than $2k$. This observation implies, in particular, that

$$d_2(z_0, z_1) = x_1 + (2k - \delta) > 2k \iff 0 \leqslant \delta < x_1 \quad (16)$$

$$d_2(z_2, z_1) = 4k - (x_1 + y_2) > 2k \iff x_1 + y_2 < 2k. \quad (17)$$

Now let $z_3 = z_2 - z_1 = (x_2 - x_1, y_2 - y_1)$ and consider the parallelogram $\mathcal{P}(z_1, z_3)$ defined by z_1 and z_3, as illustrated in Fig. 8. We will make use of the following lemma.

Lemma 3.5: The parallelogram $\mathcal{P}(z_1, z_3)$ does not contain nonzero points of Λ.

Proof: Assume to the contrary that $z^* = (\alpha, \beta)$ is such a point. It can be readily seen (cf. Fig. 8) that $0 < \alpha < x_2$ and $y_2 - y_1 < \beta < y_1$. Suppose first that $\beta \geqslant 0$. Then we have

$$\begin{aligned} &d_3(z_0, z_1, z^*) + d_3(z_0, z_2, z^*) \\ &\qquad = (\max\{x_1, \alpha\} + y_1) + (x_2 + \max\{y_2, \beta\}) < 8k \end{aligned}$$

where the last inequality follows from the fact that $x_2 + y_1 = 4k$, while $\max\{x_1, \alpha\} < x_2$ and $\max\{y_2, \beta\} < y_1$. Since this contradicts $d_3(\Lambda) = 4k$, let us assume that $\beta < 0$. But then we have

$$\begin{aligned} &d_3(z_0, z_3, z^*) + d_3(z_0, z_2, z^*) \\ &\quad = (\max\{x_2 - x_1, \alpha\} - (y_2 - y_1)) + (x_2 + y_2 - \beta) \\ &\quad = 4k + \max\{x_2 - x_1, \alpha\} - \beta < 4k + x_2 - \beta < 8k \end{aligned}$$

where the second equality follows from the fact that $x_2 + y_1 = 4k$, the first inequality follows from $\max\{x_2 - x_1, \alpha\} < x_2$, since $x_1 > 0$ and $\alpha < x_2$, while the second inequality follows from the fact that $-\beta < y_1$. Since this also contradicts the assumption $d_3(\Lambda) = 4k$, we conclude that z^* does not exist. □

It follows from Lemma 3.5 and Theorem 2.8 that $\mathcal{P}(z_1, z_3)$ is a fundamental parallelogram of Λ. Hence the volume of Λ is given by

$$V(\Lambda) = \begin{vmatrix} x_2 - x_1 & y_2 - y_1 \\ x_1 & y_1 \end{vmatrix} = x_2 y_1 - x_1 y_2 = 4k^2 - \delta^2 - x_1 y_2.$$

Thus, to complete the proof of the proposition, it remains to show that $\delta^2 + x_1 y_2 < k^2$. To this end, we introduce the quadratic function $f(\delta, x_1, y_2) = \delta^2 + x_1 y_2$ and maximize this function subject to various constraints on x_1, y_2, and δ. The constraints we have so far are (16), (17), and $y_2 \geqslant 0$. To derive one more constraint, consider the points z_1, z_2, as well as $2z_1 = (2x_1, 2y_1)$. Depending upon whether $2x_1 \leqslant x_2$ or $2x_1 > x_2$, we obtain one of the following two inequalities:

$$\begin{aligned} d_3(z_1, z_2, 2z_1) &= (x_2 - x_1) + (2y_1 - y_2) \\ &= 4k + y_1 - (x_1 + y_2) \geqslant 4k \\ &\iff \delta + x_1 + y_2 \leqslant 2k \qquad (18) \\ d_3(z_1, z_2, 2z_1) &= (2x_1 - x_1) + (2y_1 - y_2) \\ &= 4k - (2\delta - x_1 + y_2) \geqslant 4k \\ &\iff 2\delta + y_2 \leqslant x_1 \qquad (19) \end{aligned}$$

respectively. It is easy to see that the supremum of $f(\delta, x_1, y_2)$ subject to (16), (18), and $y_2 \geqslant 0$ is k^2, and this supremum is attained uniquely for $y_2 = 0$ and $\delta = x_1 = k$. Similarly, the supremum of $f(\delta, x_1, y_2)$ subject to (16), (17), (19), and $y_2 \geqslant 0$ is also k^2, and it is attained uniquely for $y_2 = 0$, $x_1 = 2k$, and $\delta = k$. Notice that the point $(\delta, x_1, y_2) = (k, k, 0)$ is ruled out by (16) while the point $(\delta, x_1, y_2) = (k, 2k, 0)$ is ruled out by (17). This completes the proof of the proposition. □

Theorem 3.6: Let Λ be any sublattice of $\mathbb{Z}^2$ with tristance $d_3(\Lambda) = t$. Set $k = \lfloor t/4 \rfloor$. Then the volume of Λ is bounded from below as follows:

$$\begin{aligned} V(\Lambda) &\geqslant 3k^2, && \text{if } t \equiv 0 \bmod 4 \qquad (20) \\ V(\Lambda) &\geqslant 3k^2 + \frac{3}{2}k + \frac{1}{2}, && \text{if } t \equiv 1 \bmod 4 \qquad (21) \\ V(\Lambda) &\geqslant 3k^2 + 3k + 1, && \text{if } t \equiv 2 \bmod 4 \qquad (22) \\ V(\Lambda) &\geqslant 3k^2 + \frac{9}{2}k + \frac{5}{2}, && \text{if } t \equiv 3 \bmod 4 \qquad (23) \end{aligned}$$

Proof: The bound (20) follows directly from Proposition 3.3 and Proposition 3.4. To prove (22), we proceed as follows. Assume to the contrary that Λ is a sublattice of $\mathbb{Z}^2$ such that $d_3(\Lambda) = 4k + 2$ and $V(\Lambda) \leqslant 3k^2 + 3k$. Let $\Lambda' = 2\Lambda$. Then by Theorem 2.9, we have

$$d_3(\Lambda') = 2d_3(\Lambda) = 8k + 4 = 4(2k+1)$$

and $V(\Lambda') = 4V(\Lambda) \leqslant 12k^2 + 12k$. But this is strictly less than $3(2k+1)^2 = 12k^2 + 12k + 3$, which contradicts (20). The remaining two bounds are proved by a similar argument. □

Theorem 3.6 implies that the lattices Λ_{4k} and Λ_{4k+2} constructed in Section III-A are optimal for all k. In other words, for $r = 2$, we have constructed optimal lattice interleavers for all even strengths t. Although this does not follow from Theorem 3.6, we conjecture that the lattices Λ_{4k+1} and Λ_{4k+3} are optimal as well. Strong evidence for this conjecture is obtained in the next subsection.

C. Computer Search

For specific values of k, the bounds (21) and (23) of Theorem 3.6 can be improved. We have verified by exhaustive computer search that the lattices Λ_{4k+1} and Λ_{4k+3} constructed in Section III-A are optimal for all $k \leqslant 250$. In other words, we have proved, by exhaustive search, the following result.

Theorem 3.7: Let Λ be any sublattice of $\mathbb{Z}^2$ with tristance $t \leqslant 1003$. Set $k = \lfloor t/4 \rfloor$. Then the volume of Λ is bounded from below as follows:

$$\begin{aligned} V(\Lambda) &\geqslant 3k^2 + 2k, && \text{if } t \equiv 1 \bmod 4 \qquad (24) \\ V(\Lambda) &\geqslant 3k^2 + 5k + 2, && \text{if } t \equiv 3 \bmod 4. \qquad (25) \end{aligned}$$

The search was performed as follows. The input for the search program is a positive integer m. The output of the program is a generator matrix of a lattice Λ with $V(\Lambda) = m$, such that $d_3(\Lambda)$ is the largest possible among all sublattices of $\mathbb{Z}^2$ with volume m, and an integer $t = d_3(\Lambda)$. We have run the search for all $m = 1, 2, \ldots, 188\,752$, which yielded tristances t between 2 and 1003. Clearly, the smallest value of m for which the exhaustive search produces a particular value of the tristance t is a lower bound on the volume of any lattice with that tristance. To reduce the number of generator matrices searched for a given m, we made use of the following lemma.

Lemma 3.8: Let Λ be a sublattice of $\mathbb{Z}^2$ with $V(\Lambda) = m$. Then there exists a lattice Λ' isomorphic to Λ and a generator matrix for Λ' of the following form:

$$\boldsymbol{G} = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \qquad (26)$$

where a is a divisor of m in the range $1 \leqslant a \leqslant \sqrt{m}$, c is another divisor of m given by $c = m/a$, and $0 \leqslant b \leqslant \lfloor c/2 \rfloor$.

Lemma 3.8 characterizes the lattices searched for each m. The proof of this lemma is left as an exercise for the reader. Obviously, for a given m, we are only interested in lattices with the largest tristance. Thus, the search is controlled by two variables: `max_tristance`, which is the largest tristance among the lattices searched so far, and `tristance` which is the current estimate of the tristance of the lattice being examined. For each m, we initialize `max_tristance` to 0, and for each new generator matrix $\boldsymbol{G}$, we initialize `tristance` to $2c$. A pseudocode description of the entire search procedure follows.

```
z_0 := (0, 0); G := ∅; max_tristance := 0;
for a ∈ {divisors of m less than √m + 1} do
{
  c := m/a; z_1 := (c, 0); tristance := 2c;
⟨*⟩for b = 1 to ⌊c/2⌋
  if gcd(b, c) ⩾ a do
  {
    for u_1 = 1 to ⌊tristance/(a + b)⌋ do
    {
      x := u_1 a; y := u_1 b mod c;
      z_2 := (x, y); z_3 := (x, y−c); t_1 = d_3(z_0, z_1, z_2);
      for u_2 = −⌊tristance/(a+b)⌋ to ⌊tristance/(a+b)⌋
      if u_2 ≠ u_1 do
      {
        z_4 = (u_2 a, u_2 b); z_5 = (u_2 a, u_2 b − c);
        t_2 = d_3(z_0, z_2, z_4);
        t_3 = d_3(z_0, z_2, z_5);
        t_4 = d_3(z_0, z_3, z_5);
        tristance := min{tristance, t_1, t_2, t_3, t_4};
        if tristance ⩽ max_tristance goto ⟨*⟩
      }
    }
    if tristance > max_tristance
    {
      G := {a, b, c}; max_tristance := tristance
    }
  }
}
```

We do not prove the correctness of this algorithm here. The key idea is that the tristance of a lattice can be computed efficiently by considering only a few points. See [7] for more details.

IV. Lattice Interleavers for Three Repetitions

This section deals with lattices with prescribed quadristance, which correspond to lattice interleavers with three repetitions. The overall organization is similar to the previous section. Thus, constructions are presented in Section IV-A, lower bounds on the attainable volume are established in Section IV-B, while results of a computer search for optimal lattices are presented in Section IV-C.

A. Constructions

We distinguish between nine cases, depending on the value of the quadristance t modulo 9. Specifically, for each $k = 1, 2, \ldots,$ we define the lattices $\Lambda_{9k}, \Lambda_{9k+1}, \ldots, \Lambda_{9k+8}$ by means of the corresponding generator matrices

$$\boldsymbol{G}_{9k} = \begin{bmatrix} k & 3k \\ 0 & 8k \end{bmatrix} \qquad \boldsymbol{G}_{9k+1} = \begin{bmatrix} k & 3k+1 \\ 0 & 8k+3 \end{bmatrix}$$

$$\boldsymbol{G}_{9k+2} = \begin{bmatrix} k & 3k+2 \\ 0 & 8k+5 \end{bmatrix} \qquad \boldsymbol{G}_{9k+3} = \begin{bmatrix} k+1 & 5k+2 \\ 1 & 8k+3 \end{bmatrix}$$

$$\boldsymbol{G}_{9k+4} = \begin{bmatrix} k+1 & 3k+1 \\ 1 & 8k+3 \end{bmatrix} \qquad \boldsymbol{G}_{9k+5} = \begin{bmatrix} k+1 & 3k+1 \\ 0 & 8k+2 \end{bmatrix}$$

$$\boldsymbol{G}_{9k+6} = \begin{bmatrix} k+1 & 3k+2 \\ 1 & 8k+6 \end{bmatrix} \qquad \boldsymbol{G}_{9k+7} = \begin{bmatrix} k+1 & 3k+2 \\ 0 & 8k+5 \end{bmatrix}$$

$$\boldsymbol{G}_{9k+8} = \begin{bmatrix} k+2 & 5k+3 \\ 1 & 8k+5 \end{bmatrix}.$$

The following theorem establishes the volume and the quadristance of $\Lambda_{9k}, \Lambda_{9k+1}, \ldots, \Lambda_{9k+8}$. This theorem is the counterpart of Theorem 3.1 of Section III-A.

Theorem 4.1:

$$d_4(\Lambda_{9k}) = 9k \qquad V(\Lambda_{9k}) = 8k^2 \tag{27}$$
$$d_4(\Lambda_{9k+1}) = 9k+1 \qquad V(\Lambda_{9k+1}) = 8k^2+3k \tag{28}$$
$$d_4(\Lambda_{9k+2}) = 9k+2 \qquad V(\Lambda_{9k+2}) = 8k^2+5k \tag{29}$$
$$d_4(\Lambda_{9k+3}) = 9k+3 \qquad V(\Lambda_{9k+3}) = 8k^2+6k+1 \tag{30}$$
$$d_4(\Lambda_{9k+4}) = 9k+4 \qquad V(\Lambda_{9k+4}) = 8k^2+8k+2 \tag{31}$$
$$d_4(\Lambda_{9k+5}) = 9k+5 \qquad V(\Lambda_{9k+5}) = 8k^2+10k+2 \tag{32}$$
$$d_4(\Lambda_{9k+6}) = 9k+6 \qquad V(\Lambda_{9k+6}) = 8k^2+11k+4 \tag{33}$$
$$d_4(\Lambda_{9k+7}) = 9k+7 \qquad V(\Lambda_{9k+7}) = 8k^2+13k+5 \tag{34}$$
$$d_4(\Lambda_{9k+8}) = 9k+8 \qquad V(\Lambda_{9k+8}) = 8k^2+16k+7 \tag{35}$$

Proof: The proof is similar to the proof of Theorem 3.1. To see that $d_4(\Lambda_{9k}) \leqslant 9k$, consider the points $(0, 0)$, $(2k, -2k)$, $(3k, k)$, $(5k, -k) \in \Lambda_{9k}$. Now let z_0, z_1, z_2, z_3 be minimal points of Λ_{9k}. By Lemma 2.11, we can assume that $z_0 = (0, 0)$ and $\mathcal{X}(z_3) \geqslant \mathcal{X}(z_2) \geqslant \mathcal{X}(z_1) \geqslant 0$. Furthermore, we can assume that z_1, z_2, z_3 are within L_1-distance $9k - 1$ from the origin, otherwise, $d_4(z_0, z_1, z_2, z_3) \geqslant 9k$ holds trivially. There are only 12 points in Λ_{9k} that satisfy these conditions, namely,

$$\begin{gathered}(0, -8k),\ (0, 8k),\ (k, -5k),\ (k, 3k),\\ (2k, -2k),\ (2k, 6k),\ (3k, k),\ (4k, -4k),\\ (4k, 4k),\ (5k, -k),\ (6k, 2k),\ (8k, 0).\end{gathered} \tag{36}$$

Given the above, there are only $\binom{12}{3} = 220$ possible choices for z_1, z_2, z_3. We have used a simple computer program to verify that $d_4(z_0, z_1, z_2, z_3) \geqslant 9k$ in each case. The proof of (28)–(35) is similar. □

B. Lower Bounds

We now prove that the lattices Λ_{9k} constructed in Section IV-A are optimal for all k. This result makes it possible to establish lower bounds on the volume of general lattices with prescribed quadristance. As in Section III-B, we distinguish between two cases, depending upon whether or not a lattice Λ contains a point whose L_1-distance from the origin is exactly one third of the quadristance of Λ.

Proposition 4.2: Let Λ be a sublattice of $\mathbb{Z}^2$ such that $d_4(\Lambda) = 9k$. Suppose that there exists a point $(\alpha, \beta) \in \Lambda$ such that $|\alpha| + |\beta| = 3k$. Then $V(\Lambda) \geqslant 9k^2$.

Proof: The proof is similar to the proof of Proposition 3.3. As in Proposition 3.3, we can assume w.l.o.g. that $\alpha \geqslant \beta \geqslant 0$. As before, we let Δ be the smallest positive real number such that the line

$$\mathcal{L}_\Delta: \quad y = (\beta/\alpha)x + \Delta \tag{37}$$

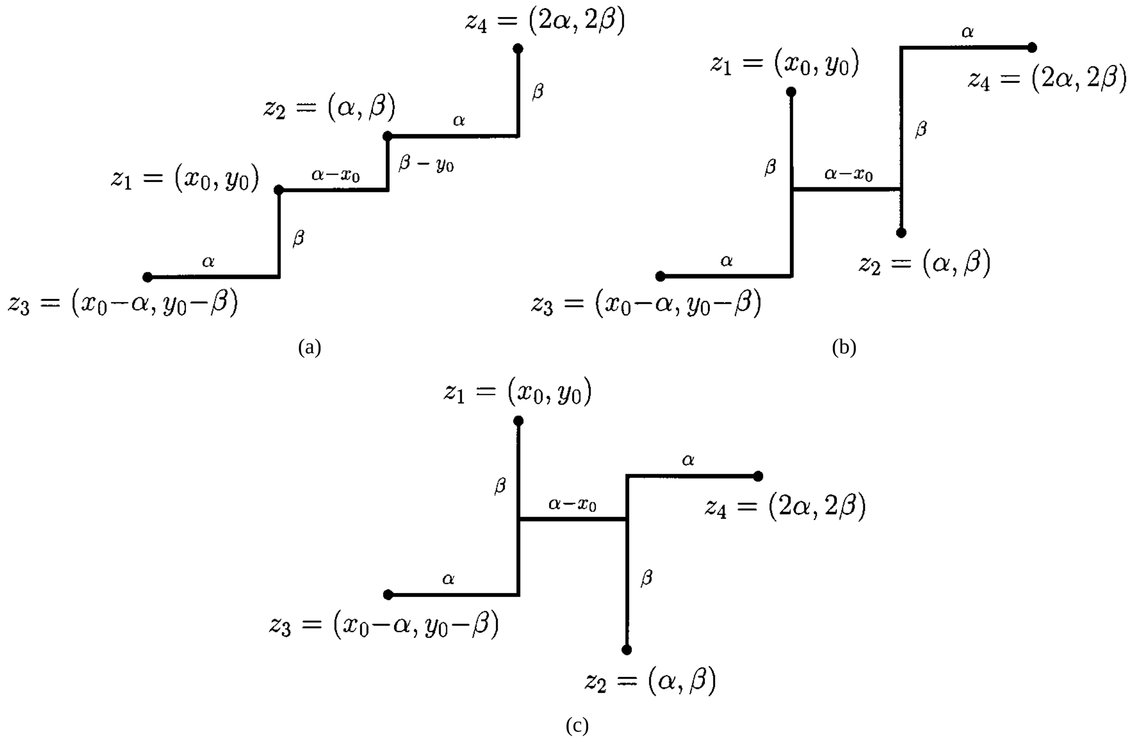

Fig. 9. Three possible configurations for z_1, z_2, z_3, z_4 (see Proposition 4.2).

contains a point of Λ. We again consider the unique lattice point $z_1 = (x_0, y_0)$ on the line $\mathcal{L}_\Delta$ such that $0 < x_0 \leqslant \alpha$. In addition, we will consider the lattice points $z_2 = (\alpha, \beta)$, $z_4 = (2\alpha, 2\beta)$, and $z_3 = z_1 - z_2 = (x_0 - \alpha, y_0 - \beta)$. As before, we start by establishing a lower bound on y_0.

Case 1: Suppose that $0 < y_0 \leqslant \beta$. Then, as illustrated in Fig. 9(a), we have

$$\begin{aligned} d_4(z_1, z_2, z_3, z_4) &= \alpha + \beta + (\alpha - x_0) \\ &\quad + (\beta - y_0) + \alpha + \beta \\ &= 9k - x_0 - y_0. \end{aligned} \tag{38}$$

Case 2: Suppose that $\beta < y_0 \leqslant 2\beta$. Then, as illustrated in Fig. 9(b), we have

$$\begin{aligned} d_4(z_1, z_2, z_3, z_4) &\leqslant \alpha + \beta + (\alpha - x_0) + \beta + \alpha \\ &= 9k - x_0 - \beta. \end{aligned} \tag{39}$$

Case 3: Suppose that $2\beta < y_0 \leqslant 3\beta$. Then, as illustrated in Fig. 9(c), we again have

$$\begin{aligned} d_4(z_1, z_2, z_3, z_4) &\leqslant \alpha + \beta + (\alpha - x_0) + \beta + \alpha \\ &= 9k - x_0 - \beta. \end{aligned} \tag{40}$$

Since $x_0, y_0 > 0$, and $\beta \geqslant 0$, we conclude that each of (38)–(40) contradicts the assumption that $d_4(\Lambda) = 9k$. Hence $y_0 > 3\beta$. Next, we establish a lower bound on Δ. As before, we distinguish between $0 < x_0 \leqslant \alpha - \beta$ and $\alpha - \beta < x_0 \leqslant \alpha$. In the latter case, we have

$$\begin{aligned} d_4(z_1, z_2, z_3, z_4) &= 2\alpha + (\alpha - x_0) + 2\beta + (y_0 - 3\beta) \\ &= 3\alpha - x_0 \frac{\alpha - \beta}{\alpha} + \Delta - \beta \\ &< 3\alpha - \frac{(\alpha - \beta)^2}{\alpha} + \Delta - \beta \\ &= 2\alpha + \beta - \frac{\beta^2}{\alpha} + \Delta \\ &= 6k - \beta - \frac{\beta^2}{\alpha} + \Delta \end{aligned} \tag{41}$$

as illustrated in Fig. 10(a). In the former case $0 < x_0 \leqslant \alpha - \beta$, we consider the lattice points z_1, z_2, and z_3 as above, along with the origin $z_0 = (0, 0)$. Then, as illustrated in Fig. 10(b), we have

$$\begin{aligned} d_4(z_0, z_1, z_2, z_3) &= 2(\alpha - x_0) + 2\beta + x_0 + (y_0 - 2\beta) \\ &\quad + \min\{x_0, y_0 - 2\beta\} \\ &\leqslant 2\alpha + y_0 \\ &= 2\alpha + \frac{\beta}{\alpha} x_0 + \Delta \\ &\leqslant 2\alpha + \beta - \frac{\beta^2}{\alpha} + \Delta \\ &= 6k - \beta - \frac{\beta^2}{\alpha} + \Delta. \end{aligned} \tag{42}$$

It follows from (41), (42) in conjunction with the fact that $d_4(\Lambda) = 9k$ that $\Delta \geqslant 3k + \beta + (\beta^2/\alpha)$ in both cases. Now consider again the parallelogram $\mathcal{P}(z_1, z_2)$ illustrated in Fig. 7. From the definition of Δ in (37) and from Theorem 2.8, we conclude that $\mathcal{P}(z_1, z_2)$ is a fundamental parallelogram of Λ. Hence the points z_1, z_2 form a basis for Λ and

$$\begin{aligned} V(\Lambda) &= \begin{vmatrix} \alpha & \beta \\ x_0 & y_0 \end{vmatrix} = \alpha\Delta \geqslant 3k\alpha + \beta\alpha + \beta^2 \\ &= 3k\alpha + 3k\beta = 9k^2 \end{aligned}$$

where the inequality follows from $\Delta \geqslant 3k + \beta + (\beta^2/\alpha)$, while the last two equalities follow from the assumption that $\alpha + \beta = 3k$. $\square$

From this point on, the proof is analogous in principle to the proof of Proposition 3.4, but considerably more complicated.

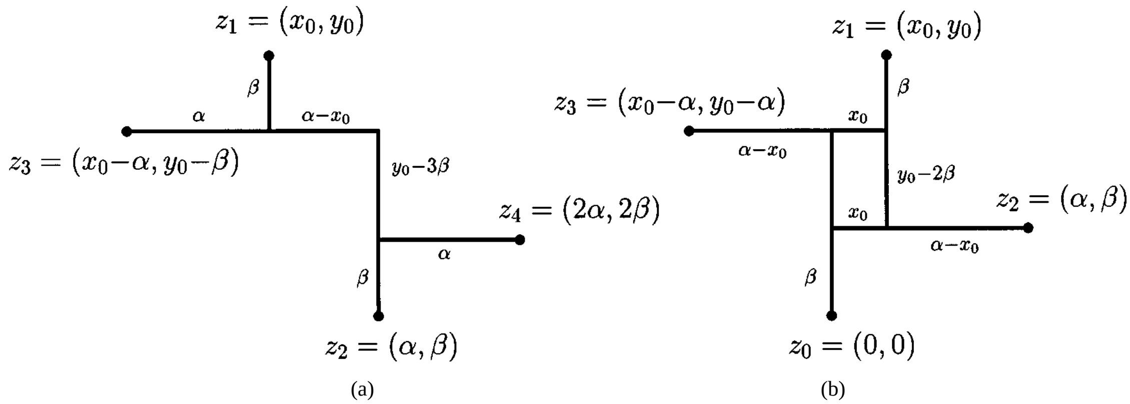

Fig. 10. Configurations of z_1, z_2, z_3, z_4 and z_0, z_1, z_2, z_3 (see Proposition 4.2).

Our goal is to show that $V(\Lambda) \geqslant 8k^2$ for any sublattice Λ of $\mathbb{Z}^2$ such that $d_4(\Lambda) = 9k$. Taking into account Proposition 4.2 and Lemma 2.10, it remains to consider only those lattices Λ that satisfy the following property.

Property P_k: The quadristance of Λ is $d_4(\Lambda) = 9k$ and all pairwise L_1-distances between points of Λ are strictly greater than $3k$.

We proceed by investigating the properties of the minimal points of a lattice Λ that satisfies property P_k. These properties are established in a series of lemmas.

Lemma 4.3: Let Λ be a sublattice of $\mathbb{Z}^2$ that satisfies property P_k and let z_0, z_1, z_2, z_3 be four minimal points of Λ. If one of the points z_0, z_1, z_2, z_3 is in a corner of the bounding rectangle $\mathcal{R}(z_0, z_1, z_2, z_3)$, then none of the other three points is in the diagonally opposite corner.

Lemma 4.4: Let Λ be a sublattice of $\mathbb{Z}^2$ that satisfies property P_k and let z_0, z_1, z_2, z_3 be minimal points of Λ. Then none of z_0, z_1, z_2, z_3 is in a corner of the bounding rectangle $\mathcal{R}(z_0, z_1, z_2, z_3)$.

Due to space limitations, Lemmas 4.3 and 4.4 are stated here without a proof. A detailed proof of both lemmas can be found in [8], which is also available online.

Lemma 4.5: Let Λ be a sublattice of $\mathbb{Z}^2$ that satisfies property P_k and let z_0, z_1, z_2, z_3 be minimal points of Λ. Then z_0, z_1, z_2, z_3 are vertices of a parallelogram.

Proof: By Lemma 4.4, each of the points z_0, z_1, z_2, z_3 belongs to a different edge of $\mathcal{R}(z_0, z_1, z_2, z_3)$. We assume w.l.o.g. that $z_0 \in \underline{\mathcal{R}(z_0, z_1, z_2, z_3)}$, $z_1 \in |\mathcal{R}(z_0, z_1, z_2, z_3)$, $z_2 \in \mathcal{R}(z_0, z_1, z_2, z_3)|$, and bring the point z_0 to the origin by an appropriate shift. This situation is summarized by the inequalities

$$x_1 < x_3 < x_2 \quad \text{and} \quad x_1 < 0 < x_2 \tag{43}$$

$$0 < y_1 < y_3 \quad \text{and} \quad 0 < y_2 < y_3 \tag{44}$$

and illustrated in Fig. 11. As can be seen from this figure, there are four cases, depending upon whether $x_3 \geqslant 0$ and/or $y_2 \geqslant y_1$. In either case, by (43), (44), and Lemma 2.2, we have

$$9k = d_4(z_0, z_1, z_2, z_3) \geqslant x_2 - x_1 + y_3. \tag{45}$$

Notice that in the context of Fig. 11, what we need to prove is that $z_1 + z_2 = z_3$. Assume to the contrary that $z_1 + z_2 = z_4 \neq z_3$. If $y_4 = y_1 + y_2 \leqslant y_3$, we consider the four distinct points z_1, z_2, z_3, z_4. Depending upon whether $y_1 \leqslant y_2$ or $y_1 > y_2$, we obtain

$$\begin{aligned} d_4(z_1, z_2, z_3, z_4) &= (x_2 - x_1) + (y_3 - y_1) \\ &\quad + \min\{y_1, |x_4 - x_3|\} \\ &\leqslant x_2 - x_1 + y_3 \leqslant 9k \\ d_4(z_1, z_2, z_3, z_4) &= (x_2 - x_1) + (y_3 - y_2) \\ &\quad + \min\{y_2, |x_4 - x_3|\} \\ &\leqslant x_2 - x_1 + y_3 \leqslant 9k \end{aligned}$$

respectively, as illustrated in Fig. 12(a) and (b). In either case, we see that the points z_1, z_2, z_3, z_4 are minimal. If $y_1 \leqslant y_2$ then z_1 is in the lower left corner of $\mathcal{R}(z_1, z_2, z_3, z_4)$. If $y_1 > y_2$ then z_2 is in the lower right corner of $\mathcal{R}(z_1, z_2, z_3, z_4)$. This contradicts Lemma 4.4. Hence assume that $y_4 = y_1 + y_2 > y_3$. In this case, we consider the point $z_5 = z_4 - z_3 = z_1 + z_2 - z_3$. Notice that $z_5 \neq z_0$ iff $z_1 + z_2 \neq z_3$. Once again, depending upon whether $y_1 \leqslant y_2$ or $y_1 > y_2$, we obtain

$$\begin{aligned} d_4(z_0, z_1, z_2, z_5) &= (x_2 - x_1) + y_2 + \min\{y_1 - y_5, |x_5|\} \\ &\leqslant x_2 - x_1 + y_3 \leqslant 9k \\ d_4(z_0, z_1, z_2, z_5) &= (x_2 - x_1) + y_1 + \min\{y_2 - y_5, |x_5|\} \\ &\leqslant x_2 - x_1 + y_3 \leqslant 9k \end{aligned}$$

as illustrated in Fig. 12(c), (d). Thus the points z_0, z_1, z_2, z_5 are minimal and, again, either z_1 or z_2 is in a corner of $\mathcal{R}(z_0, z_1, z_2, z_5)$, which contradicts Lemma 4.4. Hence $z_5 = z_0$ and $z_1 + z_2 = z_3$. □

Lemma 4.6: Let Λ be a sublattice of $\mathbb{Z}^2$ that satisfies property P_k and let z_0, z_1, z_2, z_3 be minimal points of Λ. Then z_0, z_1, z_2, z_3 are vertices of a fundamental parallelogram of Λ.

Proof: As in Lemma 4.5, we assume w.l.o.g. the situation described by (43), (44) and illustrated in Fig. 11. In view of Lemma 4.5 and Theorem 2.8, it would suffice to show that the parallelogram $\mathcal{P}(z_1, z_2)$ defined by z_0, z_1, z_2, z_3 does not contain nonzero points of Λ. Assume to the contrary that $z^* = (\alpha, \beta)$ is such a point. It is easy to see that the only point of $\mathcal{P}(z_1, z_2)$ that lies on the edge of $\mathcal{R}(z_0, z_1, z_2, z_3)$ is $z_0 = (0, 0)$. Therefore, z^* is properly inside $\mathcal{R}(z_0, z_1, z_2, z_3)$. Thus,

$$x_1 < \alpha < x_2 \quad \text{and} \quad 0 < \beta < y_3 = y_1 + y_2. \tag{46}$$

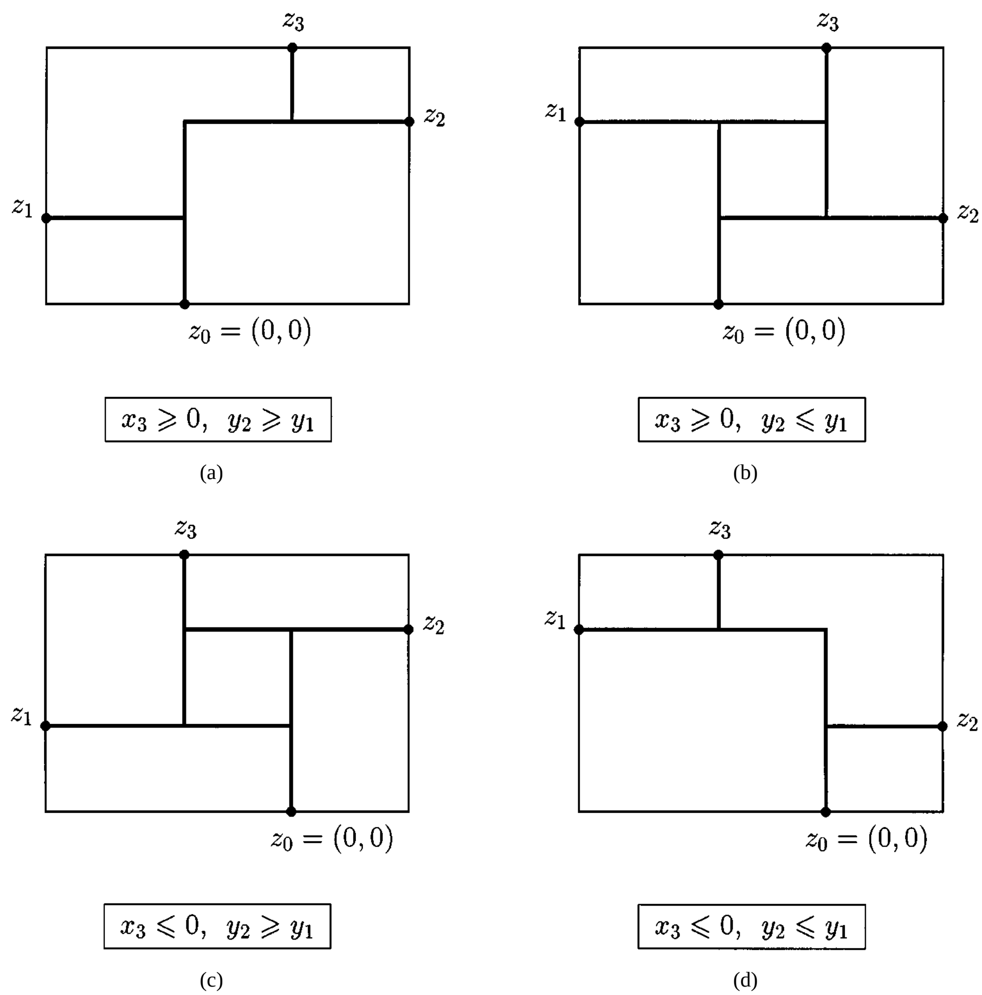

Fig. 11. Minimal spanning trees for $z_0,\ z_1,\ z_2,\ z_3$.

First, assume that $x_3 = x_1 + x_2 \geqslant 0$ and $y_1 \leqslant y_2$, as illustrated in Fig. 11(a). We distinguish between nine cases, designated by the letters $A,\ B,\ \ldots,\ I$ and illustrated in Fig. 13, depending upon where the point z^* is situated within $\mathcal{R}(z_0,\ z_1,\ z_2,\ z_3)$. In each of the nine cases, we establish a contradiction, as follows:

$$
\begin{aligned}
D,\ E\colon\ & d_4(z_1,\ z_2,\ z_3,\ z^*) = (x_2 - x_1) + (y_3 - y_1)\\
&= x_2 - x_1 + y_2 < 9k\\
E,\ F\colon\ & d_4(z_0,\ z_1,\ z_2,\ z^*) = (x_2 - x_1) + (y_2 - 0)\\
&= x_2 - x_1 + y_2 < 9k\\
A,\ B,\ C\colon\ & d_4(z_1,\ z_2,\ z_3,\ z^*)\\
&= (x_2 - x_1) + (y_3 - y_1) + \min\{\beta - y_2, |x_3 - \alpha|\}\\
&\leqslant x_2 - x_1 + \beta < 9k\\
G,\ H,\ I\colon\ & d_4(z_0,\ z_1,\ z_2,\ z^*)\\
&= (x_2 - x_1) + y_2 + \min\{|\alpha|,\ y_1 - \beta\}\\
&\leqslant x_2 - x_1 + y_1 + y_2 - \beta < 9k
\end{aligned}
$$

where the inequalities follow from (44), (46), and the fact that $x_2 - x_1 + y_1 + y_2 \leqslant 9k$ by (45). The remaining three cases in Fig. 11 are proved by a similar argument. □

Proposition 4.7: Let Λ be a sublattice of $\mathbb{Z}^2$ such that $d_4(\Lambda) = 9k$. Suppose that for all $z \in \Lambda$ we have

$$|\mathcal{X}(z)| + |\mathcal{Y}(z)| \neq 3k.$$

Then $V(\Lambda) \geqslant 8k^2$.

Proof: Let $z_0,\ z_1,\ z_2,\ z_3$ be minimal points of Λ. By Lemma 4.5, we can assume that $z_0 = (0,\ 0)$ and $z_3 = z_1 + z_2$. We also assume that $y_1 \leqslant y_2$. Thus, (43) and (44) reduce to the following:

$$x_1 < 0 < x_2 \quad \text{and} \quad 0 < y_1 \leqslant y_2. \tag{47}$$

In addition, using the fact that the L_1-distance between any two points of Λ is strictly greater than $3k$, we obtain the following conditions:

$$d_2(z_0,\ z_1) = y_1 - x_1 > 3k \tag{48}$$
$$d_2(z_0,\ z_2) = x_2 + y_2 > 3k \tag{49}$$
$$d_2(z_1,\ z_2) = x_2 - x_1 + y_2 - y_1 > 3k. \tag{50}$$

By Lemma 4.6, the parallelogram $\mathcal{P}(z_1,\ z_2)$ is a fundamental parallelogram of Λ. It follows that the volume of Λ is given by

$$V(\Lambda) = \begin{vmatrix} x_2 & y_2 \\ x_1 & y_1 \end{vmatrix} = x_2 y_1 - x_1 y_2.$$

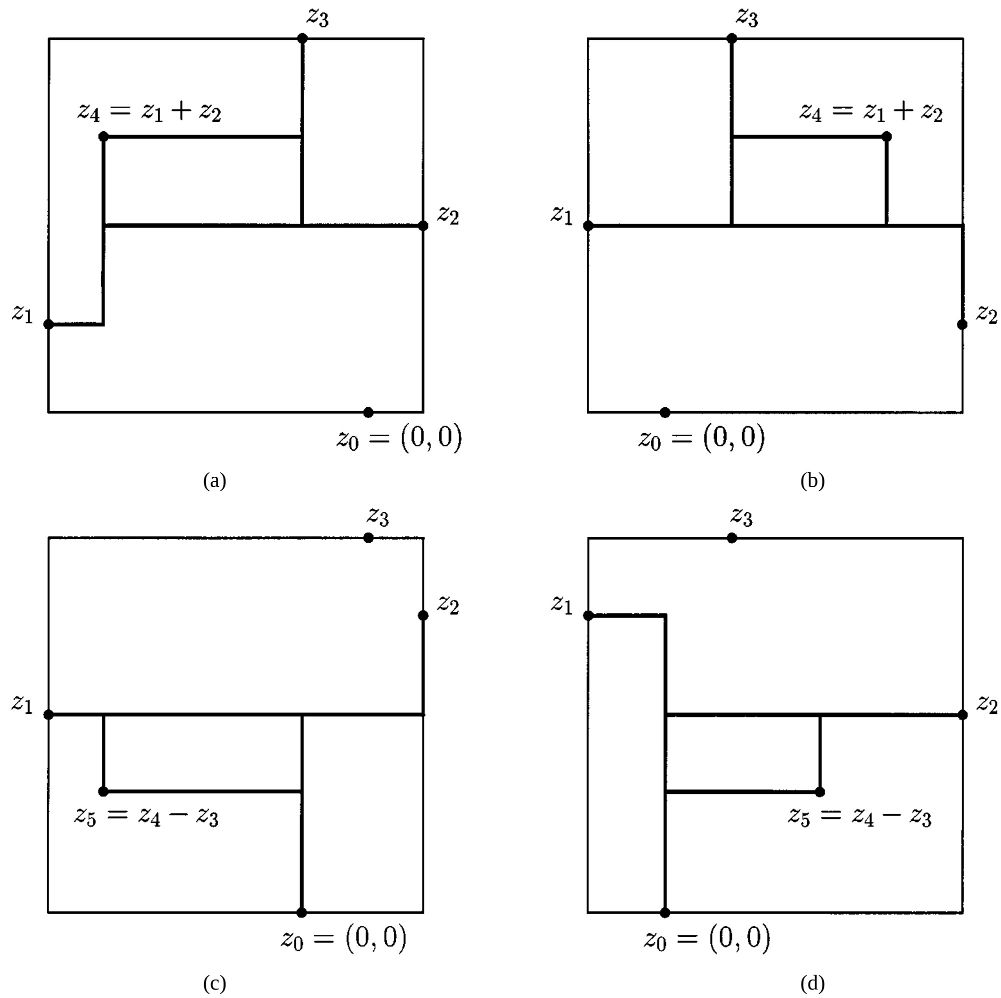

Fig. 12. Four possible configurations of minimal points z_0, z_1, z_2, z_3.

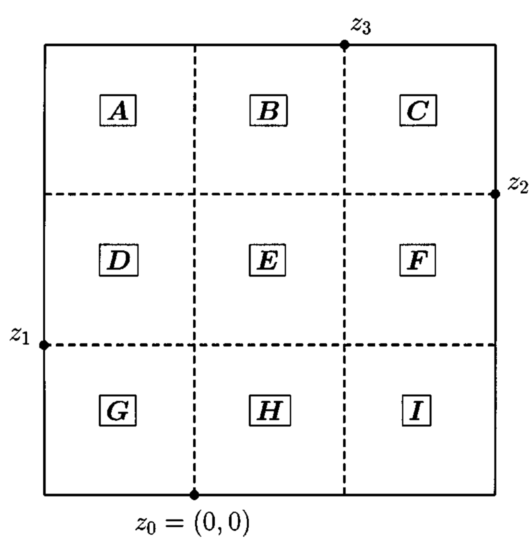

Fig. 13. Nine possible cases for an interior point (see Lemma 4.6).

As in Proposition 3.4, we minimize the quadratic function $f(x_1, x_2, y_1, y_2) = x_2y_1 - x_1y_2$ subject to various constraints on x_1, x_2, y_1, y_2. To establish further constraints, in addition to (47)–(50), we will consider three more lattice points:

$$z_4 = -z_1 = (-x_1, -y_1)$$

$$z_5 = -z_2 = (-x_2, -y_2)$$

$$z_6 = z_2 - z_1 = (x_2 - x_1, y_2 - y_1).$$

The configuration of the seven points $z_0, z_1, \ldots, z_6$ is depicted in Fig. 14(a) and (b) for $x_1 + x_2 < 0$ and $x_1 + x_2 \geqslant 0$, respectively. In general, we distinguish between the following three cases:

Case 1: $x_1 + x_2 \geqslant 0$ and

$$d_4(z_0, z_1, z_2, z_3) = x_2 - x_1 + y_2 + y_1 = 9k \quad (51)$$

Case 2: $x_1 + x_2 < 0,\ x_1 + x_2 \leqslant y_1 - y_2,$ and

$$d_4(z_0, z_1, z_2, z_3) = x_2 - x_1 + 2y_2 = 9k \quad (52)$$

Case 3: $x_1 + x_2 < 0,\ x_1 + x_2 > y_1 - y_2,$ and

$$d_4(z_0, z_1, z_2, z_3) = y_1 + y_2 - 2x_1 = 9k. \quad (53)$$

In Case 1, we derive six additional constraints by examining the quadristance of the lattice points z_0, z_2, z_3, z_4 and z_0, z_1, z_2, z_6. Specifically, referring to Fig. 14(b), we have

$$\begin{aligned} 9k &\leqslant d_4(z_0, z_2, z_3, z_4) \\ &= x_2 + (y_3 - y_4) + \min\{y_2, [x_4 - x_3]_0\} \\ &\leqslant x_2 + 2y_2 + 2y_1 \end{aligned} \quad (54)$$

$$\begin{aligned} 9k &\leqslant d_4(z_0, z_1, z_2, z_6) \\ &= (x_6 - x_1) + y_2 + \min\{x_2, [y_1 - y_6]_0\} \\ &\leqslant 2x_2 + y_2 - 2x_1 \end{aligned} \quad (55)$$

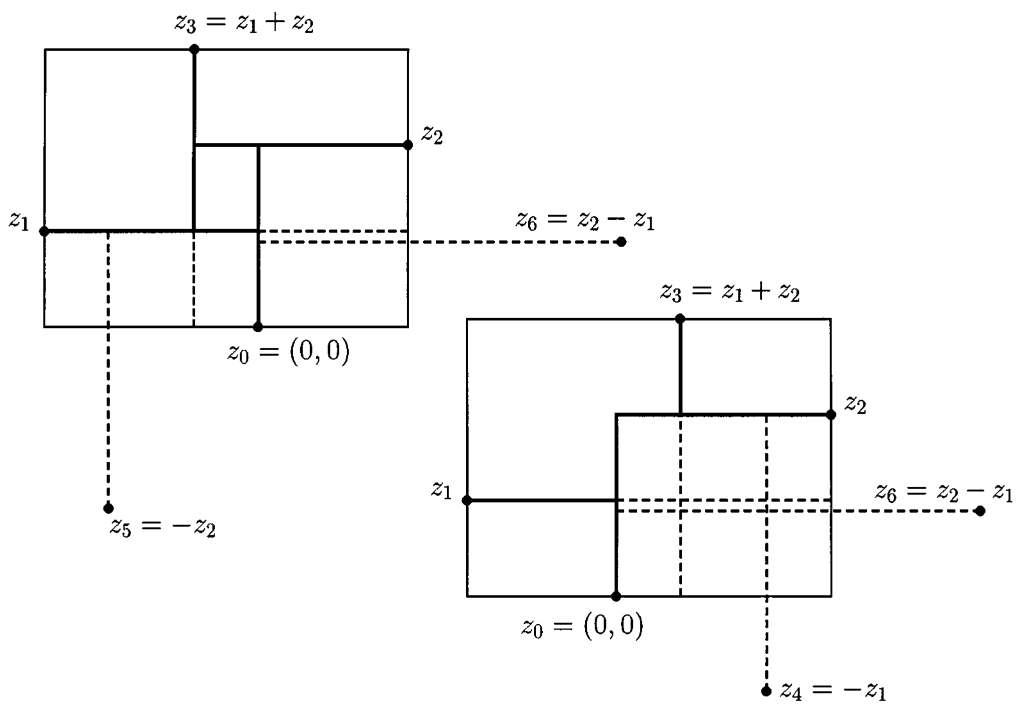

Fig. 14. Seven-point configuration for additional constraints (see Proposition 4.7).

where $[x]_0$ stands for $\max\{x, 0\}$. Each of (54) and (55) gives rise to two more inequalities, depending upon whether $x_4 - x_3 = -2x_1 - x_2 > 0$ and whether $y_1 - y_6 = 2y_1 - y_2 > 0$, respectively. Using the fact that $x_2 - x_1 + y_2 + y_1 = 9k$ by (51), we convert the six inequalities arising from (54), (55) into the following six constraints:

$$x_1 + y_1 \geqslant -y_2 \tag{56}$$
$$x_2 \geqslant x_1 + y_1 \tag{57}$$
$$2x_1 + x_2 < 0 \Rightarrow y_1 - x_1 \geqslant x_2 \tag{58}$$
$$2x_1 + x_2 \geqslant 0 \Rightarrow y_1 \geqslant -x_1 \tag{59}$$
$$y_2 - 2y_1 < 0 \Rightarrow y_1 - x_1 \geqslant y_2 \tag{60}$$
$$y_2 - 2y_1 \geqslant 0 \Rightarrow -x_1 \geqslant y_1. \tag{61}$$

We now use a simple optimization program to minimize the quadratic $f(x_1, x_2, y_1, y_2) = x_2y_1 - x_1y_2$ subject to (47)–(51) and (56)–(61). The analysis in Cases 2 and 3 is similar. In Case 2, we have

$$\begin{aligned} 9k &\leqslant d_4(z_0, z_1, z_3, z_5) \\ &= -x_1 + (y_3 - y_5) + \min\{y_1, [x_3 - x_5]_0\} \\ &\leqslant -x_1 + 2y_1 + 2y_2 \end{aligned} \tag{62}$$

in addition to (55), which holds without change in all three cases. As before, we convert the six inequalities arising from (55) and (62) into the following constraints:

$$2y_1 \geqslant x_2 \tag{63}$$
$$x_2 - y_2 \geqslant x_1 \tag{64}$$
$$x_1 + 2x_2 < 0 \Rightarrow y_1 \geqslant x_2 \tag{65}$$
$$x_1 + 2x_2 \geqslant 0 \Rightarrow y_1 + x_1 \geqslant -x_2 \tag{66}$$
$$y_2 - 2y_1 < 0 \Rightarrow 2y_1 - x_1 \geqslant 2y_2 \tag{67}$$
$$y_2 - 2y_1 \geqslant 0 \Rightarrow -x_1 \geqslant y_2 \tag{68}$$

using the fact that $x_2 - x_1 + 2y_2 = 9k$ by (52). In Case 3, we apply the same six inequalities as in Case 2, in conjunction with the fact that $y_1 + y_2 - 2x_1 = 9k$ by (53), to obtain the constraints

$$y_1 + x_1 \geqslant -y_2 \tag{69}$$
$$2x_2 \geqslant y_1 \tag{70}$$
$$x_1 + 2x_2 < 0 \Rightarrow y_2 \geqslant -x_1 \tag{71}$$
$$x_1 + 2x_2 \geqslant 0 \Rightarrow 2x_2 + y_2 \geqslant -2x_1 \tag{72}$$
$$y_2 - 2y_1 < 0 \Rightarrow x_2 - y_2 \geqslant -y_1 \tag{73}$$
$$y_2 - 2y_1 \geqslant 0 \Rightarrow x_2 \geqslant y_1. \tag{74}$$

The optimization program now shows that the minimum of $f(x_1, x_2, y_1, y_2) = x_2y_1 - x_1y_2$ subject to either (47)–(51), (56)–(61), or (47)–(50), (52), (63)–(68), or (47)–(50), (53), (69)–(74) is $8k^2$. This minimum is attained uniquely by two isomorphic lattices generated by $z_1 = (-2k, 2k)$ and $z_2 = (k, 3k)$, or by $z_1 = (-3k, k)$ and $z_2 = (2k, 2k)$. Finally, the symmetric situation where $y_1 > y_2$ can be analyzed by a similar method, and $f(x_1, x_2, y_1, y_2) \geqslant 8k^2$ in this case as well. □

Theorem 4.8: Let Λ be a sublattice of $\mathbb{Z}^2$ with quadristance $d_4(\Lambda) = t$. Set $k = \lfloor t/9 \rfloor$. Then the volume of Λ is bounded from below as follows:

$$V(\Lambda) \geqslant 8k^2, \quad \text{if } t \equiv 0 \bmod 9 \tag{75}$$
$$V(\Lambda) \geqslant 8k^2 + \tfrac{16}{9}k + \tfrac{8}{81}, \quad \text{if } t \equiv 1 \bmod 9 \tag{76}$$
$$V(\Lambda) \geqslant 8k^2 + \tfrac{32}{9}k + \tfrac{32}{81}, \quad \text{if } t \equiv 2 \bmod 9 \tag{77}$$
$$V(\Lambda) \geqslant 8k^2 + \tfrac{16}{3}k + \tfrac{8}{9}, \quad \text{if } t \equiv 3 \bmod 9 \tag{78}$$
$$V(\Lambda) \geqslant 8k^2 + \tfrac{64}{9}k + \tfrac{128}{81}, \quad \text{if } t \equiv 4 \bmod 9 \tag{79}$$
$$V(\Lambda) \geqslant 8k^2 + \tfrac{80}{9}k + \tfrac{200}{81}, \quad \text{if } t \equiv 5 \bmod 9 \tag{80}$$
$$V(\Lambda) \geqslant 8k^2 + \tfrac{32}{3}k + \tfrac{32}{9}, \quad \text{if } t \equiv 6 \bmod 9 \tag{81}$$

TABLE I
OPTIMAL LATTICE INTERLEAVERS WITH THREE REPETITIONS
Each lattice Λ is generated by (a, b) and $(0, V(\Lambda)/a)$. Lattices better than Theorem 4.1 are marked by *

t	$V(\Lambda)$	generators a	b	$\frac{t^2}{V(\Lambda)}$
4	2*	1	1	8.0000
5	3*	1	1	8.3333
6	4*	1	1	9.0000
7	5*	1	2	9.8000
8	7*	1	2	9.1429
9	8	1	3	10.1250
10	10*	1	3	10.0000
11	13	1	5	9.3077
12	15	1	11	9.6000
13	17*	1	4	9.9412
14	20*	2	4	9.8000
15	23	1	14	9.7826
16	26	2	5	9.8462
17	29*	1	12	9.9655
18	32	2	6	10.1250
19	37*	1	6	9.7568
20	40*	2	6	10.0000
21	45	1	19	9.8000
22	50	1	19	9.6800
23	53*	1	23	9.9811
24	58	1	22	9.9310
25	63	3	8	9.9206
26	68*	2	8	9.9412
27	72	3	9	10.1250
28	80*	4	8	9.8000
29	85*	1	38	9.8941
30	90*	3	9	10.0000
31	97*	1	22	9.9072
32	104	4	10	9.8462
33	109	1	30	9.9908
34	116	4	11	9.9655
35	125*	1	57	9.8000
36	128	4	12	10.1250
37	137*	1	37	9.9927
38	146*	1	27	9.8904
39	153	1	35	9.9412
40	160*	4	12	10.0000
41	169*	1	70	9.9467
42	176	1	38	10.0227
43	185	5	14	9.9946
44	194*	1	75	9.9794
45	200	5	15	10.1250
46	212*	2	46	9.9811
47	221*	1	47	9.9955
48	231	1	43	9.9740
49	241*	1	64	9.9627
50	250*	5	15	10.0000
51	259	1	46	10.0425
52	270	6	17	10.0148
53	281*	1	53	9.9964
54	288	6	18	10.1250
55	305*	1	72	9.9180
56	314*	1	129	9.9873
57	325	1	51	9.9969
58	338	1	51	9.9527
59	349*	1	136	9.9742
60	358	1	54	10.0559
61	371	7	20	10.0296
62	386*	1	81	9.9585
63	392	7	21	10.1250
64	410*	1	173	9.9902
65	425*	1	157	9.9412
66	435	1	59	10.0138
67	449*	1	67	9.9978
68	464	8	22	9.9655
69	473	1	62	10.0655
70	488	8	23	10.0410
71	505*	1	192	9.9822
72	512	8	24	10.1250
73	533*	1	73	9.9981
74	548*	2	74	9.9927
75	561	1	67	10.0267
76	578	1	67	9.9931
77	593*	1	77	9.9983
78	604	1	70	10.0728
79	621	9	26	10.0499
80	640*	8	24	10.0000
81	648	9	27	10.1250
82	674*	1	189	9.9763
83	689*	1	83	9.9985
84	703	1	75	10.0370
85	722	1	75	10.0069
86	740	10	28	9.9946
87	751	1	78	10.0786
88	770	10	29	10.0571
89	793*	1	294	9.9887
90	800	10	30	10.1250
91	829*	1	246	9.9891
92	848*	4	92	9.9811
93	861	1	83	10.0453
94	882	1	83	10.0181
95	902	11	31	10.0055
96	914	1	86	10.0832
97	935	11	32	10.0631
98	962*	1	265	9.9834
99	968	11	33	10.1250
100	1000*	10	30	10.0000
101	1021*	1	374	9.9912
102	1035	1	91	10.0522
103	1058	1	91	10.0274
104	1080	12	34	10.0148
105	1093	1	94	10.0869
106	1116	12	35	10.0681
107	1145*	1	107	9.9991
108	1152	12	36	10.1250
109	1188	12	37	10.0008
110	1210*	11	33	10.0000
111	1225	1	99	10.0580
112	1250	1	99	10.0352
113	1274	13	37	10.0228
114	1288	1	102	10.0901
115	1313	13	38	10.0724
116	1346*	1	615	9.9970
117	1352	13	39	10.1250
118	1391	13	40	10.0101
119	1417	1	104	9.9936
120	1431	1	107	10.0629
121	1458	1	107	10.0418
122	1484	14	40	10.0296
123	1499	1	110	10.0927
124	1526	14	40	10.0760
125	1565*	1	288	9.9840
126	1568	14	42	10.1250
127	1610	14	43	10.0180
128	1638	14	44	10.0024
129	1653	1	115	10.0672
130	1682	1	115	10.0476
131	1710	15	43	10.0357
132	1726	1	118	10.0950
133	1755	15	44	10.0792
134	1796*	2	134	9.9978
135	1800	15	45	10.1250
136	1845	15	46	10.0249
137	1875	15	47	10.0101
138	1891	1	123	10.0709
139	1922	1	123	10.0525
140	1952	16	46	10.0410
141	1969	1	126	10.0970
142	2000	16	47	10.0820
143	2045*	1	143	9.9995
144	2048	16	48	10.1250
145	2096	16	49	10.0310
146	2128	16	50	10.0169
147	2145	1	131	10.0741
148	2178	1	131	10.0569
149	2210	17	49	10.0457
150	2228	1	134	10.0987
151	2261	17	50	10.0845
152	2311	1	133	9.9974
153	2312	17	51	10.1250
154	2363	17	52	10.0364
155	2397	17	53	10.0229
156	2415	1	139	10.0770
157	2450	1	139	10.0608
158	2484	18	52	10.0499
159	2503	1	142	10.1003
160	2538	18	53	10.0867
161	2591	1	141	10.0042
162	2592	18	54	10.1250
163	2646	18	55	10.0412
164	2682	18	56	10.0283
165	2701	1	147	10.0796
166	2738	1	147	10.0643
167	2774	19	55	10.0537
168	2794	1	150	10.1016
169	2831	19	56	10.0887
170	2887	1	149	10.0104
171	2888	19	57	10.1250
172	2945	19	58	10.0455
173	2983	19	59	10.0332
174	3003	1	155	10.0819
175	3042	1	155	10.0674
176	3080	20	58	10.0571
177	3101	1	158	10.1029
178	3140	20	59	10.0904
179	3199	1	157	10.0159
180	3200	20	60	10.1250

$$V(\Lambda) \geqslant 8k^2 + \tfrac{112}{9}k + \tfrac{392}{81}, \qquad \text{if } t \equiv 7 \bmod 9 \qquad (82)$$
$$V(\Lambda) \geqslant 8k^2 + \tfrac{128}{9}k + \tfrac{512}{81}, \qquad \text{if } t \equiv 8 \bmod 9. \qquad (83)$$

Proof: The bound (75) follows directly from Propositions 4.2 and 4.7. The remaining bounds follow from (75) in the same way that (21)–(23) follow from (20) in Theorem 3.6. □

Theorem 4.8 is the main result of this subsection. This theorem shows that the lattices Λ_{9k} constructed in Section IV-A are optimal for all k. Although this does not follow from Theorem 4.8, we conjecture that the lattices $\Lambda_{9k+1}, \Lambda_{9k+2}, \ldots, \Lambda_{9k+8}$ are also optimal for all $k \geqslant 16$.

C. Computer Search

As in the previous section, we have conducted an exhaustive computer search for optimal lattice interleavers of strength t with $r = 3$ repetitions, where $t \leqslant 999$. For certain small values of t, we found lattices whose interleaving degree is lower than that of $\Lambda_{9k+1}, \Lambda_{9k+2}, \ldots, \Lambda_{9k+8}$. These lattices are marked by an asterisk $*$ in Table I. On the other hand, for all t in the range $144 \leqslant t \leqslant 999$, the exhaustive search confirmed the optimality of the lattices constructed in Section IV-A. The general search strategy is the same as that employed in Section III-C. The key idea is that the quadristance of a lattice can be computed efficiently by considering only a few points. We omit the details.

It follows from Theorems 4.1 and 4.8 that whenever m is of the form $8k^2$ for some positive integer k, the exhaustive search should produce a lattice with quadristance $t = 9k$. We have run the search for all $m = 1, 2, \ldots, 98\,568$, which corresponds to $k \leqslant 111$. As expected, the search produced lattices with quadristance t, for each t between 3 and $999 = 9 \cdot 111$. The results of the search are compiled in Table I up to $t = 180$. For each quadristance t, the table lists the volume $V(\Lambda) = m$ of the optimal lattice Λ and a generator matrix for Λ. The generator matrix $\boldsymbol{G}$ is of the form (26), and only the two components a and b are given in Table I. The remaining nonzero component of $\boldsymbol{G}$ can be computed as $c = m/a$. We also display in Table I the ratio $t^2/V(\Lambda)$. It follows from Theorems 4.1 and 4.8 that for optimal lattices Λ, this ratio converges (from below) to $9^2/8 = 10.125$ as $t \to \infty$. This convergence can be observed empirically in Table I.

V. Lattice Interleavers for Four Repetitions

This section deals with lattices with prescribed quintistance, which correspond to lattice interleavers with four repetitions. The results are similar to those compiled in the previous two sections for two and three repetitions, except that we do not have a lower bound. While it should be, in principle, possible to extend the proof technique of Section IV-B to the case of four repetitions, the resulting proof would be extraordinarily tedious. Thus, radically new ideas are needed to establish good lower bounds on the degree of lattice interleavers for $r \geqslant 4$. Herein, we present constructions and the results of a computer search in Sections V-A and V-B, respectively.

A. Constructions

While for two and three repetitions, we have distinguished between several cases depending on the value of the strength t modulo 4 and 9, respectively, it appears that the corresponding modulus for four repetitions is 12. Thus, we distinguish between 12 cases, and for each $k = 1, 2, \ldots,$ define the lattices $\Lambda_{12k}, \Lambda_{12k+1}, \ldots, \Lambda_{12k+11}$ by means of the corresponding generator matrices

$$\boldsymbol{G}_{12k} = \begin{bmatrix} 2k & 2k \\ 0 & 4k \end{bmatrix}$$
$$\boldsymbol{G}_{12k+1} = \begin{bmatrix} 2k & 2k \\ 0 & 4k+1 \end{bmatrix}$$
$$\boldsymbol{G}_{12k+2} = \begin{bmatrix} k+1 & 5k \\ 1 & 8k \end{bmatrix}$$
$$\boldsymbol{G}_{12k+3} = \begin{bmatrix} 2k+1 & 2k \\ 1 & 4k+1 \end{bmatrix}$$
$$\boldsymbol{G}_{12k+4} = \begin{bmatrix} k & 3k+2 \\ 0 & 8k+6 \end{bmatrix}$$
$$\boldsymbol{G}_{12k+5} = \begin{bmatrix} 2k+2 & 2k+1 \\ 1 & 4k+1 \end{bmatrix}$$
$$\boldsymbol{G}_{12k+6} = \begin{bmatrix} 2k+1 & 2k+1 \\ 0 & 4k+2 \end{bmatrix}$$
$$\boldsymbol{G}_{12k+7} = \begin{bmatrix} 2k+1 & 2k+1 \\ 0 & 4k+3 \end{bmatrix}$$
$$\boldsymbol{G}_{12k+8} = \begin{bmatrix} k+1 & 3k+2 \\ 1 & 8k+6 \end{bmatrix}$$
$$\boldsymbol{G}_{12k+9} = \begin{bmatrix} 2k+2 & 2k+1 \\ 1 & 4k+3 \end{bmatrix}$$
$$\boldsymbol{G}_{12k+10} = \begin{bmatrix} 2k+2 & 2k+2 \\ 0 & 4k+3 \end{bmatrix}$$
$$\boldsymbol{G}_{12k+11} = \begin{bmatrix} 2k+3 & 2k+2 \\ 1 & 4k+3 \end{bmatrix}. \quad (84)$$

The following theorem is the counterpart of Theorem 3.1 and Theorem 4.1 of the previous two sections. The theorem shows that $d_5(\Lambda_t) = t$ for the twelve lattices defined above, provided $k \geqslant 2$.

Theorem 5.1: For $k = 2, 3, \ldots$

$$d_5(\Lambda_{12k}) = 12k$$
$$V(\Lambda_{12k}) = 8k^2 \quad (85)$$
$$d_5(\Lambda_{12k+1}) = 12k+1$$
$$V(\Lambda_{12k+1}) = 8k^2+2k \quad (86)$$
$$d_5(\Lambda_{12k+2}) = 12k+2$$
$$V(\Lambda_{12k+2}) = 8k^2+3k \quad (87)$$
$$d_5(\Lambda_{12k+3}) = 12k+3$$
$$V(\Lambda_{12k+3}) = 8k^2+4k+1 \quad (88)$$
$$d_5(\Lambda_{12k+4}) = 12k+4$$
$$V(\Lambda_{12k+4}) = 8k^2+6k \quad (89)$$
$$d_5(\Lambda_{12k+5}) = 12k+5$$
$$V(\Lambda_{12k+5}) = 8k^2+8k+1 \quad (90)$$
$$d_5(\Lambda_{12k+6}) = 12k+6$$
$$V(\Lambda_{12k+6}) = 8k^2+8k+2 \quad (91)$$
$$d_5(\Lambda_{12k+7}) = 12k+7$$
$$V(\Lambda_{12k+7}) = 8k^2+10k+3 \quad (92)$$
$$d_5(\Lambda_{12k+8}) = 12k+8$$
$$V(\Lambda_{12k+8}) = 8k^2+11k+4 \quad (93)$$
$$d_5(\Lambda_{12k+9}) = 12k+9$$
$$V(\Lambda_{12k+9}) = 8k^2+12k+5 \quad (94)$$
$$d_5(\Lambda_{12k+10}) = 12k+10$$
$$V(\Lambda_{12k+10}) = 8k^2+14k+6 \quad (95)$$

TABLE II
OPTIMAL LATTICE INTERLEAVERS WITH FOUR REPETITIONS

t	$V(\Lambda)$	generators a	generators b	$\frac{t^2}{V(\Lambda)}$	t	$V(\Lambda)$	generators a	generators b	$\frac{t^2}{V(\Lambda)}$	t	$V(\Lambda)$	generators a	generators b	$\frac{t^2}{V(\Lambda)}$
4	1	1	1	16.0000	16	15	3	3	17.0667	70	275	5	40	17.8182
5	2	1	1	12.5000	53	160	4	11	17.5562					

$$d_5(\Lambda_{12k+11}) = 12k+11$$
$$V(\Lambda_{12k+11}) = 8k^2+16k+7. \tag{96}$$

Proof: For each Λ_{12k+i} in (85)–(96), we only need to consider those points $z \in \Lambda_{12k+i}$ for which

$$\mathcal{X}(z) \geqslant 0 \quad \text{and} \quad |\mathcal{X}(z)| + |\mathcal{Y}(z)| \leqslant 12k+i-1$$

(cf. Theorem 4.1). For Λ_{12k}, these points are

$$(0,-8k),\quad (0,-4k),\quad (0,4k),\quad (0,8k),\quad (2k,-6k),$$
$$(2k,-2k),\quad (2k,2k),\quad (2k,6k),\quad (4k,-4k),\quad (4k,0),$$
$$(4k,4k),\quad (6k,-2k),\quad (6k,2k),\quad (8k,0). \tag{97}$$

We now use a computer program to verify that

$$d_5(z_0, z_1, z_2, z_3, z_4) \geqslant 12k$$

where $z_0 = (0, 0)$ and z_1, z_2, z_3, z_4 are any four of the 14 points in (97). The remainder of the theorem is proved by a similar argument. Notice that the volumes of all the lattices are immediate from the definitions in (84), so it remains to establish their quintistance. □

We note that (85)–(96) hold for $k = 1$ as well, with a single exception: $d_5(\Lambda_{16}) = 15$. To see that $d_5(\Lambda_{16}) \leqslant 15$, consider the lattice points $(-3, -1)$, $(-2, 4)$, $(0, 0)$, $(1, 5)$, $(3, 1)$. We can also extend the construction to the case $k = 0$. Specifically, we define the lattices $\Lambda_6, \Lambda_7, \ldots, \Lambda_{11}$ by means of the corresponding generator matrices

$$\boldsymbol{G}_6 = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \qquad \boldsymbol{G}_7 = \begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix} \qquad \boldsymbol{G}_8 = \begin{bmatrix} 1 & 2 \\ 1 & 6 \end{bmatrix}$$
$$\boldsymbol{G}_9 = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} \qquad \boldsymbol{G}_{10} = \begin{bmatrix} 2 & 2 \\ 0 & 3 \end{bmatrix} \qquad \boldsymbol{G}_{11} = \begin{bmatrix} 3 & 2 \\ 1 & 3 \end{bmatrix}$$

which follow by substituting $k = 0$ into the general construction of (84). It can be readily verified by hand that for these lattices $d_5(\Lambda_t) = t$ and $V(\Lambda_t) = t - 4$, for $t = 6, 7, \ldots, 11$.

We conjecture that our construction, appropriately extended for $k = 0, 1$ as discussed above, produces optimal lattice interleavers with four repetitions for all strengths t, except for $t = 4, 5, 16, 53, 70$. Evidence in support of this conjecture is obtained, by computer search, in the next subsection.

B. Computer Search

As in the previous section, we now present the results of an exhaustive computer search for optimal lattice interleavers with four repetitions. We have employed the same general search strategy as in Sections III-A and IV-A, and conducted the search up to $m = 56\,448$. The search produced optimal lattice interleavers for all $t = 4, 5, \ldots, 1008$. The results of the search confirm that the lattices constructed in the previous subsection are optimal, at least up to $t = 1008$, with the exception of $t = 4, 5, 16, 53, 70$. The corresponding five lattices, that improve upon the construction of Section V-A, are listed in Table II. The format of this table is the same as that of Table I.

Note that Theorem 5.1 implies that $\lim_{t\to\infty} t^2/V(\Lambda)$ is at least $12^2/8 = 18$ for optimal lattices. Although we do not have a lower bound on $V(\Lambda)$ as a function of t for four repetitions, the computer search seems to indicate that, as t increases, the ratio $t^2/V(\Lambda)$ rapidly converges to 18.

VI. LATTICE INTERLEAVERS FOR MULTIPLE REPETITIONS

As the number of repetitions r increases, the task of determining optimal lattice interleavers becomes more and more difficult. Partial results for $r = 5, 6, 7, \ldots,$ are presented in this section. We start with $r = 5$, and for each $k = 1, 2, \ldots$, define the lattices $\Lambda_{18k}, \Lambda_{18k+1}, \ldots, \Lambda_{18k+17}$ by means of the corresponding generator matrices

$$\boldsymbol{G}_{18k} = \begin{bmatrix} 2k & 2k \\ 0 & 6k \end{bmatrix}$$
$$\boldsymbol{G}_{18k+1} = \begin{bmatrix} 2k & 2k \\ 0 & 6k+1 \end{bmatrix}$$
$$\boldsymbol{G}_{18k+2} = \begin{bmatrix} 2k & 2k \\ 0 & 6k+2 \end{bmatrix}$$
$$\boldsymbol{G}_{18k+3} = \begin{bmatrix} 2k & 2k+1 \\ -k & 5k+2 \end{bmatrix}$$
$$\boldsymbol{G}_{18k+4} = \begin{bmatrix} 2k+1 & 2k \\ 1 & 6k+1 \end{bmatrix}$$
$$\boldsymbol{G}_{18k+5} = \begin{bmatrix} 2k+1 & 2k+1 \\ 0 & 6k+1 \end{bmatrix}$$
$$\boldsymbol{G}_{18k+6} = \begin{bmatrix} 2k & 2k+2 \\ -k & 5k+4 \end{bmatrix}$$
$$\boldsymbol{G}_{18k+7} = \begin{bmatrix} 2k+1 & 2k+1 \\ 0 & 6k+2 \end{bmatrix}$$
$$\boldsymbol{G}_{18k+8} = \begin{bmatrix} 2k+2 & 2k \\ 1 & 6k+1 \end{bmatrix}$$
$$\boldsymbol{G}_{18k+9} = \begin{bmatrix} 2k+1 & 2k+1 \\ 0 & 6k+3 \end{bmatrix}$$
$$\boldsymbol{G}_{18k+10} = \begin{bmatrix} 2k+1 & 2k+1 \\ 0 & 6k+4 \end{bmatrix}$$
$$\boldsymbol{G}_{18k+11} = \begin{bmatrix} 2k+1 & 2k+1 \\ 0 & 6k+5 \end{bmatrix}$$
$$\boldsymbol{G}_{18k+12} = \begin{bmatrix} 2k+1 & 2k+2 \\ -k-1 & 5k+4 \end{bmatrix}$$

TABLE III
OPTIMAL LATTICE INTERLEAVERS WITH FIVE REPETITIONS

t	$V(\Lambda)$	generators a	b	$\frac{t^2}{V(\Lambda)}$	t	$V(\Lambda)$	generators a	b	$\frac{t^2}{V(\Lambda)}$	t	$V(\Lambda)$	generators a	b	$\frac{t^2}{V(\Lambda)}$
6	2	1	1	18.0000	51	99	3	9	26.2727	80	240	4	16	26.6667
8	3	1	1	21.3333	60	135	3	12	26.6667	105	416	1	79	26.5024
20	15	1	4	26.6667	65	160	1	49	26.4063	140	735	7	28	26.6667
40	60	2	8	26.6667	77	224	4	15	26.4688					

$$\boldsymbol{G}_{18k+13} = \begin{bmatrix} 2k+2 & 2k+1 \\ 1 & 6k+4 \end{bmatrix}$$

$$\boldsymbol{G}_{18k+14} = \begin{bmatrix} 2k+2 & 2k+2 \\ 0 & 6k+4 \end{bmatrix}$$

$$\boldsymbol{G}_{18k+15} = \begin{bmatrix} 2k+2 & 2k+1 \\ -1 & 6k+4 \end{bmatrix}$$

$$\boldsymbol{G}_{18k+16} = \begin{bmatrix} 2k+2 & 2k+2 \\ 0 & 6k+5 \end{bmatrix}$$

$$\boldsymbol{G}_{18k+17} = \begin{bmatrix} 2k+3 & 2k+1 \\ 1 & 6k+4 \end{bmatrix}. \qquad (98)$$

The volumes of $\Lambda_{18k}, \Lambda_{18k+1}, \ldots, \Lambda_{18k+17}$ can be computed immediately from (98). Observe that $V(\Lambda_{18k+i}) = 12k^2 + O(k)$ for all the eighteen lattices above. We conjecture that

$$d_6(\Lambda_{18k+i}) = 18k + i, \qquad \text{for } i = 0, 1, \ldots, 17 \qquad (99)$$

and all $k = 1, 2, \ldots$. While, in principle, it should be straightforward to prove this conjecture along the lines established in the proofs of Theorems 3.1, 4.1, and 5.1, we refrain from doing so for the reasons described in the remark below. Instead, the following theorem verifies this conjecture for Λ_{18k} and Λ_{18k+9} only, using an alternative, much simpler, proof technique.

Theorem 6.1:

$$d_6(\Lambda_{18k}) = 18k \qquad V(\Lambda_{18k}) = 12k^2$$

$$d_6(\Lambda_{18k+9}) = 18k + 9 \qquad V(\Lambda_{18k+9}) = 12k^2 + 12k + 3$$

Proof: Observe that $\Lambda_{18k} = 2k\Lambda^*$ while $\Lambda_{18k+9} = (2k+1)\Lambda^*$, where Λ^* is the lattice generated by $z_1 = (1, 1)$ and $z_2 = (0, 3)$. Thus, by Theorem 2.9, it would suffice to prove that $d_6(\Lambda^*) = 9$. There are 23 relevant points in Λ^*, given by

$$(0, -6), (0, -3), (0, 3), (0, 6), (1, -5),$$
$$(1, -2), (1, 1), (1, 4), (1, 7), (2, -4), (2, -1),$$
$$(2, 2), (2, 5), (3, -3), (3, 0), (3, 3), (4, -2),$$
$$(4, 1), (4, 4), (5, -1), (5, 2), (6, 0), (7, 1)$$

and the fact that the 6-dispersion of $z_0 = (0, 0)$ and any five of the 23 points above is at least 9 can be easily verified by hand. The 6-dispersion of $(0, 0)$, $(1, 1)$, $(2, 2)$, $(3, 0)$, $(4, 1)$, $(5, 2)$ is exactly 9. □

We have also conducted an exhaustive computer search for optimal lattice interleavers with five repetitions. The computer search confirmed (99) up to $k = 50$. Furthermore, the lattices constructed in (98) were found to be optimal for all $t \leqslant 900$, except $t = 5, 6, 20, 40, 51, 60, 65, 77, 80, 105, 140$.

The corresponding 11 lattices that improve upon (98) are listed in Table III. We conjecture that for all other values of t, the constructions of (98) are optimal. Finally, computer search indicates that for optimal lattice interleavers, the ratio $t^2/V(\Lambda)$ rapidly converges to $18^2/12 = 27$ as $t \to \infty$.

For six repetitions, we have very similar results. First, for each $k = 1, 2, \ldots$, we define the lattices $\Lambda_{14k}, \Lambda_{14k+1}, \ldots, \Lambda_{14k+12}$ by means of the corresponding generator matrices

$$\boldsymbol{G}_{14k} = \begin{bmatrix} k & 2k \\ 0 & 5k \end{bmatrix}$$

$$\boldsymbol{G}_{14k+1} = \begin{bmatrix} k+1 & 2k \\ 1 & 5k-1 \end{bmatrix}$$

$$\boldsymbol{G}_{14k+2} = \begin{bmatrix} k & 2k+1 \\ 0 & 5k+2 \end{bmatrix}$$

$$\boldsymbol{G}_{14k+3} = \begin{bmatrix} k & 2k+1 \\ 0 & 5k+3 \end{bmatrix}$$

$$\boldsymbol{G}_{14k+4} = \begin{bmatrix} k & 2k+2 \\ 0 & 5k+4 \end{bmatrix}$$

$$\boldsymbol{G}_{14k+5} = \begin{bmatrix} k & 2k+1 \\ -1 & 5k+2 \end{bmatrix}$$

$$\boldsymbol{G}_{14k+6} = \begin{bmatrix} k+1 & 2k+1 \\ 1 & 5k+2 \end{bmatrix}$$

$$\boldsymbol{G}_{14k+7} = \begin{bmatrix} k+1 & 2k+1 \\ 1 & 5k+3 \end{bmatrix}$$

$$\boldsymbol{G}_{14k+8} = \begin{bmatrix} k+1 & 2k+1 \\ 1 & 5k+3 \end{bmatrix}$$

$$\boldsymbol{G}_{14k+9} = \begin{bmatrix} k & 2k+2 \\ -1 & 5k+6 \end{bmatrix}$$

$$\boldsymbol{G}_{14k+10} = \begin{bmatrix} k+1 & 2k+1 \\ 0 & 5k+3 \end{bmatrix}$$

$$\boldsymbol{G}_{14k+11} = \begin{bmatrix} k+1 & 2k+2 \\ 0 & 5k+4 \end{bmatrix}$$

$$\boldsymbol{G}_{14k+12} = \begin{bmatrix} k+2 & 2k+1 \\ 1 & 5k+2 \end{bmatrix}. \qquad (100)$$

Further define $\Lambda_{14k+13} = \Lambda_{14k+14} = \Lambda_{14(k+1)}$ (notice that we also have $\Lambda_{14k+7} = \Lambda_{14k+8}$ above). It is easy to see that the volumes of all the lattices in (100) are of the form $V(\Lambda_{14k+i}) = 5k^2 + O(k)$. As in Theorem 6.1, we can show that

$$d_7(\Lambda_{14k}) = 14k \quad \text{and} \quad d_7(\Lambda_{14k+13}) = 14k + 14 \qquad (101)$$

TABLE IV
OPTIMAL LATTICE INTERLEAVERS WITH SIX REPETITIONS

t	$V(\Lambda)$	generators a	b	$\frac{t^2}{V(\Lambda)}$	t	$V(\Lambda)$	generators a	b	$\frac{t^2}{V(\Lambda)}$	t	$V(\Lambda)$	generators a	b	$\frac{t^2}{V(\Lambda)}$
9	3	1	1	27.0000	25	17	1	4	36.7647	54	77	1	34	37.8701
12	4	1	2	36.0000	32	27	1	5	37.9259	65	112	1	41	37.7232
15	7	1	2	32.1429	37	37	1	6	37.0000	96	240	4	16	38.4000
24	15	1	4	38.4000	48	60	2	8	38.4000					

for all k, using the fact that $\Lambda_{14k} = k\Lambda'$ and $\Lambda_{14k+13} = (k+1)\Lambda'$, where Λ' is the lattice generated by $(1, 2)$ and $(0, 5)$. It can be easily verified by hand that $d_7(\Lambda') = 14$. We conjecture that

$$d_7(\Lambda_{14k+i}) = 14k + i, \qquad \text{for } i = 1, 2, \ldots, 12 \tag{102}$$

for all $k = 1, 2, \ldots$, except for $d_7(\Lambda_{14k+7}) = 14k + 8$. This conjecture is confirmed by an exhaustive computer search up to $k = 50$. The computer search also shows that the lattices constructed in (100) are optimal for all $t \leqslant 700$, except for $t = 9, 12, 15, 24, 25, 32, 37, 48, 54, 65, 96$. The 11 lattices that improve upon (100) are listed in Table IV, and we conjecture that for all other values of t, the constructions of (100) are optimal. The ratio $t^2/V(\Lambda)$ appears to converge to $14^2/5 = 39.2$.

Finally, we have some partial results for seven repetitions. In particular, we introduce the family of lattices Λ_{33k} generated by

$$\boldsymbol{G}_{33k} = \begin{bmatrix} k & 8k \\ 0 & 21k \end{bmatrix}.$$

It is not difficult to prove that $d_8(\Lambda_{33}) = 33$, and therefore $d_8(\Lambda_{33k}) = 33k$ for all k by Theorem 2.9. Thus, we have a family of lattice interleavers $A(33k, 7)$ with $\deg A(33k, 7) = V(\Lambda_{33k}) = 21k^2$. Computer search shows that these lattice interleavers are optimal up to $k = 10$, and we conjecture that this is true in general. We do not have analogous results for $r = 7$ when $t \not\equiv 0 \bmod 33$. However, we were able to find optimal lattice interleavers with seven repetitions up to $t = 300$ by exhaustive computer search. The first 150 of these lattices are compiled in Table V.

For a general (large) number of repetitions r, scaled versions of the trivial lattice $\mathbb{Z}$ produce reasonable lattice interleavers. Indeed, it is easy to see that $d_{r+1}(\Lambda) \geqslant r$ for any $\Lambda \subseteq \mathbb{Z}$, and the trivial lattice $\mathbb{Z}$ attains this bound with equality. Thus, if we let $\Lambda_k = k\mathbb{Z}$, then $d_{r+1}(\Lambda_k) = kr$ and $V(\Lambda_k) = k^2$, for all $r = 1, 2, \ldots$ and all $k \in \mathbb{Z}^+$, by Theorem 2.9. This produces an infinite family of lattice interleavers $A(rk, r)$ for all r, with $\deg A(rk, r) = k^2$. For this family, we have

$$\lim_{t\to\infty} \frac{t^2}{\deg A(t, r)} = r^2. \tag{103}$$

A slightly better result for general r can be obtained by the following construction. For $r = 2, 3, \ldots$, let Ω be the lattice generated by

$$\boldsymbol{G}_\Omega = \begin{bmatrix} 1 & r \\ r & 1 \end{bmatrix}.$$

It is easy to see that $V(\Omega) = r^2 - 1$, and it can be shown that $d_{r+1}(\Omega) = r^2$. We omit the tedious proof. Scaling Ω, we obtain a family of lattices $\Lambda_k = k\Omega$ with $d_{r+1}(\Lambda_k) = kr^2$ and $V(\Lambda_k) = k^2(r^2-1)$ for all $r = 2, 3, \ldots$ and all $k \in \mathbb{Z}^+$. This, in turn, produces a family of lattice interleavers $A(r^2k, r)$ with $\deg A(r^2k, r) = k^2(r^2-1)$. For this family, we have

$$\lim_{t\to\infty} \frac{t^2}{\deg A(t, r)} = \frac{r^4}{r^2-1} = r^2\left(1 + \frac{1}{r^2-1}\right). \tag{104}$$

It is easy to see by comparing (103) and (104) that the lattices $\Lambda_k = k\Omega$ are better than $k\mathbb{Z}$ for all r. Furthermore, for small values of r, these lattices are optimal. Indeed, for $r = 2$, we have $d_3(\Lambda_k) = 4k$ and $V(\Lambda_k) = 3k^2$ which was shown to be optimal in Section III. For three repetitions, we have $d_4(\Lambda_k) = 9k$ and $V(\Lambda_k) = 8k^2$ which is also optimal (cf. Section IV). However, the result of (104) is no longer optimal for $r = 4, 5, 6, 7$, as can be seen from Table VI, which compares the limit of the ratio $t^2/\deg A(t, r)$ for $r \leqslant 7$ for various lattices.

Remark: As the number of repetitions r increases, we quickly reach a point of diminishing returns for several reasons. First, it becomes more difficult to design optimal lattices interleavers for large r. Furthermore, as r increases, the relative difference between the interleaving degree of optimal lattices and that of simple general constructions, such as $k\mathbb{Z}$ and $k\Omega$, becomes less and less significant. For example, for $r = 2$, the interleaving degree of an optimal lattice interleaver constitutes 75% of the interleaving degree of an interleaver based on $k\mathbb{Z}$, for the same strength. In contrast, for $r = 6$ the best we can do with an optimal (conjecturably) lattice interleaver is 92% of the interleaving degree of $k\mathbb{Z}$ (cf. Table VI). Finally, even for optimal lattices, we reach a point of diminishing returns in the overall savings in redundancy due to error-correction coding (cf. Sections I and VIII).

VII. INTERLEAVERS FOR AN ALTERNATIVE CONNECTIVITY MODEL

All the results so far were developed for the horizontal/vertical connectivity structure, whereby a point $(x, y) \in \mathbb{Z}^2$ is connected to two horizontal neighbors $(x-1, y)$, $(x+1, y)$ and two vertical neighbors $(x, y-1)$, $(x, y+1)$. We refer to this connectivity structure as the *grid connectivity model*, or the $+$ model for short. In some applications, a different definition of connectivity may be more appropriate. Namely, we assume that each point $(x, y) \in \mathbb{Z}^2$ is connected to its diagonal neighbors $(x-1, y-1)$, $(x-1, y+1)$, $(x+1, y-1)$, $(x+1, y+1)$ in addition to its horizontal/vertical neighbors. We refer to this connectivity structure as the *star connectivity model*, or the $*$ model for short.

The notion of a *cluster* in the star connectivity model can be defined exactly as before (cf. Definition 2.1), with the understanding that a *path* is a sequence of elements connected horizontally, vertically, or diagonally. For example, the set

TABLE V
OPTIMAL LATTICE INTERLEAVERS WITH SEVEN REPETITIONS

t	$V(\Lambda)$	generators a	generators b	$\frac{t^2}{V(\Lambda)}$	t	$V(\Lambda)$	generators a	generators b	$\frac{t^2}{V(\Lambda)}$	t	$V(\Lambda)$	generators a	generators b	$\frac{t^2}{V(\Lambda)}$
7	1	1	0	49.0000	55	60	1	13	50.4167	103	211	1	28	50.2796
8	2	1	1	32.0000	56	63	1	8	49.7778	104	212	1	38	51.0189
9	2	1	1	40.5000	57	63	3	6	51.5714	105	216	3	15	51.0417
10	2	1	1	50.0000	58	67	1	8	50.2090	106	220	2	42	51.0727
11	3	1	1	40.3333	59	69	1	15	50.4493	107	223	1	39	51.3408
12	3	1	1	48.0000	60	72	1	11	50.0000	108	228	2	42	51.1579
13	4	1	1	42.2500	61	74	1	16	50.2838	109	234	1	42	50.7735
14	4	1	1	49.0000	62	76	2	8	50.5789	110	239	1	66	50.6276
15	5	1	2	45.0000	63	79	1	9	50.2405	111	240	1	42	51.3375
16	5	1	2	51.2000	64	80	4	8	51.2000	112	245	7	14	51.2000
17	6	1	2	48.1667	65	83	1	18	50.9036	113	251	1	33	50.8725
18	7	1	2	46.2857	66	84	2	16	51.8571	114	252	6	12	51.5714
19	7	1	2	51.5714	67	88	1	20	51.0114	115	257	1	71	51.4591
20	8	1	3	50.0000	68	92	1	20	50.2609	116	264	1	42	50.9697
21	9	1	2	49.0000	69	93	1	25	51.1935	117	270	1	41	50.7000
22	10	1	3	48.4000	70	96	2	10	51.0417	118	273	1	119	51.0037
23	11	1	3	48.0909	71	100	1	18	50.4100	119	275	1	76	51.4945
24	12	1	3	48.0000	72	104	1	10	49.8462	120	283	1	43	50.8834
25	13	1	3	48.0769	73	104	1	28	51.2404	121	286	1	79	51.1923
26	14	1	3	48.2857	74	108	2	10	50.7037	122	293	1	81	50.7986
27	15	1	4	48.6000	75	111	1	20	50.6757	123	296	1	45	51.1115
28	16	1	3	49.0000	76	112	4	8	51.5714	124	301	1	84	51.0831
29	17	1	5	49.4706	77	117	1	53	50.6752	125	305	1	112	51.2295
30	18	1	4	50.0000	78	119	1	50	51.1261	126	309	1	46	51.3786
31	19	1	4	50.5789	79	124	1	23	50.3306	127	315	1	133	51.2032
32	20	2	4	51.2000	80	125	5	10	51.2000	128	320	8	16	51.2000
33	21	1	8	51.8571	81	129	1	23	50.8605	129	327	1	38	50.8899
34	23	1	5	50.2609	82	131	1	50	51.3282	130	329	1	49	51.3678
35	24	1	5	51.0417	83	135	1	25	51.0296	131	335	1	123	51.2269
36	26	1	5	49.8462	84	140	1	25	50.4000	132	336	4	32	51.8571
37	27	1	5	50.7037	85	142	1	31	50.8803	133	343	7	14	51.5714
38	28	2	4	51.5714	86	144	1	55	51.3611	134	352	2	40	51.0114
39	31	1	5	49.0645	87	150	1	35	50.4600	135	356	1	136	51.1938
40	32	1	5	50.0000	88	151	1	32	51.2848	136	364	1	48	50.8132
41	34	1	6	49.4412	89	155	1	60	51.1032	137	368	1	84	51.0027
42	35	1	6	50.4000	90	160	1	35	50.6250	138	372	2	50	51.1935
43	37	1	6	49.9730	91	165	1	25	50.1879	139	377	1	144	51.2493
44	39	1	6	49.6410	92	165	1	35	51.2970	140	384	4	20	51.0417
45	40	1	11	50.6250	93	171	1	37	50.5789	141	390	1	89	50.9769
46	43	1	6	49.2093	94	174	2	24	50.7816	142	394	1	52	51.1777
47	44	1	7	50.2045	95	175	5	10	51.5714	143	400	1	152	51.1225
48	45	3	6	51.2000	96	180	6	12	51.2000	144	403	1	92	51.4541
49	47	1	13	51.0851	97	185	1	28	50.8595	145	409	1	53	51.4059
50	49	1	18	51.0204	98	188	2	26	51.0851	146	416	2	56	51.2404
51	51	1	15	51.0000	99	189	3	24	51.8571	147	423	1	45	51.0851
52	54	1	7	50.0741	100	196	2	36	51.0204	148	425	1	92	51.5388
53	55	1	21	51.0727	101	201	1	36	50.7512	149	432	1	56	51.3912
54	57	1	21	51.1579	102	204	2	30	51.0000	150	441	1	198	51.0204

TABLE VI
LIMIT OF THE RATIO OF SQUARED STRENGTH TO INTERLEAVING DEGREE

# repetitions	1	2	3	4	5	6	7
$k\mathbb{Z}$	1.000	4.000	9.000	16.000	25.000	36.000	49.000
$k\Omega$	—	5.333	10.125	17.067	26.042	37.028	50.021
optimal	2.000	5.333	10.125	18.000	27.000	39.200	51.857

$\{(0,0),\,(1,1),\,(1,2),\,(2,3)\}$ is a cluster in the $*$ model, but not in the $+$ model. With this notion of a cluster, Definition 2.2 holds without change. Thus, the concept of an interleaving scheme $A(t,\,r)$ for the star connectivity model is well-defined.

It will be useful to introduce the notion of r-disjunction, which is the counterpart of r-dispersion for the $*$ model. Given points $z_1,\,z_2,\,\ldots,\,z_r \in \mathbb{Z}^2$, we define the *$r$-disjunction* $d_r^*(z_1,\,z_2,\,\ldots,\,z_r)$ as one less than the size of the smallest cluster in the $*$ model that contains all of these points (cf. Definition 2.3). It is easy to see that for any two distinct points $z_1 = (x_1,\,y_1)$ and $z_2 = (x_2,\,y_2)$ in $\mathbb{Z}^2$, we have

$$d_2^*(z_1, z_2) = \max\{|x_1 - x_2|,\,|y_1 - y_2|\}. \tag{105}$$

Thus, the 2-disjunction $d_2^*(z_1,\,z_2)$ is the L_∞-distance between z_1 and z_2. Furthermore, the following theorem provides an expression for the 3-disjunction of three distinct points in $\mathbb{Z}^2$.

Theorem 7.1: Let $z_1 = (x_1, y_1)$, $z_2 = (x_2, y_2)$, and $z_3 = (x_3, y_3)$ be three distinct points in $\mathbb{Z}^2$. Then

$$d_3^*(z_1, z_2, z_3) = \left\lceil \frac{\max_{1\leqslant i\leqslant 3}\{x_i+y_i\} + \max_{1\leqslant i\leqslant 3}\{x_i-y_i\}}{2} - \frac{\min_{1\leqslant i\leqslant 3}\{x_i+y_i\} + \min_{1\leqslant i\leqslant 3}\{x_i-y_i\}}{2} \right\rceil$$

Theorem 7.1 follows as a special case from Theorem 7.6 proved later in this section. This theorem is the counterpart of Theorem 2.4 for the star connectivity model.

In general, we observe that Lemmas 2.1, 2.6, 2.10, 2.11, as well as Theorems 2.3 and 2.8 hold without change for both connectivity models, whereas Lemmas 2.2, 2.5, 2.7, as well as Theorems 2.4 and 2.9 do not.

Our goal in this section is to study the minimum possible interleaving degree $\deg A(t, r)$ of interleavers for the $*$ model. For $r = 1$, the problem is trivial. Let $A(t, 1)$ be an interleaving scheme of strength t with no repetitions for the $*$ model. Let $m = \deg A(t, 1)$, and let $A_i \subset \mathbb{Z}^2$ denote the set of points in $A(t, 1)$ that are labeled by the integer i, for $i = 1, 2, \ldots, m$. Further, let $\mathcal{S} \subset \mathbb{Z}^2$ be a $t \times t$ square, namely,

$$\mathcal{S} = \left\{(x, y) \in \mathbb{Z}^2 \colon 0 \leqslant x \leqslant t-1 \text{ and } 0 \leqslant y \leqslant t-1\right\}.$$

It is easy to see from (105) that $d_2^*(z_1, z_2) \leqslant t-1$ for any $z_1, z_2 \in \mathcal{S}$. It follows from Theorem 2.3 that no two points of $\mathcal{S}$ can belong to the same set A_i, and, therefore,

$$\deg A(t, 1) \geqslant |\mathcal{S}| = t^2.$$

On the other hand, observe that $d_2^*(t\mathbb{Z}^2) = t$. Therefore, by Theorem 2.3, taking $A_1, A_2, \ldots, A_{t^2}$ as the t^2 cosets of the lattice $t\mathbb{Z}^2$ in $\mathbb{Z}^2$ achieves the lower bound $\deg A(t, 1) \geqslant t^2$ with equality. Notice that the resulting lattice interleaver is optimal among both lattice and nonlattice interleavers.

The rest of this section is concerned with $r \geqslant 2$. In Section VII-A, we construct lattice interleavers with two repetitions for the $*$ model. In Section VII-B, we establish bounds on the r-disjunction of $z_1, z_2, \ldots, z_r \in \mathbb{Z}^2$ in terms of the r-dispersion of another set of points in $\mathbb{Z}^2$. In Section VII-C, we use these results to show that the $*$ model and the $+$ model are essentially equivalent for large t.

A. Lattice Interleavers With two Repetitions for the Star Connectivity Model

As in the case of the $+$ model, the strength of a lattice interleaver for the $*$ model is equal to the r-disjunction of the underlying lattice Λ, while its interleaving degree is equal to the volume of Λ (cf. Theorem 2.3 that holds for both connectivity models). Thus, our goal in this subsection is to construct, for each t, a lattice Λ such that $d_3^*(\Lambda) = t$ and the volume of Λ is as small as possible. As in Section III-A, we distinguish between four cases, depending on the value of t modulo 4, and define the lattices $\Lambda_{4k}^*, \Lambda_{4k+1}^*, \Lambda_{4k+2}^*, \Lambda_{4k+3}^*$ by means of the corresponding generator matrices

$$\boldsymbol{G}_{4k}^* = \begin{bmatrix} k & 3k \\ 0 & 6k-1 \end{bmatrix} \qquad \boldsymbol{G}_{4k+1}^* = \begin{bmatrix} k+1 & 3k+1 \\ 1 & 6k \end{bmatrix}$$

$$\boldsymbol{G}_{4k+2}^* = \begin{bmatrix} k+1 & 3k+1 \\ 1 & 6k+2 \end{bmatrix} \qquad \boldsymbol{G}_{4k+3}^* = \begin{bmatrix} k+2 & 3k+2 \\ 2 & 6k+3 \end{bmatrix}$$

for all $k = 1, 2, \ldots$. The following theorem establishes the 3-disjunction of $\Lambda_{4k}^*, \Lambda_{4k+1}^*, \Lambda_{4k+2}^*, \Lambda_{4k+3}^*$ for $k \geqslant 2$. This theorem is the counterpart of Theorem 3.1 for the $*$ model.

Theorem 7.2: For $k = 2, 3, \ldots$

$$d_3^*(\Lambda_{4k}^*) = 4k \qquad V(\Lambda_{4k}^*) = 6k^2 - k = \frac{3t^2-2t}{8} \tag{106}$$

$$d_3^*(\Lambda_{4k+1}^*) = 4k+1 \quad V(\Lambda_{4k+1}^*) = 6k^2+3k-1 = \frac{3t^2-11}{8} \tag{107}$$

$$d_3^*(\Lambda_{4k+2}^*) = 4k+2 \quad V(\Lambda_{4k+2}^*) = 6k^2+5k+1 = \frac{3t^2-2t}{8} \tag{108}$$

$$d_3^*(\Lambda_{4k+3}^*) = 4k+3 \quad V(\Lambda_{4k+3}^*) = 6k^2+9k+2 = \frac{3t^2-11}{8}. \tag{109}$$

Proof: Let z_0, z_1, z_2 be three minimal points of Λ_{4k}^*. By Lemma 2.11, we can assume that $z_0 = (0, 0)$, $\mathcal{X}(z_1) \geqslant 0$, and $\mathcal{X}(z_2) \geqslant 0$. We need to prove that $d_3^*(z_0, z_1, z_2) \geqslant 4k$. By Lemma 2.1, this holds trivially if either

$$d_2^*(z_0, z_1) = \max\{x_1, |y_1|\} \geqslant 4k$$

or

$$d_2^*(z_0, z_2) = \max\{x_2, |y_2|\} \geqslant 4k.$$

Thus, as potential candidates for z_1 and z_2, we need to examine only those points of Λ_{4k}^* that lie in the rectangle defined by $-4k+1 \leqslant x, y \leqslant 4k-1$ and $x \geqslant 0$. There are exactly five such points

$$(k, -3k+1),\ (k, 3k),\ (2k, 1),\ (3k, -3k+2),\ (3k, 3k+1). \tag{110}$$

Given the above, there are only $\binom{5}{2} = 10$ possible choices for z_1, z_2, and it can be verified using Theorem 7.1 that $d_3^*(z_0, z_1, z_2) \geqslant 4k$ in each of the 10 cases. Finally, to see that $d_3^*(\Lambda_{4k}^*) \leqslant 4k$, consider the points

$$(0, 0),\ (k, 3k),\ (2k, 1) \in \Lambda_{4k}^*.$$

The equalities (107)–(109) can be proved by a similar argument. In a manner analogous to (110), we find that there are exactly five relevant candidates for z_1, z_2 in each of $\Lambda_{4k+1}^*, \Lambda_{4k+2}^*, \Lambda_{4k+3}^*$ that are given, respectively, by

$$(k, -3k+1), (k+1, 3k+1), (2k+1, 2), (3k+1, -3k+3), (3k+2, 3k+3) \tag{111}$$

$$(k, -3k-1), (k+1, 3k+1), (2k+1, 0), (3k+1, -3k-1), (3k+2, 3k+1) \tag{112}$$

$$(k, -3k-1), (k+2, 3k+2), (2k+2, 1), (3k+2, -3k), (3k+4, 3k+3). \tag{113}$$

Using the expression for 3-disjunction in Theorem 7.1, it can be easily verified that $d_3^*(z_0, z_1, z_2)$ is at least $4k+1, 4k+2$, and $4k+3$ in (111), (112), and (113), respectively. □

We point out that Theorem 7.2 holds for $k = 1$ as well, with a single exception: $d_3^*(\Lambda_5^*) = 4$. To see that $d_3^*(\Lambda_5^*) \leqslant 4$, consider the points $(-1, 2)$, $(0, 0)$, $(1, -2) \in \Lambda_5^*$. We conjecture that, with the exception of Λ_5^*, the lattices $\Lambda_{4k}^*, \Lambda_{4k+1}^*, \Lambda_{4k+2}^*, \Lambda_{4k+3}^*$ are optimal for all $k = 1, 2, \ldots$. As in the case of the $+$ model; this conjecture has been verified up to large values of t by an exhaustive computer search. We have applied the general search strategy developed in Section III-C, while using Theorem 7.1 to compute 3-disjunction. The search was conducted up to $t = 143$, and confirmed the optimality of $\Lambda_{4k}^*, \Lambda_{4k+1}^*, \Lambda_{4k+2}^*, \Lambda_{4k+3}^*$ except for $t = 2, 3, 5$. The lattice $\mathbb{Z}$ is trivially optimal for $t = 2$. For $t = 3$ and $t = 5$, optimal lattices are generated by $(1, 2)$, $(0, 4)$ and $(1, 3)$, $(0, 9)$, respectively.

B. A Relation Between r-Disjunction and r-Dispersion

We observe that the expression for the 3-disjunction of $z_1, z_2, z_3 \in \mathbb{Z}^2$ in Theorem 7.1 can be interpreted as follows. Consider the mapping φ: $\mathbb{R}^2 \mapsto \mathbb{R}^2$ defined by

$$\varphi(x, y) = (x + y, x - y). \tag{114}$$

Geometrically, the mapping $\varphi(\cdot)$ is tantamount to rotation by an angle of $\pi/4$ followed by scaling by a factor of $\sqrt{2}$. Let $z_1' = \varphi(z_1)$, $z_2' = \varphi(z_2)$, and $z_3' = \varphi(z_3)$. Comparing Theorem 7.1 to Theorem 2.4, it is now easy to see that

$$d_3^*(z_1, z_2, z_3) = \left\lceil \frac{d_3(z_1', z_2', z_3')}{2} \right\rceil. \tag{115}$$

In this subsection, we generalize this result for an arbitrary number of points r. For $r \geqslant 4$, the equality in (115) becomes a lower bound (cf. Proposition 7.3). However, we also prove an upper bound on $d_r^*(z_1, z_2, \ldots, z_r)$ in terms of $d_r(z_1', z_2', \ldots, z_r')$ that differs from (115) by at most $(r-2)/2$.

It will be most convenient to establish these results in a graph-theoretic context. Given a finite graph $G = (V, E)$, we say that $|V|$ is the *order* of G and $|E|$ is the *size* of G. Recall that the grid connectivity structure is reflected by the grid graph $\mathcal{G} = (V_{\mathcal{G}}, E_{\mathcal{G}})$. The vertex set of $\mathcal{G}$ is $V_{\mathcal{G}} = \mathbb{Z}^2$ and there is an edge $\{z_1, z_2\} \in E_{\mathcal{G}}$ if and only if the points $z_1, z_2 \in \mathbb{Z}^2$ are connected in the $+$ model, which happens if and only if $d_2(z_1, z_2) = 1$. The r-dispersion $d_r(z_1, z_2, \ldots, z_r)$ is then the size of a minimum spanning tree $T \subset \mathcal{G}$ for $z_1, z_2, \ldots, z_r \in \mathbb{Z}^2$. We now introduce an analogous framework for the $*$ model. Consider the *star graph* $\mathcal{G}^* = (V_{\mathcal{G}^*}, E_{\mathcal{G}^*})$, whose vertex set is $V_{\mathcal{G}^*} = V_{\mathcal{G}} = \mathbb{Z}^2$ and whose edge set is defined as follows: there is an edge $\{z_1, z_2\} \in E_{\mathcal{G}^*}$ if and only if the points $z_1, z_2 \in \mathbb{Z}^2$ are connected in the $*$ model. It should be obvious that the r-disjunction $d_r^*(z_1, z_2, \ldots, z_r)$ is then the size of a minimum spanning tree for $z_1, z_2, \ldots, z_r$ in the star graph $\mathcal{G}^*$.

Now consider the effect of the mapping $\varphi(\cdot)$ defined in (114) on the star graph $\mathcal{G}^*$. It is easy to see that $\varphi(V_{\mathcal{G}^*}) = \varphi(\mathbb{Z}^2) = D_2$, where D_2 is the two-dimensional checkerboard lattice defined by

$$D_2 = \left\{(x, y) \in \mathbb{Z}^2 : x + y \equiv 0 \bmod 2\right\}.$$

Thus, we introduce the graph $\varphi(\mathcal{G}^*) = (V_{\varphi(\mathcal{G}^*)}, E_{\varphi(\mathcal{G}^*)})$, defined as follows. The vertex set of $\varphi(\mathcal{G}^*)$ is $V_{\varphi(\mathcal{G}^*)} = D_2$ and there is an edge $\{z_1', z_2'\} \in E_{\varphi(\mathcal{G}^*)}$ if and only if $\{\varphi^{-1}(z_1'), \varphi^{-1}(z_2')\}$ is an edge in the star graph $\mathcal{G}^*$, where $\varphi^{-1}: \mathbb{R}^2 \mapsto \mathbb{R}^2$ is the inverse of $\varphi(\cdot)$, given by

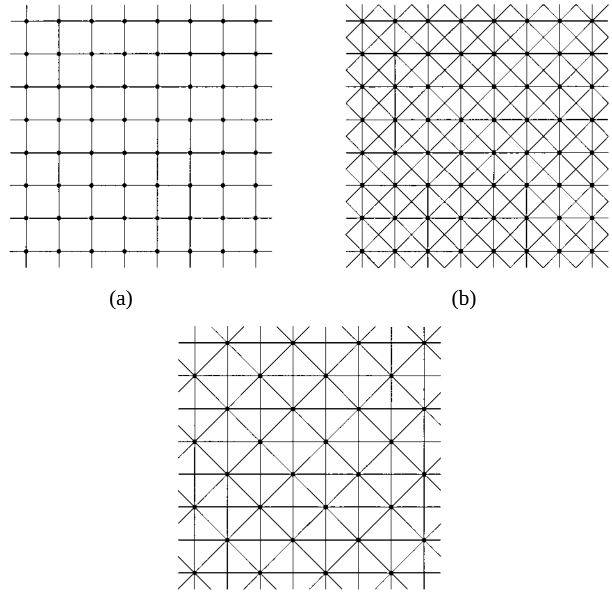

Fig. 15. (a) The grid graph $\mathcal{G}$. (b) The star graph $\mathcal{G}^*$. (c) The graph $\varphi(\mathcal{G}^*)$.

$$\varphi^{-1}(x, y) = \left(\frac{x+y}{2}, \frac{x-y}{2}\right). \tag{116}$$

Observe that, by construction, the graphs $\mathcal{G}^*$ and $\varphi(\mathcal{G}^*)$ are isomorphic. The grid graph $\mathcal{G}$, the star graph $\mathcal{G}^*$, and the graph $\varphi(\mathcal{G}^*)$ are depicted in Fig. 15(a), (b), and (c), respectively.

We next derive a different, much simpler, description for the edge set of $\varphi(\mathcal{G}^*)$. Let $z_1' = (x_1', y_1')$ and $z_2' = (x_2', y_2')$ be two distinct points of D_2. Let $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$ be the pre-images of z_1' and z_2' under $\varphi(\cdot)$, that is, $z_1 = \varphi^{-1}(z_1')$ and $z_2 = \varphi^{-1}(z_2')$. The points $z_1, z_2 \in \mathbb{Z}^2$ are connected by an edge in $\mathcal{G}^*$ if and only if $|x_1 - x_2| \leqslant 1$ and $|y_1 - y_2| \leqslant 1$. Combining this with the expression for $\varphi^{-1}(\cdot)$ in (116), we see that $z_1', z_2' \in D_2$ are connected by an edge in $\varphi(\mathcal{G}^*)$ if and only if

$$\begin{aligned} |(x_1' + y_1') - (x_2' + y_2')| &\leqslant 2 \\ |(x_1' - y_1') - (x_2' - y_2')| &\leqslant 2. \end{aligned} \tag{117}$$

After a straightforward manipulation, the condition in (117) simplifies to $|x_1' - x_2'| + |y_1' - y_2'| \leqslant 2$, or equivalently $d_2(z_1', z_2') \leqslant 2$. But it is easy to see that the L_1-distance between any two distinct points in D_2 is at least 2. Thus, there is an edge $\{z_1', z_2'\}$ in $\varphi(\mathcal{G}^*)$ if and only if $d_2(z_1', z_2') = 2$.

The key conclusion from the foregoing characterization of $E_{\varphi(\mathcal{G}^*)}$ is that $\varphi(\mathcal{G}^*)$ is precisely the *power graph* of the grid graph $\mathcal{G}$. That is, each edge in $\varphi(\mathcal{G}^*)$ corresponds to a path in $\mathcal{G}$ consisting of exactly two distinct edges. This immediately leads to the following result.

Proposition 7.3: Given any r points $z_1, z_2, \ldots, z_r \in \mathbb{Z}^2$, let $z_1', z_2', \ldots, z_r' \in D_2$ denote their images under $\varphi(\cdot)$. Then

$$d_r(z_1', z_2', \ldots, z_r') \leqslant 2 d_r^*(z_1, z_2, \ldots, z_r).$$

Proof: Suppose that T^* is a minimum spanning tree for $z_1, z_2, \ldots, z_r$ in the star graph $\mathcal{G}^*$. By definition, the size of T^* is $d_r^*(z_1, z_2, \ldots, z_r)$. Since the graphs $\mathcal{G}^*$ and $\varphi(\mathcal{G}^*)$ are isomorphic, the image of T^* under $\varphi(\cdot)$ is a minimum spanning tree for $z'_1, z'_2, \ldots, z'_r$ in $\varphi(\mathcal{G}^*)$. Let us denote this spanning tree by $\varphi(T^*)$. Since $\varphi(\mathcal{G}^*)$ is a power graph of $\mathcal{G}$, we can replace each edge in $\varphi(T^*)$ by a corresponding pair of edges in $\mathcal{G}$ to produce a spanning tree for $z'_1, z'_2, \ldots, z'_r$ in the grid graph $\mathcal{G}$. The size of this spanning tree is exactly twice the size of T^*. Hence,

$$d_r(z'_1, z'_2, \ldots, z'_r) \leqslant 2d^*_r(z_1, z_2, \ldots, z_r). \qquad \square$$

To prove an upper bound on $d^*_r(z_1, z_2, \ldots, z_r)$ in terms of $d_r(z'_1, z'_2, \ldots, z'_r)$, we first need to establish a certain property of minimal spanning trees in the grid graph $\mathcal{G}$. Note that the degree of every vertex in $\mathcal{G}$ is 4, and, therefore, the degree of any vertex in any subgraph of $\mathcal{G}$ is at most 4.

Lemma 7.4: Suppose that T is a minimal spanning tree for $z_1, z_2, \ldots, z_r \in \mathbb{Z}^2$ in the grid graph $\mathcal{G}$. Let $\eta_1, \eta_2, \eta_3, \eta_4$ denote the number of vertices in T of degree $1, 2, 3, 4$, respectively. Then $\eta_3 + 2\eta_4 \leqslant r - 2$.

Proof: First observe that T has at most r leaves. Indeed, if there is a leaf in T that is not one of the r points $z_1, z_2, \ldots, z_r$, then we could remove this leaf along with the single edge incident upon it, to obtain a smaller spanning tree for $z_1, z_2, \ldots, z_r$. Thus, $\eta_1 \leqslant r$. Now, the order of T is

$$|V| = \eta_1 + \eta_2 + \eta_3 + \eta_4$$

while its size is

$$|E| = \frac{\eta_1 + 2\eta_2 + 3\eta_3 + 4\eta_4}{2}.$$

It is well known that T is a tree if and only if $|V| - |E| = 1$. With $|V|$ and $|E|$ expressed in terms of $\eta_1, \eta_2, \eta_3, \eta_4$ as above, this condition is equivalent to $\eta_1 - \eta_3 - 2\eta_4 = 2$. Thus, we have $\eta_3 + 2\eta_4 = \eta_1 - 2 \leqslant r - 2$. $\square$

Proposition 7.5: Given any r points $z_1, z_2, \ldots, z_r \in \mathbb{Z}^2$, let $z'_1, z'_2, \ldots, z'_r \in D_2$ denote their images under $\varphi(\cdot)$. Then

$$2d^*_r(z_1, z_2, \ldots, z_r) \leqslant d_r(z'_1, z'_2, \ldots, z'_r) + r - 2.$$

Proof: Suppose that T is a minimal spanning tree for $z'_1, z'_2, \ldots, z'_r$ in the grid graph $\mathcal{G}$. Given T, we will construct a spanning tree for $z'_1, z'_2, \ldots, z'_r$ in the graph $\varphi(\mathcal{G}^*)$. To do so, observe that each edge in $\mathcal{G}$, and hence also in T, connects a point of D_2 with a point of D^*_2, where $D^*_2 = (1, 0) + D_2$ is the coset of D_2 in $\mathbb{Z}^2$. We now examine each vertex v in $V_T \cap D^*_2$, where V_T is the vertex set of T. Observe that the degree of v in T is either 2, 3, or 4, and all the neighbors of v are points of D_2 located at L_1-distance 2 from each other. Thus, if v has two neighbors $v_1, v_2 \in D_2$, we can replace the two edges incident upon v by a single edge $\{v_1, v_2\} \in E_{\varphi(\mathcal{G}^*)}$. If v has three neighbors $v_1, v_2, v_3 \in D_2$, we replace the three edges incident upon v by two edges $\{v_1, v_2\}, \{v_2, v_3\} \in E_{\varphi(\mathcal{G}^*)}$. Finally, if v has four neighbors v_1, v_2, v_3, v_4, we replace the corresponding four edges by the three edges $\{v_1, v_2\}$, $\{v_2, v_3\}$, and $\{v_3, v_4\}$ in $E_{\varphi(\mathcal{G}^*)}$. Let $T^* \subset \varphi(\mathcal{G}^*)$ denote the graph that results by applying the foregoing procedure to every vertex of $V_T \cap D^*_2$. It is easy to see that each of the three types of operations described above preserves connectivity and does not introduce cycles. Thus, since T is a tree, so is T^*. The vertex set of T^* is precisely $V_T \cap D_2$. Since $z'_1, z'_2, \ldots, z'_r \in V_T \cap D_2$ by definition, these r points are among the vertices of T^*. Hence, T^* is a spanning tree for $z'_1, z'_2, \ldots, z'_r$ in $\varphi(\mathcal{G}^*)$.

We next count the number of edges in $T = (V_T, E_T)$ and $T^* = (V_{T^*}, E_{T^*})$. As in Lemma 7.4, let η_i denote the number of vertices of degree i in V_T, for $i = 1, 2, 3, 4$. Similarly, let $\eta'_2, \eta'_3, \eta'_4$ denote the number of vertices of degree 2, 3, 4, respectively, in $V_T \cap D^*_2$. Then

$$|E_T| = \frac{1}{2}\sum_{v \in V_T} \deg(v) = \sum_{v \in V_T \cap D^*_2} \deg(v) = 2\eta'_2 + 3\eta'_3 + 4\eta'_4 \tag{118}$$

where $\deg(v)$ stands for the degree of a vertex $v \in V_T$ in T. Notice that $\eta'_3 + 2\eta'_4 \leqslant \eta_3 + 2\eta_4 \leqslant r - 2$ by Lemma 7.4. Referring to the construction process of T^*, we have

$$|E_{T^*}| = \eta'_2 + 2\eta'_3 + 3\eta'_4 = \frac{|E_T|}{2} + \frac{\eta'_3 + 2\eta'_4}{2} \leqslant \frac{|E_T| + r}{2} - 1 \tag{119}$$

where the second equality follows from (118). As the graph $\varphi(\mathcal{G}^*)$ is isomorphic to the star graph $\mathcal{G}^*$, the size of T^* is an upper bound on the size $d^*_r(z_1, z_2, \ldots, z_r)$ of a minimum spanning tree for $z_1, z_2, \ldots, z_r$ in $\mathcal{G}^*$. In conjunction with (119), this completes the proof of the proposition. $\square$

Theorem 7.6: Given any r points $z_1, z_2, \ldots, z_r \in \mathbb{Z}^2$, let $z'_1, z'_2, \ldots, z'_r \in D_2$ denote their images under $\varphi(\cdot)$. Then

$$\left\lceil \frac{d_r(z'_1, z'_2, \ldots, z'_r)}{2} \right\rceil \leqslant d^*_r(z_1, z_2, \ldots, z_r) \leqslant \left\lfloor \frac{d_r(z'_1, z'_2, \ldots, z'_r) + r - 2}{2} \right\rfloor.$$

Proof: The theorem follows immediately from Propositions 7.3 and 7.5. $\square$

Theorem 7.6 is the main result of this subsection. Note that for $r \leqslant 3$, the upper and lower bounds on $d^*_r(z_1, z_2, \ldots, z_r)$ in Theorem 7.6 coincide. This proves (115) and, thereby, also Theorem 7.1.

C. Asymptotic Equivalence of the Star and the Grid Models

Let $\Delta_r(t)$ denote the least possible interleaving degree of a lattice interleaver for the $+$ model with $r - 1$ repetitions and strength at least t. Let $\Delta^*_r(t)$ denote the analogous quantity for the $*$ model. Notice that, by definition, $\Delta_r(t)$ and $\Delta^*_r(t)$ are nondecreasing functions of t. If Λ is a sublattice of $\mathbb{Z}^2$ such that $d_r(\Lambda) \geqslant t$ and $V(\Lambda) = \Delta_r(t)$, we say that Λ is *optimal for the* $+$ *model*. Similarly, if $\Lambda \subseteq \mathbb{Z}^2$ is such that $d^*_r(\Lambda) \geqslant t$ and $V(\Lambda) = \Delta^*_r(t)$, we say that Λ is *optimal for the* $*$ *model*.

The results of the previous subsection make it possible to establish upper and lower bounds on $\Delta^*_r(t)$ in terms of $\Delta_r(t)$. Under a certain assumption, these bounds coincide as $t \to \infty$. Furthermore, our proofs of these bounds are constructive, so that, for large strengths t, one can obtain an asymptotically optimal lattice for the $*$ model from an optimal lattice for the $+$ model, and vice versa.

Theorem 7.7: Let t and r be positive integers, with $t \geqslant r-1$. Then $\Delta^*_r(t) \leqslant 2\Delta_r(t)$.

Proof: Let Λ be an optimal lattice for the $+$ model, so that $d_r(\Lambda) \geqslant t$ and $V(\Lambda) = \Delta_r(t)$. Define $\Lambda^* = \varphi^{-1}(2\Lambda)$. It should be obvious that Λ^* is a sublattice of $\mathbb{Z}^2$ and

TABLE VII
OVERALL CODING REDUNDANCY AND RATE, USING LATTICE INTERLEAVERS WITH MDS CODES

# repetitions	1	2	3	4	5	6	7	$\geqslant 8$
$\Delta_r(t)$	$\frac{1}{2}t^2$	$\frac{3}{16}t^2$	$\frac{8}{81}t^2$	$\frac{1}{18}t^2$	$\frac{1}{27}t^2$	$\frac{5}{196}t^2$	$\frac{7}{363}t^2$	$\frac{r^2-1}{r^4}t^2$
redundancy	τt^2	$0.75\,\tau t^2$	$0.59\,\tau t^2$	$0.44\,\tau t^2$	$0.37\,\tau t^2$	$0.31\,\tau t^2$	$0.27\,\tau t^2$	$\frac{2(r^2-1)}{r^3}\tau t^2$
rate	0.25	0.44	0.56	0.67	0.72	0.77	0.80	$1-\frac{3(r^2-1)}{2r^3}$

$V(\Lambda^*) = 2V(\Lambda)$. We next show that $d_r^*(\Lambda^*) \geqslant d_r(\Lambda)$. Indeed, let $z_1, z_2, \ldots, z_r$ be any r points in Λ^*, and let $z_1', z_2', \ldots, z_r'$ be their images under $\varphi(\cdot)$. By construction, we have $z_1', z_2', \ldots, z_r' \in 2\Lambda$, and therefore,

$$d_r^*(z_1, z_2, \ldots, z_r) \geqslant \frac{d_r(z_1', z_2', \ldots, z_r')}{2} \geqslant t$$

where the first inequality follows from Proposition 7.3 while the second inequality follows from the fact that $d_r(2\Lambda) = 2d_r(\Lambda)$ by Theorem 2.9. Hence, $d_r^*(\Lambda^*) \geqslant t$ and $\Delta_r^*(t) \leqslant V(\Lambda^*)$. It follows that $\Delta_r^*(t) \leqslant 2\Delta_r(t)$, and the lattice $\Lambda^* = \varphi^{-1}(2\Lambda)$ achieves this bound with equality. □

Theorem 7.8: Let t and r be positive integers, with $t \geqslant r-1$. Then $\Delta_r(2t - r + 2) \leqslant 2\Delta_r^*(t)$.

Proof: Let Λ be an optimal lattice for the $*$ model, so that $d_r^*(\Lambda) \geqslant t$ and $V(\Lambda) = \Delta_r^*(t)$. Define $\Lambda' = \varphi(\Lambda)$. Then $V(\Lambda') = 2V(\Lambda)$ and $d_r(\Lambda') \geqslant 2d_r^*(\Lambda) - r + 2$. Indeed, let $z_1', z_2', \ldots, z_r'$ be any r points in Λ', and let $z_1, z_2, \ldots, z_r$ be their pre-images under $\varphi(\cdot)$. By construction $z_1, z_2, \ldots, z_r \in \Lambda$, and, therefore,

$$d_r(z_1', z_2', \ldots, z_r') \geqslant 2d_r^*(z_1, z_2, \ldots, z_r) - r + 2$$

in view Proposition 7.5. It follows that $\Delta_r(2t-r+2) \leqslant 2\Delta_r^*(t)$, and the lattice $\Lambda' = \varphi(\Lambda)$ attains this bound with equality. □

Theorems 7.7 and 7.8 can be used to convert an asymptotically optimal lattice for one model into an asymptotically optimal lattice for the other model. Combining the two theorems, we have

$$\frac{\Delta_r(2t-r+2)}{2} \leqslant \Delta_r^*(t) \leqslant 2\Delta_r(t). \tag{120}$$

Now suppose that $\lim_{t\to\infty} \Delta_r(t)/t^2 = \gamma_r$, where γ_r is a constant. In other words, we assume that as t increases, the ratio $\Delta_r(t)/t^2$ converges to a limit. It was shown by Blaum, Bruck, and Vardy [3] that this is true for $r = 2$, and we have proved that this is also true for $r = 3, 4$ in Sections III and IV, respectively. Empirical evidence for $r = 5, 6, 7, 8$ is presented in Section VI (cf. Table VI). With this assumption, the bounds in (120) take the following asymptotic form:

$$2\gamma_r t^2 - 2\gamma_r (r-2)t + \frac{\gamma_r (r-2)^2}{2} + o(t^2) \leqslant \Delta_r^*(t) \leqslant 2\gamma_r t^2 + o(t^2) \tag{121}$$

where $o(t^2)$ is a function of t such that $\lim_{t\to\infty} o(t^2)/t^2 = 0$. It is evident from (121) that the two bounds on $\Delta_r^*(t)$ coincide asymptotically, and the ratio $\Delta_r^*(t)/t^2$ converges to $2\gamma_r$ for $t \to \infty$. Thus, $\lim_{t\to\infty} \Delta_r^*(t)/\Delta_r(t) = 2$ for all r, and the two connectivity models are asymptotically equivalent.

Example: Let us consider the case $r = 3$. The lattice Λ_{4k} generated by (k, k) and $(0, 3k)$ is optimal for the $+$ model, as we have seen in Section III. Applying the transformation of Theorem 7.7, we obtain the lattice $\Lambda^* = \varphi^{-1}(2\Lambda_{4k})$ generated by $(k, 3k)$ and $(0, 6k)$. According to the results of this subsection, the lattice Λ^* is asymptotically optimal for the $*$ model. Indeed, the four lattices $\Lambda_{4k}^*, \Lambda_{4k+1}^*, \Lambda_{4k+2}^*, \Lambda_{4k+3}^*$ constructed in Section VII-A all converge to Λ^* as $k \to \infty$. Conversely, starting with Λ_{4k}^* and applying the transformation of Theorem 7.8, we obtain the lattice $\Lambda' = \varphi(\Lambda_{4k}^*)$ generated by $(2k+1, 2k-1)$ and $(2, 6k-2)$. The volume of Λ' is $12k^2 - 2k$, and its tristance is at least $t = 8k - 1$ by Theorems 7.2 and 7.8. Thus, $V(\Lambda') = (3t-1)(t+1)/16$. Referring to the results of Section III, it is easy to see that the lattice Λ' becomes optimal for the $+$ model as $k, t \to \infty$.

VIII. DISCUSSION AND OPEN PROBLEMS

With the results of Sections III–VI at hand, it is now possible to give a general overview of the savings in redundancy that can be obtained using (lattice) interleaving schemes with repetitions. This overview, presented in what follows, continues the discussion started with a specific example in Section I.

Consider the following situation. Suppose we have a two-dimensional array of area $\mathcal{S}$ and would like to correct up to τ error clusters of size up to t. For the sake of comparison, we will use MDS codes (say, shortened RS codes) along with optimal lattice interleavers of strength t with r repetitions. Thus, the interleaving degree is $\Delta_r(t)$, and we need $\Delta_r(t)$ codes that correct up to τr errors each. For MDS codes, the redundancy of each code is $2\tau r$, and hence the overall redundancy of the error-correction/interleaving scheme is $2\tau r\Delta_r(t)$. The overall information rate is given by $1 - 2\tau r\Delta_r(t)/\mathcal{S}$. Table VII summarizes these results for $r = 1, 2, \ldots, 7$ and $r \geqslant 8$. In each case, we assume that t is sufficiently large, so that an asymptotic form of $\Delta_r(t)$ can be used. The results reported for $r = 1, 2, 3$ are exact (cf. Sections III and IV), while the results for $r \geqslant 4$ represent the best available upper bounds (cf. Sections V and VI). In order to illustrate these results concretely, the overall information rate is given for a specific example: $\mathcal{S} = 2^{18}$, $t = 256$, and $\tau = 3$.

A well-known limitation of MDS codes is that their length is bounded by a function of the size of their alphabet. In particular, for RS codes of length n over GF(q), we have $n \leqslant q + 1$. In our context, this condition translates into $\mathcal{S} \leqslant q\Delta_r(t)$. Thus, the area of the array cannot be too large, unless the size of the

TABLE VIII
OVERALL CODING REDUNDANCY AND RATE, USING LATTICE INTERLEAVERS WITH BINARY CODES

# repetitions	1	2	3	4	5	6	7	$\geqslant 8$
$\Delta_r(t)$	$\frac{1}{2}t^2$	$\frac{3}{16}t^2$	$\frac{8}{81}t^2$	$\frac{1}{18}t^2$	$\frac{1}{27}t^2$	$\frac{5}{196}t^2$	$\frac{7}{363}t^2$	$\frac{r^2-1}{r^4}t^2$
redundancy	$3.5\tau t^2$	$3.16\tau t^2$	$2.77\tau t^2$	$2.26\tau t^2$	$1.99\tau t^2$	$1.73\tau t^2$	$1.58\tau t^2$	$\frac{\tau t^2(r^2-1)\log_2\frac{2^6r^4}{r^2-1}}{r^3}$
rate	0.56	0.61	0.65	0.72	0.75	0.78	0.80	$1-\frac{\tau(r^2-1)\log_2\frac{2^6r^4}{r^2-1}}{2^6r^3}$

clusters we need to correct is also large (since $\Delta_r(t) \sim t^2$). An alternative is to use binary error-correcting codes, which can be arbitrarily long. It is known [19] that for each positive integer e, there exist binary linear codes of length n, minimum distance $d \geqslant 2e+1$, and redundancy $e\log_2 n$. Using such binary codes in place of RS codes in the interleaving scheme discussed above, we find that the overall redundancy due to error-correction coding is

$$\tau r\Delta_r(t)\log_2 n = \tau r\Delta_r(t)\log_2 \frac{\mathcal{S}}{\Delta_r(t)}.$$

As before, we summarize the results for binary codes in Table VIII for $r = 1, 2, \ldots, 7$ and $r \geqslant 8$. For the sake of comparison, we assume that $\mathcal{S}/t^2 = 2^6$ in Table VIII. The overall information rate is given in Table VIII for the case where $\mathcal{S} = 2^{20}$, $t = 128$, and $\tau = 8$. Observe that it would not have been possible to code for these parameters (with two or more repetitions) using RS codes over GF(256).

In summary, this work presents solutions to a number of problems concerning two-dimensional lattice interleavers with repetitions. Nevertheless, it is fair to say that we know much less than we would like to, and many questions concerning multidimensional interleaving schemes remain open. We conclude the paper with a list of open problems and suggestions for future research.

Computation of Lattice r-Dispersion: As mentioned in Section II-A, it was proved in [10] that for large r, the problem of computing the r-dispersion of $z_1, z_2, \ldots, z_r \in \mathbb{Z}^2$ becomes NP-hard. Hence, it is obviously NP-hard to compute the minimum r-dispersion of an infinite subset $\mathcal{S} \subset \mathbb{Z}^2$. While this is certainly true if $\mathcal{S}$ is arbitrary, this need not be true if $\mathcal{S}$ is a lattice. Thus, we propose the following problem. Given a two-dimensional lattice $\Lambda \subset \mathbb{Z}^2$ and a positive integer r, devise an efficient algorithm to compute the minimum r-dispersion of Λ or prove that this task is NP-hard.

Lattice Interleavers With Two and Three Repetitions: In Sections III and IV of this paper, we made a number of conjectures that have been verified by exhaustive computer search up to large values of t. The proof of these conjectures is obviously open for future work. Thus, we suggest the following problems. Prove that the lattices Λ_{4k+1} and Λ_{4k+3} constructed in Section III-A are optimal for all k. Similarly, prove that the lattices $\Lambda_{9k+1}, \Lambda_{9k+2}, \ldots, \Lambda_{9k+8}$ constructed in Section IV-A are optimal for all k, except for the cases marked by an asterisk in Table I. The latter problem appears to be substantially more difficult, since one would have to account for the exceptions in any such proof.

Lower Bounds on the Degree of Lattice Interleavers: As mentioned in Section V, it would be extremely tedious to extend the proof technique of Propositions 4.2 and 4.7 to the case of four repetitions. Thus, we propose the following problem. Develop new methods for establishing lower bounds on the degree of lattice interleavers. In particular, use these methods to prove that if $d_5(\Lambda) = 6k$ for a lattice $\Lambda \subseteq \mathbb{Z}^2$, then $V(\Lambda) \geqslant 2k^2$ (cf. Theorem 5.1).

Constructions of Lattice Interleavers With Multiple Repetitions: In [3] and in Sections III–VI of this paper, we have constructed the lattices $\Lambda_{2k}, \Lambda_{4k}, \Lambda_{9k}, \Lambda_{12k}, \Lambda_{18k}, \Lambda_{14k}, \Lambda_{33k}$ generated by

$$\begin{bmatrix} k & k \\ 0 & 2k \end{bmatrix} \quad \begin{bmatrix} k & k \\ 0 & 3k \end{bmatrix} \quad \begin{bmatrix} k & 3k \\ 0 & 8k \end{bmatrix} \quad \begin{bmatrix} 2k & 2k \\ 0 & 4k \end{bmatrix}$$

$$\begin{bmatrix} 2k & 2k \\ 0 & 6k \end{bmatrix} \quad \begin{bmatrix} k & 2k \\ 0 & 5k \end{bmatrix} \quad \begin{bmatrix} k & 8k \\ 0 & 21k \end{bmatrix}$$

respectively, that are apparently optimal for $r = 1, 2, \ldots, 7$, respectively (as confirmed by analysis for $r = 1, 2, 3$, and by computer search for $r \geqslant 4$). Is there a general pattern here? How should this construction be continued for $r \geqslant 8$? We pose these questions as open problems for future research.

Asymptotics of Interleaving Degree as a Function of Strength: We conjecture that for all $r = 1, 2, \ldots$, the least possible interleaving degree of a lattice interleaver with r repetitions and strength t becomes proportional to t^2 for $t \to \infty$. In other words, using the notation of Section VII-C, we conjecture that the limit $\lim_{t\to\infty} t^2/\Delta_r(t)$ exists. We leave the proof of this conjecture as an open problem. If the limit does exist, it must be between r^2+1 and $2r^2$, where the lower bound follows from (104) while the upper bound follows from the results of [3] and Theorem 2.10.

Lower Bounds on the Degree of General (Nonlattice) Interleavers: All the best known interleaving schemes constructed so far belong to the class of lattice interleavers. It is known [3] that better interleaving schemes without repetitions do not exist. On the other hand, much less is known for $r \geqslant 2$. Certain bounds for $r = 2$, based on considering the maximum tristance within a rectangular shape, were derived in [2]. These bounds appear to be weak, however, and we suggest the following as an open problem. Improve upon the lower bounds of [2] on the interleaving degree of general (nonlattice) interleaving schemes with two repetitions. A closely related problem is this: What is the largest set $\mathcal{S}_t \subset \mathbb{Z}^2$ such that the tristance between any three points of $\mathcal{S}_t$ is at most t? Obviously, the same question can be asked for an arbitrary r-dispersion. Nontrivial answers would immediately lead to nontrivial lower bounds on the degree of general interleavers with r repetitions.

Optimality of Lattice Interleavers: In general, we conjecture that interleaving schemes based on lattices are always optimal, and propose the following as a research problem. Prove this conjecture, or construct, for any r and t, an interleaver of strength t with r repetitions whose interleaving degree is strictly less than the degree of the best possible lattice interleaver with the same parameters.

Multidimensional Interleavers With and Without Repetitions: As pointed out in [3], correction of three-dimensional error clusters has applications in holographic recording. Applications in four-dimensional optical storage devices (where the fourth dimension is the wavelength) are conceivable in the future. Thus, it would be interesting to extend the results obtained in this paper to three and higher dimensions, and develop a general framework for the study of multidimensional interleaving schemes with and without repetitions. Notwithstanding potential applications to holographic storage, this problem appears to have some inherent intellectual interest.

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