

*Proof of Theorem 1:* Let  $x_k = A_k A_{-k}$ . Then  $x_k \in \{0, 1\}$ . The equations shown in (7) are a system of equations in  $N - 1$  variables which has the following coefficient matrix

$$B = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ \alpha^{-1} & \alpha^{-2} & \cdots & \alpha^{-(N-1)} \\ \alpha^{-2} & \alpha^{-4} & \cdots & \alpha^{-2(N-1)} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha^{-(N-1)} & \alpha^{-(N-1)2} & \cdots & \alpha^{-(N-1)^2} \end{bmatrix}$$

The submatrix consisting of the first  $N - 1$  rows is the Vandermonde matrix, so the rank of  $B$  is  $N - 1$ . It follows that  $x_k = 0$ ,  $k = 1, 2, \dots, N - 1$ .  $\square$

## II. LINEAR SPANS OF THE SEQUENCES IN $C$

In this section, we will derive a decomposition of a sequence in  $C$  as  $r$ th-order linear sequences where  $r \mid n$  and give an achievable upper bound for the linear spans of two-level correlation sequences.

Let  $G$  be the set of all coset leaders modulo  $N$ . We denote by  $f_{\alpha^v}(x)$  the minimal polynomial of  $\alpha^v$  for  $v \in G$ . If  $f_{\alpha^v}(x) \neq f_{\alpha^{-v}}(x)$ , then we call  $v$  a *nonreciprocal-symmetry coset leader modulo  $N$* . Otherwise, we call  $v$  a *reciprocal-symmetry coset leader modulo  $N$* . Notice that if  $v$  is a nonreciprocal-symmetry coset leader modulo  $N$ , so is  $p - v$ . Let  $P$  be the set of all nonreciprocal symmetry coset leaders modulo  $N$  which satisfy  $v + 2^j u \not\equiv 0 \pmod{N}$  for  $u, v$  in  $P$ . Let  $S$  be the set of all reciprocal-symmetry coset leaders modulo  $N$ . Let  $\bar{P} = \{k \mid -k \in P\}$ . Then

$$G = P \cup \bar{P} \cup S. \tag{10}$$

The following corollary follows from Theorem 1.

*Corollary 1:* If  $\{a_t\} \in C$ , then

$$A_k A_{-k} = 0 \quad \text{if } k \in P \quad \text{and} \quad A_k = 0 \quad \text{if } k \in S. \tag{11}$$

Moreover

$$a_t = \sum_{k \in P} Tr(A_k \alpha^{-rk^t}), \quad t = 0, 1, \dots, N - 1 \tag{12}$$

where  $r_k \in P$  or  $r_k \in \bar{P}$ , and  $A_k \in \{0, 1\}$ .

The linear span of  $\{a_t\}$ , denoted by  $LS(\{a_t\})$ , is the number of nonzero values in the spectrum of  $\{a_t\}$ . We denote by  $P_c$  the set of coset leaders in  $P$  together with their conjugates, and denote by  $S_m$  the set of all reciprocal-symmetry coset leaders modulo  $2^m - 1$ , where  $m$  is a proper factor of  $n$ .

*Theorem 2:* If  $\{a_t\} \in C$ , the linear span of  $\{a_t\}$  is bounded by

$$LS(\{a_t\}) \leq |P_c| \tag{13}$$

where if  $n$  odd

$$|P_c| = \frac{1}{2} \sum_{d|n} \sum_{m|d} m \mu(m) 2^{d/m} \tag{14}$$

if  $n$  even

$$|P_c| = \frac{1}{2} \sum_{d|n} \sum_{m|d} m (\mu(m) 2^{d/m} - \delta_m |S_m|) \tag{15}$$

where  $\delta_m$  is zero if  $m$  is odd and is one if  $m$  is even.

*Proof:* From Corollary 1, the formula (13) is immediate. Now we will find the number of coset leaders in  $P$ . Note that  $|G|$  is the number of irreducible polynomials over  $GF(2)$  whose degrees divide  $n$ . Golomb [3] showed that  $|S| \neq 0$  if and only if  $n$  is even. So, if  $n$  is odd, then  $|S| = 0$ . From [2], we get the formulae (14) and (15).  $\square$

*Remark 1:* If  $N$  is a prime number and  $\{a_t\}$  is a quadratic sequence

$$LS(\{a_t\}) = |P_c| = \frac{1}{2}(N - 1) \tag{16}$$

which achieves this upper bound.

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## Linear Complexity of de Bruijn Sequences— Old and New Results

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**Abstract**—The linear complexity of a de Bruijn sequence is the degree of the shortest linear recursion which generates the sequence. It is well known that the complexity of a binary de Bruijn sequence of length  $2^n$  is bounded below by  $2^{n-1} + n$  and above by  $2^n - 1$  for  $n \geq 3$ . We briefly survey the known knowledge in this area. Some new results are also presented, in particular, it is shown that for each interval of length  $2^{\lfloor \log n \rfloor + 1}$ , in the above range, there exist binary de Bruijn sequences of length  $2^n$  with linear complexity in the interval.

**Index Terms**—Complexity distribution, de Bruijn sequences, Games and Chan algorithm, linear complexity, minimal complexity.

## I. INTRODUCTION

De Bruijn sequences have many applications in communication systems [13]. They have many desirable properties such as long period and low predictability and can be used as stream ciphers [17] in cryptographic applications. A comprehensive survey on those properties can be found in [13]. Outline of past work about construction of those sequences and other related problems can be found in [9]. The (linear) complexity  $C(S)$  of a sequence  $S$  is one of the measures of its predictability [15]. This paper deals with the complexity distribution of de Bruijn sequences.

Let  $s_1 s_2 \cdots s_{k-1}$  denote a string of  $k$  binary digits. A cyclic, or closed, string is called a *sequence* and is denoted by  $S =$

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$[s_0 s_1 \cdots s_{k-1}]$ , where  $k = l(S)$  is the *length* of the sequence (or the string). For a string  $S$  let  $S^r$  denote concatenation of  $r$  occurrences of  $S$ . The *weight* of  $S$ ,  $W(S)$ , is the number of "1's" in  $S$ . The *order* of a sequence  $S = [s_0 s_1 \cdots s_{k-1}]$  is the least integer  $n$  such that the  $n$ -tuples  $V_i = (s_i, s_{i+1}, \dots, s_{i+n-1})$ ,  $0 \leq i \leq k-1$ , with subscripts taken modulo  $k$ , are all distinct. Such sequences can be viewed as  $k$ -cycles from a feedback shift-register of  $n$ -stages, where the  $n$ -tuples  $V_i$  are successive *states* of the register (or of the sequence). Two sequences  $S_1$  and  $S_2$  are said to be *equivalent*,  $S_1 \simeq S_2$ , if one is a cyclic shift of the other. The *complement*  $cS$  and the *reverse*  $rS$  of a string  $S = s_0 s_1 \cdots s_{k-1}$  are defined by  $cS = \bar{s}_0 \bar{s}_1 \cdots \bar{s}_{k-1}$ , where  $\bar{s}_i$  is the binary complement of  $s_i$ , and  $rS = s_{k-1} \cdots s_1 s_0$ . Note that the operators  $c$  and  $r$  commute.  $S$  is called a *CR-sequence* if  $cS \simeq rS$ , or equivalently  $crS \simeq S$ . A sequence  $S$  of length  $2^n$  and order  $n$  is called a *de Bruijn sequence*. Note, that each of the possible  $2^n$   $n$ -tuples appears exactly once as a state of  $S$ . The set of all de Bruijn sequences of order  $n$  will be denoted by  $DS(n)$ . It is well known [3] that the cardinality of  $DS(n)$  is  $2^{2^n-1-n}$ .

Every sequence  $S = [s_0 s_1 \cdots s_{k-1}]$  satisfies a linear recursion of degree  $m \leq k$

$$s_{i+m} + \sum_{j=1}^m a_j s_{i+m-j} = 0, \quad \text{for all } i \geq 0. \quad (1)$$

In terms of a *shift operator*  $\mathbf{E}$ , defined by

$$\mathbf{E}[s_0, s_1, \dots, s_{k-1}] = \mathbf{E}[s_1, \dots, s_{k-1}, s_0] \quad (2)$$

i.e.,  $\mathbf{E}s_i = s_{i+1}$ , for all  $i \geq 0$  the linear recursion takes the form

$$\left( \mathbf{E}^m + \sum_{j=1}^m a_j \mathbf{E}^{m-j} \right) s_i = 0, \quad \text{for all } i \geq 0. \quad (3)$$

Let  $f(\mathbf{E})s_i = 0$ ,  $i \geq 0$ , be the linear recursion of least degree satisfied by  $S$ . Then the *complexity*  $C(S)$  of  $S$  is defined as the degree of  $f(\mathbf{E})$  viewed as a polynomial in  $\mathbf{E}$ . An efficient algorithm to compute the complexity of sequences with length  $2^n$  was given by Games and Chan [12].

The input to the Games and Chan algorithm is a sequence  $S$  of length  $l(S) = 2^n$ . If  $S \neq 0^{2^n}$ , the complexity of  $S$  is computed recursively as follows. Initially, set  $c_n = 0$  and  $A_n = S$ . At a typical step of the algorithm the left half of  $A_m$ ,  $L(A_m) = [a_0 \cdots a_{2^{m-1}-1}]$ , is added to the right half,  $R(A_m) = [a_{2^{m-1}} \cdots a_{2^m-1}]$ , the result being a sequence  $B_m$ , of length  $2^{m-1}$ . If  $B_m = 0^{2^{m-1}}$ ,  $A_m$  is replaced by  $A_{m-1} = L(A_m)$  and the complexity is left unchanged, i.e.,  $c_{m-1} = c_m$ . If  $B_m \neq 0^{2^{m-1}}$ ,  $A_m$  is replaced by  $A_{m-1} = B_m$  and  $c_m$  is replaced by  $c_{m-1} = c_m + 2^{m-1}$ . The complexity of  $S$  is given by  $C(S) = c_0 + 1$ .

Chan, Games and Key [2] proved that the complexity of a de Bruijn sequence of length  $2^n$ ,  $n \geq 3$ , is bounded below by  $2^{n-1} + n$  and above by  $2^n - 1$ . They gave complete complexity distribution tables for de Bruijn sequences of order  $n$ , where  $4 \leq n \leq 6$ . They had four conjectures on  $\gamma(c, n)$ , the number of de Bruijn sequences of order  $n$  and complexity  $c$ .

Description:

- C.1) For  $n \geq 3$ ,  $\gamma(2^{n-1} + n + 1, n) = 0$ .
- C.2) The lower bound for the complexities of de Bruijn sequences of order  $n$  is attained for all  $n$ , that is,  $\gamma(2^{n-1} + n, n) > 0$ .
- C.3) At least half of the de Bruijn sequences of order  $n$  have maximum complexity, that is,  $\gamma(2^n - 1, n) \geq 2^{2^n-1-n-1}$ .
- C.4) For  $n \geq 4$ ,  $\gamma(c, n) \equiv 0 \pmod{4}$ .

Games [10] proved C.1), i.e., he showed that there are no de Bruijn sequences of order  $n$  and complexity  $2^{n-1} + n + 1$ . Etzion and Lempel [8] showed how to construct de Bruijn sequences of order  $n$  and minimal complexity  $2^{n-1} + n$ , thus proving C.2). For most of the other values of  $c$  between  $2^{n-1} + n$  and  $2^n - 1$ , it is not known whether  $\gamma(c, n) > 0$ , although we conjecture.

*Conjecture 1:* For  $n \geq 3$  and for all  $c$  in the range between  $2^{n-1} + n$  and  $2^n - 1$ , except for  $2^{n-1} + n + 1$ , there exist de Bruijn sequences of order  $n$  and complexity  $c$ , that is,  $\gamma(c, n) > 0$ .

C.3) is an open problem and it is not known if it is true for  $n = 7$ , since it is not possible yet to make an exhaustive computer search on all the de Bruijn sequences of order seven and maximal complexity 127. Etzion [5] found that C.4) is not true in general, but there are values of  $c$  and  $n$  for which  $\gamma(c, n) \equiv 0 \pmod{4}$ .

V.1) For even  $n \geq 4$ ,  $\gamma(c, n) \equiv 0 \pmod{4}$  [2], [9].

V.2) For even  $c$  and  $n \geq 3$ ,  $\gamma(c, n) \equiv 0 \pmod{4}$  [7].

V.3) For all  $n \geq 4$  and  $c$  in the range  $2^{n-1} + n \leq c \leq 2^{n-1} + 2^{n-2}$ ,  $\gamma(c, n) \equiv 0 \pmod{4}$  [4].

V.4) For all  $n \geq 4$ ,  $\gamma(2^n - 1, n) \equiv 0 \pmod{8}$  [7].

V.5) For all  $k \geq 3$ ,  $\gamma(2^{2k} - 1, 2k) \equiv 0 \pmod{16}$  [7].

For later reference, we also state the following known facts.

*Fact 1* [2], [9]: If  $S$  is a sequence whose length is a power of two then  $C(S) = c$  if and only if  $(\mathbf{E} + 1)^{c-1} S = 1^{l(S)}$ .

*Fact 2* [8]: Let  $F(n)$  denote a maximal set of pairwise inequivalent sequences of period  $2^{\lfloor \log n \rfloor + 1}$  and complexity  $n + 1$ . Every  $S \in F(n)$  satisfies  $(\mathbf{E} + 1)^n S = 1^{l(S)}$ , the cardinality of  $F(n)$  is  $|F(n)| = 2^{n - \lfloor \log n \rfloor - 1}$  and each of the  $2^n$  binary  $n$ -tuples appears exactly once in one of the members of  $F(n)$ .

An interesting application of de Bruijn sequences with minimal complexity and the sequences of  $F(n)$ , in the design of perfect maps, can be found in [6].

*Fact 3:* Let  $S_1$  and  $S_2$  be two sequences of length  $2^m$ . If  $C(S_1) < C(S_2)$  then  $C(S_1 + S_2) = C(S_2)$ .

The main objective of this paper is to survey the known knowledge in this area and to present some new related results. In Section II we will give some new results on the distribution of complexities of de Bruijn sequences. In Section III we modify the construction in [8] to obtain a sufficient condition for the existence of de Bruijn sequences of order  $n$  with complexity  $2^{n-1} + n + \sum_{i=\lfloor \log n \rfloor + 1}^{n-3} \alpha_i 2^i$  for  $n \geq 9$  and any choice of  $\alpha_i \in \{0, 1\}$ . In Section IV we will prove that the sufficient condition of Section III may be realized, and show that the attained values of complexities for de Bruijn sequences are dense.

## II. COMPLEXITY DISTRIBUTION

For  $n = 7$  it is difficult to compute  $\gamma(c, 7)$  for every  $c$ , since there are  $2^{57}$  de Bruijn sequences of order seven. By using computer search we found that  $\gamma(71, 7) = 477240$ ,  $\gamma(73, 7) = 688$ ,  $\gamma(74, 7) = 696$ ,  $\gamma(75, 7) = 5760$ , and  $\gamma(76, 7) = 1232$ .

Etzion and Lempel [7] defined two operators on every  $S \in DS(n)$ .  $zS$  (respectively,  $uS$ ) denotes the sequence obtained from  $S$ , by interchanging the positions of the unique runs of  $n$  and  $n - 2$  "0's" (respectively, "1's"). It is readily verified that  $zS, uS \in DS(n)$ , and it was proved [7] that  $C(S) = 2^n - 1$  if and only if  $C(zS) = 2^n - 1$  and  $C(uS) = 2^n - 1$ . They defined the group  $G = \{e, r, z, u, rz, ru, zu, rzu\}$ , where  $e$  is the identity operator, and proved that if  $k \geq 3$  and  $S \in DS(2k)$  then  $(G \cup Gc)S$  consists of 16 pairwise inequivalent de Bruijn sequences of order  $n$ . For odd  $n \geq 5$  and for  $S \in DS(n)$ , we can conclude from [7] that one of the following cases holds.

*Case 1:*  $(G \cup Gc)S$  consists of sixteen pairwise inequivalent de Bruijn sequences, each of them is not a CR-sequence.

Case 2:  $GS = GcS$  consists of eight pairwise de Bruijn sequences, four of them are CR-sequences.

Games [11] mentioned the following lemma on  $\delta(c, n)$ , the number of de Bruijn CR-sequences of order  $n$  and complexity  $c$ .

*Lemma 1:* For  $n \geq 5$ ,  $\delta(2^n - 1, n) \equiv 0 \pmod{2^{(n+1)/2}}$ .

Now, we can extend V.4) and V.5) to the following theorem.

*Theorem 1:* For  $n \geq 5$ ,  $\gamma(2^n - 1, n) \equiv 0 \pmod{16}$ .

*Proof:* From Lemma 1 we have that the number of sequences which belong to Case 2 is congruent to 0 modulo  $2 \cdot 2^{(n+1)/2}$ . Together with Case 1 and V.5) we have that for  $n \geq 5$ ,  $\gamma(2^n - 1, n) \equiv 0 \pmod{16}$ .

**Q.E.D.**

### III. SUFFICIENT CONDITIONS FOR THE EXISTENCE OF DE BRUIJN SEQUENCES WITH LOW COMPLEXITY

Although we did not prove Conjecture 1, we were able to prove that the complexity distribution of de Bruijn sequences cannot be too sparse. In this section we derive a sufficient condition for the existence of de Bruijn sequences of order  $n$  with complexity  $2^{n-1} + n + \sum_{i=\lceil \log n \rceil+1}^{n-3} \alpha_i 2^i$  for any selection of  $\alpha_i \in \{0, 1\}$ .

The companion  $U'$  of a state  $U = (u_1, u_2, \dots, u_{n-1}, u_n)$  is defined by  $U' = (u_1, u_2, \dots, u_{n-1}, \bar{u}_n)$ . Two sequences,  $S_1$  and  $S_2$  are said to be *adjacent* if they are state-disjoint and there exists a state  $U$  on  $S_1$  whose companion  $U'$  is on  $S_2$ . The following lemma is a well-known observation [13].

*Lemma 2:* Two adjacent sequences  $S_1$  and  $S_2$ , with  $U$  on  $S_1$  and  $U'$  on  $S_2$ , are joined into a single sequence when the predecessors of  $U$  and  $U'$  are interchanged.

The following theorem is a straightforward generalization of the results from [8].

*Theorem 2:* If the sufficient condition, stated below, holds for a given  $n$  then there exists a de Bruijn sequence of order  $n$  with complexity  $2^{n-1} + n + g$ .

#### A. The Sufficient Condition

Consider a set  $F$  of  $2^r$  sequences of length  $2^{n-r}$ , which contains all the binary  $n$ -tuples as states. Then it is possible to choose one state in each of the sequences of  $F$ , designated as the first state of the sequence, and it is possible to arrange the members of  $F$  in pairs  $P_i = (A_i, B_i)$ ,  $1 \leq i \leq 2^{r-1}$ , so that properties p.1)–p.4) hold.

- p.1) For each pair  $P_i$ , the first state of  $A_i$  is the companion of the first state of  $B_i$ .
- p.2) For each  $i$ ,  $A_i + B_i = A_1 + B_1$ , where the sum of the sequences is their bitwise sum.
- p.3)  $C(A_1 + B_1) = n + g$ .
- p.4) The graph  $(V, E)$ , where  $V = \{v_i | 1 \leq i \leq 2^{r-1}\}$  and  $\{v_i, v_j\} \in E$  if and only if  $A_i$  and  $A_j$  have a pair of companion states in the same position (relative to their respective first states), is a connected graph.

The proof of Theorem 2 is the same as the proof of Theorem 2 in [8].

*Example 1:* Let  $n = 7$ , we take sequences of length 16. These sequences are listed below in four pairs that satisfy p.1)–p.4). It is easy to verify that this arrangement satisfies p.1)–p.3). To check p.4), let  $POC(i, j)$  denote a position in  $A_i$  and  $A_j$  which implies  $\{v_i, v_j\} \in E$  according to p.4). It is easy to see now that the three edges implied by  $POC(1, 2) = 4, POC(1, 3) = 7, POC(1, 4) = 10$

form a tree of  $(V, E)$ , thus validating p.4)

$$\begin{aligned} (A_1, B_2) &= ([1000000010110100], [1000001100011101]) \\ (A_2, B_2) &= ([1111000010010111], [1111001100111110]) \\ (A_3, B_3) &= ([0111001010110010], [0111000100011011]) \\ (A_4, B_4) &= ([0101000011110100], [0101001101011101]). \end{aligned}$$

The join is performed by applying Lemma 2 to the seven states

$$\begin{aligned} &(0000101), (0101101), (1101001), (0011001) \\ &(1000111), (0111011), (1000000) \end{aligned}$$

and their companions. The generated de Bruijn sequence is one of the combinations to form a de Bruijn sequence with complexity 74 from those eight sequences.

If  $g = 0$  and  $F = F(n)$  then this sufficient condition becomes the same as the one for constructing de Bruijn sequences with minimal complexity [8]. By making some modifications in the sufficient condition of Theorem 2, we can obtain a stronger result.

*Theorem 3:* If the sufficient condition, stated below, holds for a given  $n$  then there exists a de Bruijn sequence  $S$  of order  $n$  with  $C(S) = 2^{n-1} + n + \sum_{i=\lceil \log n \rceil+1}^{n-3} \alpha_i 2^i$  for any selection of  $\alpha_i \in \{0, 1\}$ .

#### B. The Sufficient Condition

Consider a set  $F(n)$  as defined in Fact 2 and let  $k = 2^{n-\lceil \log n \rceil-3}$ . Then it is possible to choose one state (of size  $n$ ) in each of the sequences  $F(n)$ , designated as the first state of the sequence, and it is possible to arrange the members of  $F(n)$  in pairs  $P_i = (A_i, B_i)$ ,  $1 \leq i \leq 2k$ , so that properties q.1)–q.8) hold.

- q.1) For each pair  $P_i$ ,  $1 \leq i \leq 2k$ , the first state of  $A_i$  is the companion of the first state of  $B_i$ .
- q.2) For each  $i$ ,  $1 \leq i \leq 2k$ ,  $A_i + B_i = A_1 + B_1$ .
- q.3)  $C(A_1 + B_1) = n$ .
- q.4) For each  $i$ ,  $1 \leq i \leq k$ ,  $A_i + A_{i+k} = A_1 + A_{k+1}$ .
- q.5)  $C(A_1 + A_{k+1}) = n$ .
- q.6) For each  $i$ ,  $1 \leq i \leq k$ ,  $C(A_i + B_{k+i}) < n$ .
- q.7) For each  $i$ ,  $1 \leq i \leq k$ ,  $A_i$  and  $A_{k+i}$  have a pair of companion states in position  $2^{\lceil \log(n-1) \rceil} + 1$ .
- q.8) The graph  $(V(n), E(n))$ , where  $V(n) = \{v_i | 1 \leq i \leq k\}$  and  $\{v_i, v_j\} \in E(n)$  iff  $A_i$  and  $A_j$  have a pair of companion states in the same position is a connected graph.

*Example 2:* Let  $n = 9$ . By Fact 2,  $|F(n)| = 32$ , so that there are 32 sequences of length 16 with complexity ten. These sequences are listed below in eight pairs that satisfy q.1)–q.8). It is easy to verify that this arrangement satisfies q.1)–q.7). To check q.8), it is easy to see now that the seven edges implied by  $POC(1, 2) = 4, POC(2, 3) = 1, POC(3, 4) = 5, POC(2, 5) = 3, POC(5, 6) = 6, POC(3, 7) = 2, POC(5, 8) = 7$  form a tree of  $(V(9), E(9))$ , thus validating q.8), as shown in (4) at the bottom of the next page.

Before we prove the theorem we remark that properties q.1)–q.8) imply many other properties, e.g., we have the following result.

*Lemma 3:* For each  $i$ ,  $1 \leq i \leq k$ ,  $A_i + B_{k+i} = A_1 + B_{k+1}$ .

*Proof:*  $A_i + B_{k+i} = A_1 + B_{k+1} + A_1 + B_{k+1} + A_i + B_{k+i}$  and by q.4) we have that  $A_i + B_{k+i} = A_1 + B_{k+1} + A_1 + B_{k+1} + A_i + B_{k+i} + A_i + A_{k+i} + A_1 + A_{k+1} = A_1 + B_{k+1} + (A_1 + A_1) + (A_i + A_i) + (A_{k+1} + B_{k+1} + A_{k+i} + B_{k+i}) = A_1 + B_{k+1}$ .

**Q.E.D.**

*Proof of Theorem 3:* Given an arrangement of a set  $F(n)$  that satisfies q.1)–q.8), let  $(V(n), T)$  denote a tree of  $(V(n), E(n))$ . We join the members of  $F(n)$  to form a single sequence  $S$  by applying Lemma 2 as follows.

First, we form  $S_1$  by joining all the  $A_i$ 's,  $1 \leq i \leq k$ , sequences via the companion pairs that define the edges of  $(V(n), T)$ . Then, we form  $S_2$  by joining all the  $B_i$ 's,  $1 \leq i \leq k$ , sequences via the corresponding companion pairs whose existence is guaranteed by q.2). We form  $S_3$  by joining all the  $A_i$ 's,  $k+1 \leq i \leq 2k$ , sequences via the companion pairs whose existence is guaranteed by q.4) and the edges of  $(V(n), T)$ . Finally, we form  $S_4$  by joining all the  $B_i$ 's,  $k+1 \leq i \leq 2k$ , sequences via the corresponding companion pairs whose existence is guaranteed by q.2) and q.4). We designate the first states of  $S_1$ ,  $S_2$ ,  $S_3$ , and  $S_4$ , to be the first states of  $A_1$ ,  $B_1$ ,  $A_{k+i}$ , and  $B_{k+i}$ , respectively. It is easy to verify that, under this convention, the following holds:

- r.1) Two states occupying the same position in an  $(A_i, B_i)$  pair are also located opposite each other in either  $S_1$  and  $S_2$  (respectively,  $S_3$  and  $S_4$ ).
- r.2) Two states occupying the same position in  $(A_i, A_{k+i})$  (respectively,  $(B_i, B_{k+i})$ ), are also located opposite each other in either  $S_1$  and  $S_3$  (respectively,  $S_2$  and  $S_4$ ).
- r.3) The position of each state in  $S_1$  and  $S_3$  (respectively,  $S_2$  and  $S_4$ ) is congruent to its original  $A_i$ -position (respectively,  $B_i$ -position) modulo  $2^{\lceil \log n \rceil + 1}$ .

We can write  $S_1$  as  $x_1 \hat{S}_1 = x_1 y_1 x_2 y_2 \cdots x_k y_k$ , where  $l(x_i) = 2^{\lceil \log(n-1) \rceil}$ ,  $1 \leq i \leq k$ . The length of the  $y_i$ 's depend whether  $n$  is a power of two or not. If  $n = 2^m$  for some  $m$ , then  $l(y_i) = 3 \cdot 2^{\lceil \log(n-1) \rceil}$ ,  $1 \leq i \leq k$ , and otherwise  $l(y_i) = 2^{\lceil \log(n-1) \rceil}$ ,  $1 \leq i \leq k$ . Similarly, we can write  $S_3$  as  $a_1 \hat{S}_3 = a_1 b_1 a_2 b_2 \cdots a_k b_k$ , where  $l(a_i) = l(x_i)$  and  $l(y_i) = l(b_i)$ ,  $1 \leq i \leq k$ . The first states of  $\hat{S}_1$  and  $\hat{S}_3$  are companions as guaranteed by q.7). As a result, it follows from Lemma 2 that  $S = \hat{S}_1 S_2 x_1 \hat{S}_3 S_4 a_1$  is a de Bruijn sequence, and by q.2), q.4), r.2), and r.3)

$$\hat{S}_1 S_2 x_1 + \hat{S}_3 S_4 a_1 \simeq S_1 S_2 + S_3 S_4 = (A_1 + A_{k+1})^{2k}.$$

Also, by q.5) we have that  $C(A_1 + A_{k+1}) = n$ . Due to this form of  $S$ , it follows directly from the Games and Chan algorithm that  $C(S) = 2^{n-1} + n$ . This is an alternative method for constructing de Bruijn sequences with minimal complexity and the solution if all the  $\alpha_i$ 's are "0's."

$S_1 S_2$  may be written in the form  $x_1 y_1 x_2 y_2 \cdots x_k y_k x_1 z_1 x_2 z_2 \cdots x_k z_k$  as a consequence of q.1), r.1),

and r.3). Similarly,  $S_3 S_4$  may be written in the form  $a_1 b_1 a_2 b_2 \cdots a_k b_k a_1 c_1 a_2 c_2 \cdots a_k c_k$ . Now, note that by q.2), q.4), r.2), and r.3), we have that  $x_i y_i + a_i b_i = x_i z_i + a_i c_i = A_1 + A_{k+1}$ ,  $1 \leq i \leq k$  (let  $\hat{d} \triangleq A_1 + A_{k+1}$ ). Similarly, by q.2), r.2), r.3), and Lemma 3, we have that  $x_i y_i + a_i c_i = x_i z_i + a_i b_i = A_1 + B_{k+1}$ ,  $1 \leq i \leq k$  (let  $\hat{e} \triangleq A_1 + B_{k+1}$ ). Finally, define  $\hat{f} \triangleq \hat{d} + \hat{e}$ . By q.5) we have that  $C(\hat{d}) = n$ , and by q.7) we have that  $C(\hat{e}) < n$ , and hence by Fact 3  $C(\hat{f}) = n$ . Recall that the sequence

$$S = y_1 x_2 y_2 \cdots x_k y_k x_1 z_1 x_2 z_2 \cdots x_k z_k x_1 b_1 a_2 b_2 \cdots a_k b_k a_1 c_1 a_2 c_2 \cdots a_k c_k a_1$$

is a de Bruijn sequence (note that the first  $n-1$  bits of  $y_1 x_2$  and  $b_1 a_2$  are identical). Now, note that since by q.1) and r.1) the first  $n-1$  bits of  $a_i b_i$  and  $a_i c_i$  are identical, it follows that the sequence

$$S'_3 S'_4 = a_1 b_1 a_2 u_2 \cdots a_k u_k a_1 c_1 a_2 v_2 \cdots a_k v_k$$

where  $u_i$  and  $v_i$  are either  $b_i$  or  $c_i$ ,  $u_i \neq v_i$ , includes the same states as  $S_3 S_4$ . Therefore,

$$S' = y_1 x_2 y_2 \cdots x_k y_k x_1 z_1 x_2 z_2 \cdots x_k z_k x_1 b_1 a_2 u_2 \cdots a_k u_k a_1 c_1 a_2 v_2 \cdots a_k v_k a_1$$

is a de Bruijn sequence.  $u_i$  and  $v_i$  should be chosen in such a way that  $C(S) = 2^{n-1} + n + \sum_{i=\lceil \log n \rceil + 1}^{n-3} \alpha_i 2^i$ ,  $\alpha_i \in \{0, 1\}$  (the procedure for choosing them is given below). We now show how the computation of the complexity of  $S'$  with the Games and Chan algorithm should look like. Let  $S' = L(S')R(S')$ , where  $L(S')$  and  $R(S')$  are the left and right halves of  $S'$ , as defined in the Introduction. Define  $T_1 = L(S') + R(S')$ . To obtain the desired complexity of  $S'$  the minimal period of  $T_1$  must be  $2^{r+1}$ . Then, we can write  $T_1 = (T_2)^{2^{n-r-2}}$ , where  $l(T_2) = 2^{r+1}$  and  $T_2$  also have minimal period  $2^{r+1}$ .  $T_2$  is a concatenation of strings of the form  $\hat{d}$  and  $\hat{e}$ . Let  $T_2 = L(T_2)R(T_2)$  and define  $T_3 = L(T_2) + R(T_2)$ . Clearly,  $T_3$  is a concatenation of sequences of the form  $\hat{f}$  and  $0^{2^{\lceil \log n \rceil + 1}}$ . Based on these facts we have the following procedure to generate the de Bruijn sequence  $S'$  with the desired complexity  $2^{n-1} + n + \sum_{i=\lceil \log n \rceil + 1}^{n-3} \alpha_i 2^i$ ,  $\alpha_i \in \{0, 1\}$ , i.e., we just take the steps of the Games and Chan algorithm backwards, starting with the

$$\begin{aligned}
(A_1, B_1) &= ([00101010 \ 10000000], [00101010 \ 01111111]) \\
(A_2, B_2) &= ([11011010 \ 10001111], [11011010 \ 01110000]) \\
(A_3, B_3) &= ([01011010 \ 11110000], [01011010 \ 00001111]) \\
(A_4, B_4) &= ([10100010 \ 11110111], [10100010 \ 00001000]) \\
(A_5, B_5) &= ([00111010 \ 10010000], [00111010 \ 01101111]) \\
(A_6, B_6) &= ([11000110 \ 10010011], [11000110 \ 01101100]) \\
(A_7, B_7) &= ([10011010 \ 11001111], [10011010 \ 00110000]) \\
(A_8, B_8) &= ([11000100 \ 10010001], [11000100 \ 01101110]) \\
(A_9, B_9) &= ([11010101 \ 10000000], [11010101 \ 01111111]) \\
(A_{10}, B_{10}) &= ([00100101 \ 10001111], [00100101 \ 01110000]) \\
(A_{11}, B_{11}) &= ([10100101 \ 11110000], [10100101 \ 00001111]) \\
(A_{12}, B_{12}) &= ([01011101 \ 11110111], [01011101 \ 00001000]) \\
(A_{13}, B_{13}) &= ([11000101 \ 10010000], [11000101 \ 01101111]) \\
(A_{14}, B_{14}) &= ([00111001 \ 10010011], [00111001 \ 01101100]) \\
(A_{15}, B_{15}) &= ([01100101 \ 11001111], [01100101 \ 00110000]) \\
(A_{16}, B_{16}) &= ([00111011 \ 10010001], [00111011 \ 01101110])
\end{aligned} \tag{4}$$

sequence  $\hat{f}$  whose complexity is  $n$ .

```

len := 2⌊log n⌋+1;
j := ⌊log n⌋ + 1;
let r be the largest integer such that αr = 1;
S :=  $\hat{f}$ ; {S is of length len; C(S) = n}
while len <> 2r do
begin
    if αj = 1 then S := Si
        else S := SS;
    len := 2 · len;
    { S is of length len; C(S) = n + ∑i=⌊log n⌋+1j αi2i }
    j := j + 1;
end;
{ S is of length 2r; C(S) = n + ∑i=⌊log n⌋+1r-1 αi2i }.
    
```

Now, let  $t = 2^{r-\lfloor \log n \rfloor - 1}$  and  $S = w_1 w_2 \cdots w_t$ , where  $l(S) = 2^r$  and  $l(w_i) = 2^{\lfloor \log n \rfloor + 1}$ .

```

for i := 1 to 2r-⌊log n⌋-1 do
    if wi = 02⌊log n⌋+1 then ui :=  $\hat{d}$ 
        else {wi =  $\hat{f}$ } ui :=  $\hat{e}$ ;
    
```

```

S :=  $\hat{d}^t u_1 u_2 \cdots u_t$ ;
{ S is of length 2r+1; C(S) = n + ∑i=⌊log n⌋+1r αi2i }
    
```

for  $len := r + 2$  to  $n - 2$  do  $S := SS$ ;

```

{ S is of length len; C(S) = n + ∑i=⌊log n⌋+1r αi2i }
    
```

Now, let  $S = m_1 m_2 \cdots m_k$ .  $\{l(S) = 2^{n-2}; l(m_i) = 2^{\lfloor \log n \rfloor + 1}\}$ .

for  $i := 1$  to  $k$  do

```

    if mi =  $\hat{d}$  then begin ui := bi; vi := ci end
        else {pi =  $\hat{e}$ } begin ui := ci; vi := bi end
    
```

Note, that  $m_1 = \hat{d}$  and hence  $u_1 = b_1$  and  $v_1 = c_1$ . Now, let

$$S'_3 S'_4 = a_1 b_1 a_2 u_2 \cdots a_k u_k a_1 c_1 a_2 v_2 \cdots a_k v_k$$

and by the Games and Chan algorithm

$$S' = y_1 x_2 y_2 \cdots x_k y_k x_1 z_1 x_2 z_2 \cdots x_k z_k x_1 b_1 a_2 u_2 \cdots a_k u_k a_1 c_1 a_2 v_2 \cdots a_k v_k a_1$$

is a de Bruijn sequence with the desired linear complexity.

**Q.E.D.**

#### IV. VALIDITY OF THE SUFFICIENT CONDITION

In this section we show that for every  $n \geq 9$ , there exists a valid set  $F(n)$ . That is, there exists an arrangement for the set  $F(n)$  that satisfies q.1)–q.8). This is done by demonstrating a recursive construction which is similar to the construction in [8]. For the recursive construction we need the  $D$ -morphism operator  $D = E+1$  defined in [16] for de Bruijn graphs and its inverse  $D^{-1}$

as defined by Lempel [16]. When applied to a sequence,  $D$  can be viewed as being equivalent to the operator  $E + 1$ . That is, for  $S = [s_0, s_1, s_2, \dots, s_{t-1}]$

$$DS = (E + 1)S = [s_0 + s_1, s_1 + s_2, \dots, s_{t-2} + s_{t-1}, s_{t-1} + s_0].$$

When applied to individual states,  $D$  effects a two-to-one map from  $B^n$  (the set of all binary  $n$ -tuples) onto  $B^{n-1}$ . Thus,  $D^{-1}$  actually consists of two maps  $D_0^{-1}$  and  $D_1^{-1}$  which, when applied to a sequence  $S = [s_0, s_1, s_2, \dots, s_{t-1}]$  of even weight yield a pair of complementary sequences

$$D_0^{-1}S = \left[ 0, s_0, s_0 + s_1, \dots, \sum_{i=0}^{t-2} s_i \right],$$

$$D_1^{-1}S = \left[ 1, 1 + s_0, 1 + s_0 + s_1, \dots, 1 + \sum_{i=0}^{t-2} s_i \right].$$

while when  $W(S)$  is odd, the images under  $D_0^{-1}$  and  $D_1^{-1}$  are self-dual and are cyclic shifts of one another

$$D_0^{-1}S = \left[ 0, s_0, s_0 + s_1, \dots, \sum_{i=0}^{t-2} s_i, 1, 1 + s_0, 1 + s_0 + s_1, \dots, 1 + \sum_{i=0}^{t-2} s_i \right]$$

and

$$D_1^{-1}S = \left[ 1, 1 + s_0, 1 + s_0 + s_1, \dots, 1 + \sum_{i=0}^{t-2} s_i, 0, s_0, s_0 + s_1, \dots, \sum_{i=0}^{t-2} s_i \right].$$

It also follows from the definition of  $D$  (see [2]) that if  $f(E)S = 0$  and  $E + 1$  is a factor of  $f(E)$  then  $C(DS) = C(S) - 1$ . Hence, by Fact 1, we obtain the following.

**Fact 4:** Let  $l(S)$  be a power of two. Then  $C(D^{-1}S) = C(S) + 1$ , where  $D^{-1}$  stands for either  $D_0^{-1}$  or  $D_1^{-1}$ .

We start with a set  $F(n)$ ,  $n = 2^m$ ,  $m \geq 3$ , arranged to satisfy the sufficient condition of Theorem 2 to obtain de Bruijn sequences with minimal complexity  $2^{n-1} + n$ . The existence of such valid set was proven in [8]. This set does not have to satisfy the sufficient condition of Theorem 3. The next step is to find a valid set  $F(n+1)$  which satisfies the sufficient condition of Theorem 3. This is done with the following construction.

**Construction 1:** Given a positive integer  $n = 2^m$ ,  $m \geq 3$  with the set  $F(n)$  which satisfies the sufficient condition of Theorem 2 with  $g = 0$ , construct a valid set  $F(n+1)$  by applying  $F(n+1) = D_0^{-1}F(n) \cup D_1^{-1}F(n)$ , where  $D_i^{-1}F(n) = \cup_{S \in F(n)} D_i^{-1}S$ ,  $i = 0, 1$ .

**Lemma 4 [8]:** The set  $F(n+1)$  obtained via Construction 1 satisfies the defining properties given in Fact 2.

To continue with the arrangement of the set  $F(n)$ , we need the following fact and lemmas.

**Fact 5 [2]:** Let  $S$  be a sequence of length  $2^{m+1}$ . Then  $C(S) = 2^m + 1$  if and only if  $S = [X\bar{X}]$  for some  $X$ .

**Lemma 5 [8]:** If the first states of  $S_1$  and  $S_2$  are companions, then the first states of  $D_i^{-1}S_1$  and  $D_i^{-1}S_2$  are companions,  $i \in \{0, 1\}$ .

**Lemma 6 [8]:** The  $m$ th state of  $D_i^{-1}S_1$  is the companion of either the  $m$ th state of  $D_i^{-1}S_2$  or the  $m$ th state of  $D_i^{-1}S_2$ ,  $i \in \{0, 1\}$  if and only if the  $m$ th states of  $S_1$  and  $S_2$  are companions.

*Lemma 7 [8]:* If  $l(S_1) = l(S_2)$  and  $W(S_1) \equiv W(S_2) \pmod{2}$ , then  $D_0^{-1}(S_1 + S_2) = D_i^{-1}S_1 + D_i^{-1}S_2$ ,  $i \in \{0, 1\}$ .

Now, given the pair  $P_i = (A_i, B_i)$  of  $F(n)$  it follows by q.1) and Lemma 5, that the pairs  $P_{i0} = (D_0^{-1}A_i, D_0^{-1}B_i)$  and  $P_{i1} = (D_1^{-1}A_i, D_1^{-1}B_i)$  of  $F(n+1)$  satisfy q.1). By Lemma 7, Construction 1 preserves q.2) and, by Fact 4 and Lemma 7, Construction 1 preserves q.3). Now, for either  $j = 0$  or  $j = 1$  let  $P'_i = (D_j^{-1}A_i, D_j^{-1}B_i)$  and  $P'_{k+i} = (D_{1-j}^{-1}B_i, D_{1-j}^{-1}A_i)$ . By Fact 5,  $A_i = [X\bar{X}]$  for some  $X$  and by q.1),  $B_i = [X'\bar{X}']$ . Therefore,  $D_j^{-1}A_i + D_{1-j}^{-1}B_i = [1^{2^{\lfloor \log n \rfloor}} 0^{2^{\lfloor \log n \rfloor}}]$  and since  $C([1^{2^{\lfloor \log n \rfloor}} 0^{2^{\lfloor \log n \rfloor}}]) = 2^{\lfloor \log n \rfloor} + 1$ , it follows that q.4), q.5), and q.7) are satisfied.  $D_0^{-1}A_i + D_1^{-1}A_i = D_0^{-1}B_i + D_1^{-1}B_i = [1^{2^{\lfloor \log n \rfloor + 1}}]$  and hence q.6) is satisfied. By Lemma 6, q.8), and Construction 1, the existence of a tree  $T(n)$  for  $F(n)$ , implies the existence of a corresponding pair of trees  $T_1(n+1)$  and  $T_2(n+1)$  isomorphic to  $T(n)$ . In the construction of these two trees we will decide for each  $i$ , which  $j$  is taken for  $P'_i = (D_j^{-1}A_i, D_j^{-1}B_i)$ . Thus, q.8) is satisfied.

*Construction 2:* Given a positive integer  $n \geq 9$  which is not a power of 2, construct a valid set  $F(n+1)$  by repeatedly applying the recursion  $F(t+1) = D_0^{-1}F(t) \cup D_1^{-1}F(t)$ , beginning with the valid set  $F(2^{\lfloor \log n \rfloor + 1})$  obtained by Construction 1.

The proof that the set  $F(n+1)$  of Construction 2 satisfies q.1)–q.8) is similar to the one in [8], with some additional proofs for the properties not needed in [8] as given after Construction 1.

*Theorem 4:* For every  $n \geq 4$  there exists a de Bruijn sequence of order  $n$  and complexity  $2^{n-1} + n + \sum_{i=\lfloor \log n \rfloor + 1}^{n-3} \alpha_i 2^i$  for any selected  $\alpha_i \in \{0, 1\}$ .

The following two theorems, from [2] and [4], respectively, are useful for proving our main result.

*Theorem 5:*  $\gamma(2^{n-1} + c, n) \geq 2\gamma(c, n-1)$ .

*Theorem 6:*  $\gamma(2^{n-1} + 2^{n-2}, n) > 0$ .

Combining Theorems 4–6 and computer search which shows that conjecture 1 is true for  $n = 7$  and  $n = 8$ , we have the following theorem.

*Theorem 7:* For any given  $t$ ,  $2^{n-1} + n \leq t \leq 2^n - 2^{\lfloor \log n \rfloor + 1}$  there exists a de Bruijn sequence of order  $n$  and linear complexity  $c$ , where  $t \leq c < t + 2^{\lfloor \log n \rfloor + 1}$ .

## V. CONCLUSIONS

We have proven that in each interval of length  $2^{\lfloor \log n \rfloor + 1}$ , between  $2^{n-1} + n$  and  $2^n - 1$ , there is some integer  $c$  and de Bruijn sequences of length  $2^n$  and linear complexity  $c$ . It is still desired to prove that for every  $c$  between  $2^{n-1} + n$  and  $2^n - 1$ ,  $c \neq 2^{n-1} + n + 1$  there is a de Bruijn sequences of length  $2^n$  and linear complexity  $c$ .

Another direction in this area is to find the complexity distribution of nonbinary de Bruijn sequences. A pioneer work in this direction can be found in [1]. More results can be found in [14].

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## Every Binary $(2^m - 2, 2^{2^m - 2 - m}, 3)$ Code Can Be Lengthened to Form a Perfect Code of Length $2^m - 1$

Tim Blackmore

**Abstract**—We answer a problem posed by Etzion and Vardy by showing that a binary code of length  $N = 2^m - 2$  with  $2^{N-m}$  codewords and minimum distance three can always be lengthened to form a perfect code of length  $2^m - 1$ .

**Index Terms**—Lengthened code, perfect code.

## I. INTRODUCTION

We write  $\mathbb{F}_2$  for the binary field and  $\mathbb{F}_2^n$  for the set of binary vectors of length  $n$ . Throughout, by distance we mean Hamming distance. For  $v, w \in \mathbb{F}_2^n$  and a code  $C \subseteq \mathbb{F}_2^n$  we write  $d(v, w)$  for the distance

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