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Codes and Anticodes in the Grassman Graph

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Perfect codes and optimal anticodes in the Grassman graph $G_q(n, k)$ are examined. It is shown that the vertices of the Grassman graph cannot be partitioned into optimal anticodes, with a possible exception when $n = 2k$. We further examine properties of diameter perfect codes in the graph. These codes are known to be similar to Steiner systems. We discuss the connection between these systems and "real" Steiner systems.

Key Words: anticodes; codes; Steiner systems; tiling.

1. INTRODUCTION

In his pioneer work on association schemes Delsarte [2] proved the following result.

THEOREM 1. *Let $\mathscr{X}$ and $\mathscr{Y}$ be subsets of the vertex set $\mathscr{V}$ of a distance regular graph Γ, such that the nonzero distances occurring between vertices in $\mathscr{X}$ do not occur between vertices of $\mathscr{Y}$. Then*

$$|\mathscr{X}|\,|\mathscr{Y}| \leqslant |\mathscr{V}|. \tag{1}$$

In particular, Theorem 1 holds when $\mathscr{X}$ is a code $\mathscr{C}$ with minimum distance $D+1$, and $\mathscr{Y}$ is an anticode $\mathscr{A}$ with maximum distance D. An *anticode* with diameter D is a set of codewords such that the distance between any two codewords of the anticode is at most D. An anticode $\mathscr{A}$ is called *optimal* if it is the largest anticode among all the anticodes with the same parameters (length and maximal distance) as $\mathscr{A}$. If $\mathscr{A}$ is a sphere, where $D = 2e$, then (1) becomes the well known sphere packing bound, and the code $\mathscr{C}$ is called an e-perfect code. Recently, Ahlswede *et al.* [1] have

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generalized the notion of perfect codes, and called any code $\mathscr{C}$, which meets the bound (1), a D-diameter perfect code. This definition is a generalization of the e-perfect code notion since any e-perfect code is a $2e$-diameter perfect code.

Ahlswede *et al.* examined three distance regular graphs, which they considered to be the most interesting graphs in this discussion. These graphs are the Hamming graph, the Johnson graph, and the Grassman graph. Nontrivial diameter perfect codes are known in all these graphs. In the Hamming graph, in addition to the Hamming and Golay codes, the extended Hamming and extended Golay codes are diameter perfect, as well as all MDS codes. In the Johnson graph no nontrivial e-perfect codes are known [3], but all Steiner systems are diameter perfect codes.

Let $\left[{V \atop k} \right]$ denote the set of all k-dimensional subspaces of a vector space V over $GF(q)$. The vertex set of the Grassman graph $G_q(n, k)$ consists of all k-dimensional subspaces of $GF(q)^n$, i.e., $\left[{V \atop k} \right]$, where $V = GF(q)^n$. Two such subspaces are adjacent, i.e., connected by an undirected edge, if and only if they intersect in a $(k-1)$-dimensional subspace. Martin and Zhu [6] proved that there are no nontrivial e-perfect codes in the Grassman graph. The size of optimal anticodes in the Grassman graph was determined by Frankl and Wilson [5] in their work on t-intersecting families. Ahlswede *et al.* [1] have observed that the results of Frankl and Wilson together with Theorem 1 imply that in $G_q(n, k)$ only "Steiner system type" diameter perfect codes can exist. Here $\mathscr{F} \subseteq \left[{V \atop k} \right]$ is called a *Steiner structure* $S[t, k, n]_q$ if the elements of $\mathscr{F}$ are k-dimensional subspaces (called *blocks*), and each t-dimensional subspace of V is contained in exactly one block from $\mathscr{F}$. Trivial Steiner structures in $G_q(n, k)$ are $S[t, t, n]_q$ and $S[t, n, n]_q$. Ahlswede *et al.* [1] mentioned that $S[1, k, n]_q$ exists when k divides n. Other Steiner structures are not known.

If an e-perfect code exists then we can partition the corresponding graph into optimal anticodes with diameter $2e$ (spheres with radius e centered around the codewords). Ahlswede *et al.* [1] asked the following natural question: Does the existence of a D-diameter perfect code in all cases implies a partition of the graph by optimal anticodes as for e-perfect codes? In the Hamming graph this is certainly true as implied by [1]. Ahlswede *et al.* [1] proved that the Johnson graph cannot be partitioned into optimal anticodes of diameter D, if the optimal anticode is not a sphere. All Steiner systems are diameter perfect codes in the Johnson graph [1] and only trivial e-perfect codes are Steiner systems [3]. Hence, for all known diameter perfect codes which are not e-perfect codes, in the Johnson graph, there is no partition of the graph into the corresponding optimal anticodes.

The remainder of this paper is organized as follows. In Section 2 we discuss Steiner structures in the Grassman graph. Necessary conditions for the existence of such structures are derived, some constructions of systems from known systems are given, and interesting connections with the well known Steiner systems are presented. In Section 3 we examine partitions of the Grassman graph into optimal anticodes. We first summarize the results of Frankl and Wilson [5] concerning optimal anticodes. Later we show the main result. There is no partition of the Grassman graph $G_q(n, k)$ into optimal anticodes, unless $n=2k$. When $n=2k$, Frankl and Wilson conjectured that there can be only two different types of optimal anticodes. If their conjecture is correct then also in this case there is no partition of the Grassman graph $G_q(n, k)$ into optimal anticode. Finally, we show that if a D-diameter perfect code exists in $G_q(n, k)$, where $k \geqslant 2D$ if $n \geqslant 2k$, then there exists a tiling with maximal anticodes of another Grassman graph.

2. STEINER STRUCTURES

For a real number $b \neq 1$, and all nonnegative integers k, the b-ary *Gaussian binomial coefficient* $\left[{x \atop k} \right]_b$ is defined by

$$\begin{bmatrix} x \\ 0 \end{bmatrix}_b = 1$$

$$\begin{bmatrix} x \\ k \end{bmatrix}_b = \frac{(b^x - 1)(b^{x-1} - 1) \cdots (b^{x-k+1} - 1)}{(b^k - 1)(b^{k-1} - 1) \cdots (b-1)}, \qquad k = 1, 2, \ldots,$$

where x is a real number. In our discussion, b is a power of a prime, and x is an integer, $x \geqslant k$.

Recall that $\mathscr{F} \subseteq \left[{V \atop k} \right]$ is called a *Steiner structure* $S[t, k, n]_q$ if the blocks of $\mathscr{F}$ are k-dimensional subspaces, and each t-dimensional subspace of V is contained in exactly one block from $\mathscr{F}$. The first three results generalize the known results on Steiner systems to Steiner structures.

LEMMA 1. *The total number of blocks in an* $S[t, k, n]_q$ *is*

$$\frac{\begin{bmatrix} n \\ t \end{bmatrix}_q}{\begin{bmatrix} k \\ t \end{bmatrix}_q}.$$

Proof. The total number of t-dimensional subspaces of an n-dimensional subspaces is $\left[\begin{smallmatrix} n \\ t \end{smallmatrix}\right]_q$. Each block of $S[t, k, n]_q$ contains $\left[\begin{smallmatrix} k \\ t \end{smallmatrix}\right]_q$ t-dimensional subspaces. Since each t-dimensional subspace is contained in exactly one block, it follows that the total number of blocks in $S[t, k, n]_q$ is

$$\frac{\begin{bmatrix} n \\ t \end{bmatrix}_q}{\begin{bmatrix} k \\ t \end{bmatrix}_q}. \quad \blacksquare$$

THEOREM 2. *If $S[t, k, n]_q$ exists, $t \geqslant 2$, then $S[t-1, k-1, n-1]_q$ exists.*

Proof. Let U_n be an n-dimensional vector space over $GF(q)$ and let $S \subseteq \left[\begin{smallmatrix} U_n \\ k \end{smallmatrix}\right]$ be an $S[t, k, n]_q$. U_n can be written as $U_n = U_{n-1} + U_1$, where $U_{n-1} \in \left[\begin{smallmatrix} U_n \\ n-1 \end{smallmatrix}\right]$ is an $(n-1)$-dimensional subspace and $U_1 \in \left[\begin{smallmatrix} U_n \\ 1 \end{smallmatrix}\right]$. We claim that the set S' defined below is an $S[t-1, k-1, n-1]_q$,

$$S' \triangleq \{ W \cap U_{n-1} \mid W \in S,\ U_1 \subseteq W \}.$$

Clearly, $S' \subseteq \left[\begin{smallmatrix} U_{n-1} \\ k-1 \end{smallmatrix}\right]$ and for $Y \in \left[\begin{smallmatrix} U_{n-1} \\ t-1 \end{smallmatrix}\right]$, $Y + U_1$ is a t-dimensional subspace. Hence, $Y + U_1$ is contained in exactly one block $W \in S$. Therefore, Y is contained in exactly one element of S', the subspace $W \cap U_{n-1}$. Thus S' is an $S[t-1, k-1, n-1]_q$. $\blacksquare$

COROLLARY 1. *If $S[t, k, n]_q$ exists, then for all $0 \leqslant i \leqslant t-1$,*

$$\frac{\begin{bmatrix} n-i \\ t-1 \end{bmatrix}_q}{\begin{bmatrix} k-i \\ t-1 \end{bmatrix}_q},$$

must be integers.

Trivial Steiner structures $S[t, n, n]_q$ and $S[t, t, n]_q$ exist for all $t \leqslant n$. The only known nontrivial structures are $S[1, k, n]_q$, where k divides n. These structures are partitions of the n-dimensional space V (excluding the allzero vector) into k-dimensional subspaces (excluding the allzero vectors). Such partitions are obtained from any q, k, and n, such that k divides n. One method to construct such partitions may be deduced from perfect byte correcting codes [4]. Let $n = sk$ and let $\xi \in GF(q^n)$ be a root of a primitive polynomial of degree s over $GF(q^k)$. Denote

$$r = \frac{q^n - 1}{q^k - 1} = 1 + q^k + q^{2k} + \cdots + q^{(s-1)k},$$

and for each i, $0 \leqslant i \leqslant r-1$, we define

$$H_i = \{\xi^i, \xi^{r+i}, \xi^{2r+i}, ..., \xi^{(q^k-2)r+i}\}.$$

One can easily verify that $H_0 \cup \{0\} = GF(q^k)$ and that the H_i's, when viewed as sets of length n vectors over $GF(q)$, form the desired partition. Each such set is a block in $S[1, k, n]_q$.

Next, we show two intervals in which only trivial Steiner structures exist.

LEMMA 2. *If $n \leqslant 2k-t$, then only trivial Steiner structures $S[t, k, n]_q$ exist for which $k=n$.*

Proof. Let S be an $S[t, k, n]_q$ and $n \leqslant 2k-t$. Assume that S contains two different elements, W_1, W_2. Since W_1 and W_2 are two different blocks in the structure, it follows that $\dim(W_1 \cap W_2) \leqslant t-1$. Hence,

$$\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2) \geqslant 2k-t+1 > n,$$

a contradiction. Therefore S contains at most one element, i.e. $k=n$. ∎

THEOREM 3. *If $2k-t < n < 2k$, then only trivial $S[t, k, n]_q$ exist for which $t=k$.*

Proof. Let $S \subseteq \left[{V \atop k}\right]$ be an $S[t, k, n]_q$ and $2k-t<n<2k$. For two subspaces W_1 and W_2 of dimension k, if $\dim(W_1+W_2) \leqslant 2k-t$ then $\dim(W_1 \cap W_2) \geqslant t$ and hence each $Y \in \left[{V \atop 2k-t}\right]$ contains no more than one block of S.

On the other hand, each block of S is contained in exactly $\left[{n-k \atop k-t}\right]_q$ $(2k-t)$-dimensional subspaces of V. The number of $(2k-t)$-dimensional subspaces of V is $\left[{n \atop 2k-t}\right]_q$. Therefore,

$$\begin{bmatrix} n-k \\ k-t \end{bmatrix}_q \cdot |S| = \begin{bmatrix} n-k \\ k-t \end{bmatrix}_q \frac{\begin{bmatrix} n \\ t \end{bmatrix}_q}{\begin{bmatrix} k \\ t \end{bmatrix}_q} \leqslant \begin{bmatrix} n \\ 2k-t \end{bmatrix}_q,$$

which implies

$$\frac{\prod_{i=1}^{n-k}(q^i-1)\prod_{i=1}^{2k-t}(q^i-1)}{\prod_{i=1}^{k}(q^i-1)\prod_{i=1}^{n-t}(q^i-1)} \leqslant 1.$$

Since $2k-t<n<2k$, we have

$$\frac{(q^{n-t+1}-1)(q^{n-t+2}-1)\cdots(q^{2k-t}-1)}{(q^{n-k+1}-1)(q^{n-k+2}-1)\cdots(q^{k}-1)}\leqslant 1,$$

which can hold only if $k=t$ and the structure is trivial. ∎

We can summarize the previous two theorems in the following bound.

COROLLARY 2. *If a nontrivial* $S[t,k,n]_q$ *exists, then* $n\geqslant 2k$.

The bound of Corollary 2 is tight for $t=1$ as the construction given before shows. This bound will be useful later when we turn to deal with tiling optimal anticodes in the Grassman graph.

It is well known that if $S(t,k,n)$ and $S(t,n,v)$ exist then by substitution of $S(t,k,n)$ on each block of $S(t,n,v)$ we obtain an $S(t,k,v)$, Similarly we have

THEOREM 4. *If* $S[t,k,n]_q$ *and* $S[t,n,v]_q$ *exist, then* $S[t,k,v]_q$ *exists also.*

The next theorem provides a tool to obtain new Steiner systems from Steiner structures.

THEOREM 5. *Let* S *be an* $S[t,k,r]_q$ *and let* H *be the set of column vectors of the* $r\times n$ *parity check matrix of an* $[n,k',d]$ *code over* $GF(q)$, *where* $r=n-k'$ *and* $d-1\geqslant t$. *If there exists an integer* $v\geqslant t$ *such that*

$$S' \triangleq \{B\cap H \mid B\in S,\ |B\cap H|\geqslant t\}\subseteq\binom{H}{v},$$

then S' *is a Steiner system* $S(t,v,n)$.

Proof. Every $d-1$ columns in the parity check matrix of an $[n,k',d]$ code are linearly independent. Therefore, each set of t columns from H is contained in exactly one block of S and thus, also in exactly one block of S'. Since all the blocks of S' are of the same size v, it follows that S' is a Steiner system $S(t,v,n)$. ∎

Theorem 5 can be used to obtain some Steiner systems. In particular it can be used to obtain the Steiner systems related to projective and affine geometries.

THEOREM 6. *If* $S[2,k,n]_q$ *exists, then* $S(2,(q^k-1)/(q-1),(q^n-1)/(q-1))$ *exists.*

Proof. We take H to be the set of columns from the parity check matrix of the $[(q^n-1)/(q-1), (q^n-1)/(q-1)-n, 3]$ Hamming code over $GF(q)$ and use Theorem 5. ∎

COROLLARY 3. *For all $n \geqslant 2$ and, q, a power of a prime, a Steiner system $S(2, q+1, (q^n-1)/(q-1))$ exists.*

Proof. Use Theorem 6 with $S[2, 2, n]_q$. ∎

THEOREM 7. *If $S[2, k, n]_q$ exists, then $S(2, q^{k-1}, q^{n-1})$ exists.*

Proof. We take H to be the set of columns from the parity check matrix of the $[q^{n-1}, q^{n-1}-n, d \geqslant 3]$ code over $GF(q)$, whose columns consist of all the vectors ending with a 1, and use Theorem 5. ∎

COROLLARY 4. *For all $n \geqslant 2$ and a power of a prime q, a Steiner system $S(2, q, q^{n-1})$ exists.*

Proof. Use Theorem 7 with $S[2, 2, n]_q$. ∎

THEOREM 8. *If $S[3, k, n]_2$ exists, then $S(3, 2^{k-1}, 2^{n-1})$ exists.*

Proof. We take H to be the set of columns from the parity check matrix of the $[2^{n-1}, 2^{n-1}-n, 4]$ binary extended Hamming code and use Theorem 5. ∎

The existence of the trivial $S[3, 3, n]_2$ systems for all $n \geqslant 3$, together with Theorem 8 produces another set of well known systems.

COROLLARY 5. *For all $n \geqslant 3$, a Steiner system $S(3, 4, 2^{n-1})$ exists.*

3. NONEXISTENCE OF TILINGS WITH OPTIMAL ANTICODES

3.1. *The Size of an Optimal Anticode*

Let V be an n-dimensional vector space over $GF(q)$ and let $n \geqslant k \geqslant t$. A family, $\mathcal{F}$, of k-dimensional subspaces of V, i.e., $\mathcal{F} \subseteq \left[{V \atop k}\right]$ is said to be t-intersecting if $\dim(F \cap F') \geqslant t$ holds for all $F, F' \in \mathcal{F}$. It is easy to verify that every anticode of diameter D in the Grassman graph $G_q(n, k)$ is equivalent to a $(k-D)$-intersecting family.

There are two types of trivial t-intersecting families.

- If $n \leqslant 2k-t$ then a t-intersecting family is of size $\left[{n \atop k}\right]_q$ and contains all the k-dimensional subspaces of V.

- If $t=k$ or $k=n$ then a t-intersecting family contains just one block of dimension k.

Tiling $G_q(n,k)$ with trivial t-intersecting families is simple. Therefore, from now on we assume, $t<k<n$ and $n>2k-t$.

The size and structure of nontrivial optimal t-intersecting families were determined almost completely by Frankl and Wilson [5].

THEOREM 9. *Let $\mathcal{F} \subseteq \left[{V \atop k}\right]$ be a t-intersecting family. Then*

$$|\mathcal{F}| \leqslant \max\left\{\begin{bmatrix} n-t \\ k-t \end{bmatrix}_q, \begin{bmatrix} 2k-t \\ k \end{bmatrix}_q\right\}.$$

Frankl and Wilson [5] have analyzed the two possible solutions implied by Theorem 9. They proved that the bound is attained with equality by optimal anticodes.

- If $2k-t<n<2k$ then the bound is

$$|\mathcal{F}| \leqslant \begin{bmatrix} 2k-t \\ k \end{bmatrix}_q,$$

and it is attained in a unique way by taking some $Y \in \left[{V \atop 2k-t}\right]$ and defining

$$\mathcal{F} = \left\{F \in \begin{bmatrix} V \\ k \end{bmatrix} \,\middle|\, \dim(F \cap Y) \geqslant k\right\}.$$

These anticodes will be called *anticodes* (*t-intersecting families*) *of type* I.

- If $2k<n$ then the bound is

$$|\mathcal{F}| \leqslant \begin{bmatrix} n-t \\ k-t \end{bmatrix}_q,$$

and it is attained in a unique way by taking some $Y \in \left[{V \atop t}\right]$ and defining

$$\mathcal{F} = \left\{F \in \begin{bmatrix} V \\ k \end{bmatrix} \,\middle|\, \dim(F \cap Y) \geqslant t\right\}.$$

These anticodes will be called *anticodes* (*t-intersecting families*) *of type* II.

- If $2k=n$ then the bound is

$$|\mathcal{F}| \leqslant \begin{bmatrix} n-t \\ k-t \end{bmatrix}_q = \begin{bmatrix} 2k-t \\ k \end{bmatrix}_q,$$

and it is attained by both type I and type II anticodes. Frankl and Wilson [5] have conjectured that no other anticodes attain this bound.

For the two types of anticodes we call Y, the center of the anticode, and denote it by $\Omega(\mathscr{F}) \triangleq Y$.

3.2. *Tilings with Anticodes of Type* I

Let $P(Y)$ denote the power set of a set Y.

THEOREM 10. *There is no tiling of $G_q(n, k)$ for $2k - t < n \leqslant 2k$ with nontrivial type* I *anticodes.*

Proof. Assume the contrary, that

$$\{\mathscr{F}_0, \mathscr{F}_1, ..., \mathscr{F}_m\} \subseteq P\left(\begin{bmatrix} V \\ k \end{bmatrix}\right),$$

is a set of optimal type I t-intersecting families, which forms a tiling of $G_q(n, k)$. We will now examine the set,

$$S \triangleq \{\Omega(\mathscr{F}_i) \mid 0 \leqslant i \leqslant m\} \subseteq \begin{bmatrix} V \\ 2k-t \end{bmatrix}.$$

Since $\{\mathscr{F}_0, \mathscr{F}_1, ..., \mathscr{F}_m\}$ is a partition of $\left[\begin{smallmatrix} V \\ k \end{smallmatrix}\right]$, it follows by the definition of type I anticodes that for every $F \in \left[\begin{smallmatrix} V \\ k \end{smallmatrix}\right]$ there is exactly one index i such that $F \subseteq \Omega(\mathscr{F}_i)$. Therefore, S is an $S[k, 2k-t, n]_q$.

$k > t$ implies that $k \neq 2k - t$ and since also $2k - t \neq n$, we have that S is not a trivial structure. Therefore, by Corollary 2,

$$n \geqslant 2(2k - t),$$

and since we also have $n \leqslant 2k$ it follows that

$$2k \geqslant 2(2k - t),$$

or

$$t \geqslant k$$

which is a contradiction. Thus, there is no tiling of $G_q(n, k)$ for $2k - t < n \leqslant 2k$ with nontrivial type I anticodes. ∎

3.3. *Tilings with Anticodes of Type* II

LEMMA 3. *Any nontrivial tiling of the graph $G_q(n, k)$, $n \geqslant 2k$, with optimal anticodes, requires at least three anticodes.*

Proof. We need only consider type II anticodes since they attain the bound on an anticode size when $n \geqslant 2k$. The number of t-intersecting families in a tiling is

$$\frac{\begin{bmatrix} n \\ k \end{bmatrix}_q}{\begin{bmatrix} n-t \\ k-t \end{bmatrix}_q} = \frac{(q^{n-t+1}-1)(q^{n-t+2}-1)\cdots(q^n-1)}{(q^{k-t+1}-1)(q^{k-t+2}-1)\cdots(q^k-1)}.$$

Since $1 \leqslant t < k < n$ and $q \geqslant 2$ we have

$$\frac{(q^{n-t+1}-1)(q^{n-t+2}-1)\cdots(q^n-1)}{(q^{k-t+1}-1)(q^{k-t+2}-1)\cdots(q^k-1)} > 2^t,$$

which implies

$$\frac{\begin{bmatrix} n \\ k \end{bmatrix}_q}{\begin{bmatrix} n-t \\ k-t \end{bmatrix}_q} \geqslant 3. \quad \blacksquare$$

THEOREM 11. *There is no set of* $\lceil \begin{bmatrix} n \\ k \end{bmatrix}_q / (2 \cdot \begin{bmatrix} n-t \\ k-t \end{bmatrix}_q) \rceil$ *pairwise disjoint nontrivial type* II *anticodes in* $G_q(n, k)$ *for* $n \geqslant 2k$.

Proof. Assume

$$\{\mathscr{F}_0, \mathscr{F}_1, ..., \mathscr{F}_{m-1}\} \subseteq P\left(\begin{bmatrix} V \\ k \end{bmatrix}\right),$$

is a set of pairwise disjoint type II anticodes in $G_q(n, k)$, $n \geqslant 2k$, where

$$m \geqslant \frac{1}{2} \cdot \frac{\begin{bmatrix} n \\ k \end{bmatrix}_q}{\begin{bmatrix} n-t \\ k-t \end{bmatrix}_q}.$$

By Lemma 3 we have that $m \geqslant 2$.

Let S be the set of centers of the anticodes in the tiling, i.e.,

$$S \triangleq \{\Omega(\mathscr{F}_i) \mid 0 \leqslant i \leqslant m-1\} \subseteq \begin{bmatrix} V \\ t \end{bmatrix}.$$

If $k \geqslant 2t$ then there exists $F \in [{V \atop k}]$ such that

$$\Omega(\mathscr{F}_0) \subseteq F \qquad \text{and} \qquad \Omega(\mathscr{F}_1) \subseteq F.$$

Therefore, by the definition of type II anticodes we have that $F \in \mathscr{F}_0 \cap \mathscr{F}_1$, which contradicts the fact that $\mathscr{F}_0 \cap \mathscr{F}_1 = \emptyset$. Thus, $k \leqslant 2t-1$.
Assume now that for some $0 \leqslant i < j \leqslant m-1$,

$$\dim(\Omega(\mathscr{F}_i) \cap \Omega(\mathscr{F}_j)) \geqslant 2t-k.$$

Hence,

$$\begin{aligned}\dim(\Omega(\mathscr{F}_{\mathrm{i}}) &+ \Omega(\mathscr{F}_{\mathrm{j}}))\\ &= \dim(\Omega(\mathscr{F}_{\mathrm{i}})) + \dim(\Omega(\mathscr{F}_{\mathrm{j}})) - \dim(\Omega(\mathscr{F}_{\mathrm{i}}) \cap \Omega(\mathscr{F}_{\mathrm{j}}))\\ &\leqslant t + t - (2t-k) = k,\end{aligned}$$

which implies that there exists some $F \in [{V \atop k}]$ such that

$$\Omega(\mathscr{F}_i) + \Omega(\mathscr{F}_j) \subseteq F.$$

Therefore, $F \in \mathscr{F}_i \cap \mathscr{F}_j$, which again contradicts the fact that $\mathscr{F}_i \cap \mathscr{F}_j = \emptyset$. Hence, $k \leqslant 2t-1$ and for all $0 \leqslant i < j \leqslant m-1$ we have

$$\dim(\Omega(\mathscr{F}_i) \cap \Omega(\mathscr{F}_j)) \leqslant 2t-k-1.$$

Each element of S contains exactly $[{t \atop 2t-k}]_q$ elements of $[{V \atop 2t-k}]$. Since $\dim(\Omega(\mathscr{F}_i) \cap \Omega(\mathscr{F}_j)) \leqslant 2t-k-1$, we have that each $F \in [{V \atop 2t-k}]$ is contained in at most one element of S. Therefore,

$$|S| \cdot \begin{bmatrix} t \\ 2t-k \end{bmatrix}_q \leqslant \begin{bmatrix} n \\ 2t-k \end{bmatrix}_q. \tag{2}$$

Recall that

$$|\mathrm{S}| \geqslant \tfrac{1}{2} \cdot \frac{\begin{bmatrix} n \\ k \end{bmatrix}_q}{\begin{bmatrix} n-t \\ k-t \end{bmatrix}_q}, \tag{3}$$

and hence by Eqs. (2) and (3) we have that

$$\frac{(q^{n-t+1}-1)(q^{n-t+2}-1)\cdots(q^{n+k-2t}-1)}{(q^{t+1}-1)(q^{t+2}-1)\cdots(q^k-1)} \leqslant 2. \tag{4}$$

Hence

$$n + k - 2t \leqslant k.$$

and therefore,

$$n \leqslant 2t < 2k,$$

a contradiction. Thus, the theorem is proved. ∎

COROLLARY 6. *There is no tiling of* $G_q(n, k)$, *for* $n \geqslant 2k$, *with nontrivial type* II *anticodes.*

The proof of Corollary 6 may be obtained by using the previous section and the duality presented in the next subsection. But we cannot omit the proof as we still need Theorem 11 in the next subsection.

3.4. *Tilings with Anticodes of Type* I *and* II

When $n = 2k$ at least two types of optimal anticodes exist. We have already proved that no tilings exists when $n = 2k$ which consist only of type I anticodes or only of type II anticodes. We will now prove that there is no tiling of $G_q(n, k)$, $n = 2k$, which uses any combination of the two types of anticodes.

We first examine the set of dual subspaces of the elements of a t-intersecting family. Given $\mathscr{F} \subseteq \left[{V \atop k} \right]$ a t-intersecting family, we define

$$\mathscr{F}^{\perp} \triangleq \{ F^{\perp} \mid F \in \mathscr{F} \} \subseteq \begin{bmatrix} V \\ n-k \end{bmatrix}.$$

LEMMA 4. *If* $\mathscr{F}$ *is a t-intersecting family of either type* I *or type* II *then* $\mathscr{F}^{\perp}$ *is a* $(n - 2k + t)$*-intersecting family of the other type.*

Proof. Given $F_1^{\perp}, F_2^{\perp} \in \mathscr{F}^{\perp}$ we know that

$$\dim(F_1 \cap F_2) \geqslant t.$$

Thus,

$$\dim(F_1 + F_2) \leqslant 2k - t.$$

One can easily verify that $(F_1 + F_2)^{\perp} = F_1^{\perp} \cap F_2^{\perp}$ and hence

$$\dim(F_1^{\perp} \cap F_2^{\perp}) \geqslant n - 2k + t,$$

which proves the first part of the lemma, i.e., $\mathscr{F}^{\perp}$ is a $(n - 2k + t)$-intersecting family.

We distinguish now between two cases.

Case 1. If $\mathscr{F}$ is of type I, then for all $F \in \mathscr{F}$,

$$F \subseteq \Omega(\mathscr{F}).$$

Therefore, for all $F^{\perp} \in \mathscr{F}^{\perp}$,

$$\Omega(\mathscr{F})^{\perp} \subseteq F^{\perp}.$$

Clearly, $F^{\perp}$ has dimension $k' = n - k$ and $\Omega(\mathscr{F})^{\perp}$ has dimension $t' = n - (2k - t)$. Therefore, $0 < t' < k'$ and hence $\mathscr{F}^{\perp}$ is a t'-intersecting family of type II in $G_q(n, k')$, and by the construction of type II anticode $\Omega(\mathscr{F})^{\perp}$ is the center of $\mathscr{F}^{\perp}$, i.e.,

$$\Omega(\mathscr{F})^{\perp} = \Omega(\mathscr{F})^{\perp}.$$

Case 2. If $\mathscr{F}$ is of type II then by a similar argument we have that $\mathscr{F}^{\perp}$ is an intersecting family of type I. ▮

Note that if $n \neq 2k$ then $\mathscr{F}$ is an optimal anticode if and only if $\mathscr{F}^{\perp}$ is an optimal anticode.

LEMMA 5. *Let T be a tiling of $G_q(n, k)$, $n = 2k$, with r_1 type* I *anticodes and r_2 type* II *anticodes. There exists a tiling of $G_q(n, k)$ which consists of r_2 type* I *anticodes and r_1 type* II *anticodes.*

Proof. Let $T = \{\mathscr{A}_0, \mathscr{A}_1, ..., \mathscr{A}_{r_1-1}, \mathscr{A}'_0, \mathscr{A}'_1, ..., \mathscr{A}'_{r_2-1}\}$. We claim that the set

$$T^{\perp} \triangleq \{\mathscr{A}_0^{\perp}, \mathscr{A}_1^{\perp}, ..., \mathscr{A}_{r_1-1}^{\perp}, \mathscr{A}_0'^{\perp}, \mathscr{A}_1'^{\perp}, ..., \mathscr{A}_{r_2-1}^{\perp}\}.$$

is also a tiling of $G_q(n, k)$, where $\mathscr{A}_i^{\perp}, \mathscr{A}_j'^{\perp}$ stand for a set of all the dual vector spaces of the elements in the respective anticodes for all i and j.

Let $\mathscr{B} \in T$ be some anticode. Clearly, $\mathscr{B}$ is a t-intersecting family, and by Lemma 4, $\mathscr{B}^{\perp}$ is also a t-intersecting family of the opposite type.

Both types of anticodes are exactly of the same size when $n = 2k$. To prove that $T^{\perp}$ is also a tiling of $G_q(n, k)$ we have to show that the elements of $T^{\perp}$ are pairwise disjoint.

Let $\mathscr{B}_1{}^{\perp}, \mathscr{B}_2{}^{\perp} \in T^{\perp}$ be two anticodes of any of the two types. We denote their intersection by

$$\mathscr{C} \triangleq \mathscr{B}_1{}^{\perp} \cap \mathscr{B}_2{}^{\perp} \subseteq \begin{bmatrix} V \\ n-k \end{bmatrix}.$$

If $\mathscr{C} \neq \varnothing$, then let $F \in \mathscr{C}$ be some $(n-k)$-dimensional subspace of V. By definition, $F^{\perp} \in \mathscr{B}_1 \cap \mathscr{B}_2$ which contradicts the fact that T is a tiling. ∎

COROLLARY 7. *There is no nontrivial tiling of $G_q(n, k)$, $n = 2k$, which consists only of type* I *and type* II *anticodes.*

Proof. Let us assume that such a tiling T exists. In either T or $T^{\perp}$, at least half of the elements are type II anticodes, which is impossible by Theorem 11. ∎

4. TILINGS WITH MAXIMAL ANTICODES

An anticode $\mathscr{C}$, over a space V, with diameter D is called *maximal* if for each $u \in V \backslash \mathscr{C}$ there exists a codeword $c \in \mathscr{C}$ such that the distance between u and c is greater than D. In other words, any addition of a word to $\mathscr{C}$ will destroy the maximum distance. Note, that type I anticodes when defined for $n > 2k$ are also maximal, as are type II anticodes when defined for $n < 2k$.

In this section we will assume that $k \geqslant 2D$ if $n \geqslant 2k$.

DEFINITION 1. Let C be a diameter perfect code in $G_q(n, k)$ with minimal distance $D+1$. For each $X \in C$ we define the following set,

$$\Gamma(X) \triangleq \{\Omega(\mathscr{A}) \mid X \in \mathscr{A}, \mathscr{A} \in \mathscr{A}_D\},$$

where $\mathscr{A}_D$ is the set of all the optimal anticodes of diameter D in $G_q(n, k)$, which are all of either type I or all of type II.

Note that for $n \neq 2k$, the set $\mathscr{A}_D$ is unique, while for the case of $n = 2k$ there are two sets, $\mathscr{A}^{I}_{D}$ and $\mathscr{A}^{II}_{D}$ which are sets of anticodes of type I and II, respectively.

THEOREM 12. *For all $X \in C$, $\Gamma(X)$ is a maximal anticode in $G_q(n, f(k))$. where*

$$f(k) = \begin{cases} k - D, & n \geqslant 2k, k \geqslant 2D \\ k + D, & n \leqslant 2k. \end{cases}$$

In addition,

$$\Omega(\Gamma(X)) = X,$$

and $\Gamma(X)$ is of the opposite type of the anticodes in the set $\mathscr{A}_D$ which created it.

Proof. If $n \geqslant 2k$ the anticodes in $\mathscr{A}_D$ are of type II and for all $X \in C$,

$$\Gamma(X) = \left\{ W \in \begin{bmatrix} V \\ t \end{bmatrix} \;\middle|\; W \subseteq X \right\},$$

where $t = k - D = f(k)$. This is exactly a maximal anticode of type I in $G_q(n, f(k))$, i.e., a t'-intersecting family in $G_q(n, k')$, with $k' = t = k - D$ and $t' = 2t - k = k - 2D$. It is also obvious that $\Omega(\Gamma(X)) = X$. The case where $n \leqslant 2k$ is handled similarly. ▮

In the next theorem we use the following result from [1].

THEOREM 13. *C is a D-diameter perfect code in $G_q(n, k)$ with minimum distance $D+1$ if and only if each optimal anticode, with diameter D, in $G_q(n, k)$ contains a codeword.*

THEOREM 14. *The set $\mathbf{\Gamma} \triangleq \{\Gamma(X)\}_{X \in C}$ is a partition of $G_q(n, f(k))$ into maximal anticodes.*

Proof. We first want to prove that if $\Gamma(X_1), \Gamma(X_2) \in \mathbf{\Gamma}$, $\Gamma(X_1) \neq \Gamma(X_2)$, then $\Gamma(X_1) \cap \Gamma(X_2) = \varnothing$. Let us assume the contrary, that is, that there exist $\Gamma(X_1), \Gamma(X_2) \in \mathbf{\Gamma}$, $\Gamma(X_1) \neq \Gamma(X_2)$ such that $\Gamma(X_1) \cap \Gamma(X_2) \neq \varnothing$.

Let $\Omega(\mathscr{A}) \in \Gamma(X_1) \cap \Gamma(X_2)$, then $X_1 \in \mathscr{A}$ and also $X_2 \in \mathscr{A}$. Since $\mathscr{A}$ is of diameter D, the distance between X_1 and X_2 is at most D which contradicts the minimum distance of the the code C.

To finish the proof, we also have to show that

$$\bigcup_{X \in C} \Gamma(X) = \begin{bmatrix} V \\ f(k) \end{bmatrix}.$$

If $Y \in \left[\begin{smallmatrix} V \\ f(k) \end{smallmatrix}\right]$, then $Y = \Omega(\mathscr{A})$ and hence $\mathscr{A}$ is an optimal anticode in $G_q(n, k)$ with diameter D. By Theorem 13, the code C is diameter perfect in $G_q(n, k)$ with minimal distance $D+1$ if and only if each optimal anticode in $G_q(n, k)$ with diameter D contains a codeword. Therefore, there exists a codeword $X \in C$ such that $X \in \mathscr{A}$, and then $Y = \Omega(\mathscr{A}) \in \Gamma(X)$ as required.

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