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Optimal constant weight codes over Z_k and generalized designs[1]

Tuvi Etzion

Computer Science Department, Technion — Israel Institute of Technology, Haifa 32000, Israel

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Abstract

We consider optimal constant weight codes over arbitrary alphabets. Some of these codes are derived from good codes over the same alphabet, and some of these codes are derived from block design. Generalizations of Steiner systems play an important role in this context. We give several construction methods for these generalizations. An interesting class of codes are those which form generalized Steiner systems and their supports form ordinary Steiner systems. Finally, we consider classes of codes which are MDS constant weight codes.

1. Introduction

Recently, lot of research was done on constructions of nonbinary codes, which are either not linear or with elements not over some Galois field, e.g. [10, 19]. As in the binary case constant weight codes might have an important role in these codes. There are many well-known binary codes which contain optimal constant weight codes as subcodes, or the best-known constant weight codes as subcodes. Some of these optimal codes form t-designs. This was the first motivation for this paper. One of the good examples for codes which contain t-designs is the binary Hamming code. It is well known that the codewords of weight 3 in the binary Hamming code form a Steiner triple system, and the codewords of weight 4, in the extended binary Hamming code form a Steiner quadruple system. Obviously, we can ask what can be said about the codewords of weights 3 and 4 in the nonbinary Hamming code. The codewords of weight 7 in the binary Golay code form a Steiner system $S(4,7,23)$ and the codewords of weight 8 in the extended binary Golay code form a Steiner system $S(5,8,24)$. The supports (the nonzero coordinates) of the codewords of weight 5 in the

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ternary Golay code form the Steiner system $S(4,5,11)$, and the supports of the codewords of weight 6 in the ternary extended Golay code form the Steiner system $S(5,6,12)$. But, if we consider the codewords rather than their supports, what can be said on the constant weight codes formed from the codewords of weights 5 and 6 in the ternary Golay code? MDS codes over $GF(q)$ were extensively studied, and the same is true for their combinatorial equivalent orthogonal arrays. The codewords of minimum weight in an orthogonal array form an optimal constant weight code. For length n and weight w there are $\binom{n}{w}(q-1)$ codewords in the code, and its minimum Hamming distance is w. Codes with these parameters will be called MDS constant weight codes. Can we give constructions for these MDS constant weight codes, with parameters such that an orthogonal array with the equivalent parameters cannot exist?

We will give a known generalization of t-design and as a special case we modify this generalization for Steiner systems. Some generalized Steiner systems will be constructed from known nonbinary codes. But, as in the binary case we will try to find constructions for generalized Steiner systems which cannot be obtained from codes over nonbinary alphabet. As the most interesting cases we will consider generalized Steiner triple systems, generalized Steiner quadruple systems, and generalized 2-designs. Our results can be compared with some similar designs as the H-design and the group divisible design.

The rest of the paper is organized as follows. In Section 2, we give the definition for the generalized designs, and we will find the necessary conditions for the existence of generalized designs. In Section 3, we will consider the trivial cases, and we will see that cases which seems 'trivial' are not always trivial. In Section 4, we will consider generalized Steiner triple systems over arbitrary alphabet, and we give a complete solution over alphabet with 3 and 4 letters. In Section 5, we will consider generalized Steiner quadruple systems and in Section 6 we consider generalized 2-designs. In Section 7, we will consider the designs derived from the ternary Golay code and we consider an interesting question of double designs, which are codes for which their codewords form a generalized Steiner system, and the supports of the codewords form a Steiner system. In Section 8, we will discuss MDS constant weight codes. We conclude in Section 9 with various open problems which are derived from our discussion.

2. Generalized t-designs

The definition for a generalized t-design over an arbitrary alphabet will be an obvious generalization of the known definition of t-design. Let X be a v-set, whose elements are called *points*. A t-design is a collection of distinct w-subsets (called *blocks*) of X with the property that any t-subset of X is contained in exactly λ blocks. When $\lambda = 1$ the design is a Steiner system $S(t,w,v)$. This is also a binary constant weight code of length v, weight w, and minimum Hamming distance $2(w-t+1)$.

A generalization of t-design is the H-design which was introduced by Hanani [11] (the notation of H-design is due to Mills [17]). An $H(m,q,w,t)$ design is a triple (X,G,B), where X is a set of points whose cardinality is mq, and $G=\{G_1,G_2,\dots,G_m\}$ is a partition of X into m sets of cardinality q; the members of G are called groups. A *transverse* of G is a subset of X that meets each group in at most one point. The set B contains w-element transverses of G, called *blocks*, with the property that each t-element transverse of G is contained in precisely one block. When $q=1$ then $H(m,1,w,t)$ is just a Steiner system $S(t,w,m)$. When $w=m$ this $H(m,q,w,t)$ is equivalent to orthogonal array $OA(t,w,q)$. An $OA(t,n,q)$ is an $q^t \times n$ matrix M, with entries from a set with q elements, such that the matrix generated by any t columns contains each ordered t-tuple exactly once as a row.

From the H-design $H(m,q,w,t)$ we can form a constant weight code as follows. Let $G_i=\{\alpha_{1i},\alpha_{2i},\dots,\alpha_{qi}\}$, where $0\notin G_i$; the code has a codeword for each block. The length of each codeword is m and the code is over an alphabet with $q+1$ letters. Assume $\{a_1,a_2,\dots,a_w\}$ is a block in B (this block will be denoted by $\{[i_1,j_1],[i_2,j_2],\dots,[i_w,j_w]\}$, where $a_s=\alpha_{j_s i_s}$). We form the following codeword: coordinate i, $1\leqslant i\leqslant m$, has value j, $1\leqslant j\leqslant q$, if for some r, $1\leqslant r\leqslant w$, $a_r=\alpha_{ji}$; all the other coordinates have zero value. The minimum Hamming distance of the code is at least $1+w-t$, which is usually the actual distance in the known constructions for these designs. We will usually call our codewords as blocks, but sometimes they will be denoted by m-tuples.

In the code which is related to $H(m,q,w,t)$ we want that the minimum Hamming distance will be large as possible. Since in each t coordinates we must have all q^t possibilities of alphabet letters, it follows that for $q>1$ there are two codewords for which the first t coordinates are nonzeros, and they differ in exactly one coordinate in these t coordinates. The minimum Hamming distance between these two codewords is at most $1+2(w-t)$. An $H(m,q,w,t)$ which forms a code with minimum Hamming distance at least $1+2(w-t)$ will be called a generalized Steiner system $GS(t,w,m,q)$. Note, that this is a generalization of a Steiner system, since in a Steiner system the minimum Hamming distance is $2+2(w-t)$. Of course, $H(m,q,w,t)$ might have any distance between $1+w-t$ and $1+2(w-t)$. An $H(m,q,w,t)$ with minimum Hamming distance d will be denoted by $GS_d(t,w,m,q)$.

What are the necessary conditions for the existence of $GS_d(t,w,n,k)$? Note, that if Q is a $GS_d(t,w,n,k)$ then the set of blocks $\{\{x_1,x_2,\dots,x_{w-i}\}:\{[0,1],[1,1],\dots,[i-1,1],x_1,x_2,\dots,x_{w-i}\}\in Q\}$ forms a $GS_d(t-i,w-i,n-i,k)$; for $i=1$ this system is called the derived system. Now, in a $GS_d(t-i,w-i,n-i,k)$ each block covers $\binom{w-i}{t-i}$ subsets of size $t-i$. We have to cover a total of $\binom{n-i}{t-i}k^{t-i}$ subsets of size $t-i$. Hence, the first necessary condition for the existence of $GS_d(t,w,n,k)$ is that for each $i, 0\leqslant i\leqslant t-1$, $\binom{w-i}{t-i}$ divides $\binom{n-i}{t-i}k^{t-i}$.

H-designs were studied in a few contexts. Hartman et al. [13] have studied $H(n,k,4,3)$ designs in the context of covering triples by quadruples. Orthogonal array were studied by many authors, and they are used in more places in this paper. They are also called MDS codes and will be discussed in Section 8. Raghavarao [20] gives the known results related to designs and in [16] we can find the relevant results for codes. $H(n,k,w,2)$ were extensively studied; they are a special case of group divisible designs, and will be denoted by $GDD(n,k,w,2)$. Hanani [12] has proved that the necessary conditions for the existence of $H(n,k,3,2)$ are also sufficient. But, in almost all constructions for $H(n,k,w,t)$ the designs are not generalized Steiner systems, i.e., they do not have the required minimum Hamming distance. In the sections which follow we will give constructions for generalized Steiner system. We will use Z_k for the additive group *modulo* k consisting of the integers $\{0,1,\ldots,k-1\}$ and ZZ_k for the additive group *modulo* k consisting of the integers $\{1,2,\ldots,k\}$.

3. Trivial generalized Steiner systems

In this section we will consider the cases which seems to be trivial for generalized Steiner systems. We consider generalized Steiner systems $GS(t,w,n,k)$.

The first trivial case is when $t = w$. A $GS(t,t,n,k)$ is just a collection of all possible $\binom{n}{t}t^k$ blocks of size t over an alphabet with $k+1$ letters. The code has minimum Hamming distance 2 if $k=1$ and minimum Hamming distance 1 if $k>1$.

For $k=1$ the next trivial case is when $w=n$, where the design consists of one block which contains all the points. As said before, for $k>2$, $H(n,k,n,t)$ is equivalent to an orthogonal array $OA(t,n,k)$. If we let the set of size k consists of the elements $\{1,2,\ldots,k\}$ we obtain the code which form $GS_d(t,n,n,k)$, where $d=1+n-t$. But, although the constant weight codes which are obtained are optimal they are not generalized Steiner systems $GS(t,n,n,k)$ since the minimum Hamming distance is $1+n-t$ rather than $1+2(n-t)$ as needed.

For $k=1$ the last trivial case is when $t=1$, where the $GS(t,w,n,k)$ exists if w divides n, and it consists of n/w parallel blocks (a set of blocks are called parallel if the intersection between any two of them is empty and there union is all the set of points). For $k>1$ and $t=1$ the situation is much more complicated. The necessary condition for the existence of a $GS(1,w,n,k)$ is that w divides nk. Given k and w, we believe that there exists an n_0 such that this condition is sufficient for any $n \geqslant n_0$. Let $A(n,d,w)$ be the maximum size of a binary code with constant weight w and minimum Hamming distance d. It is easy to prove the following theorem.

Theorem 1. *Given $k>1$ and w, if $GS(1,w,n,k)$ exists then $n \leqslant wA(n,2w-2,w)/k$ and $n \geqslant 1+(w-1)k$.*

Proof. Since the minimum Hamming distance of the constant weight code derived from $GS(1,w,n,k)$ is $2w-1$, it follows that the supports of the code form a binary constant weight code of weight w and distance $2w-2$, and hence the size of $GS(1,w,n,k)$ is at most $A(n,2w-2,w)$, and the number of nonzero entries in the code is at most $wA(n,2w-2,w)$. Since each coordinate must have exactly one codeword which each of the k nonzero alphabet letters we must have $n \leqslant wA(n,2w-2,w)/k$.

Now, there are k codewords which share the first coordinate, each one having a different letter in the first coordinate, and other than the first coordinate they do not share any of the other $w - 1$ coordinates. Hence, we have $n \geqslant 1 + (w - 1)k$. □

A related constructive result is the following theorem

Theorem 2. *A generalized Steiner system $GS(1, w, n, k)$ exists if and only if there exists a binary constant weight code of length n, weight w, and minimum Hamming distance $2w - 2$ with M codewords, such that each coordinate has $k = wM/n$ nonzeros entries.*

Proof. Given a generalized Steiner system $GS(1, w, n, k)$ we form the binary constant weight code from the supports of the codewords of $GS(1, w, n, k)$. Given a binary constant weight code of length n, weight w, and minimum Hamming distance $2w - 2$ with M codewords, such that each coordinate has $k = wM/n$ nonzeros entries, we generate $GS(1, w, n, k)$ by assigning the k nonzero alphabet letters arbitrarily to the k nonzero entries in each coordinate, in a way that each coordinate will have each one of the k nonzero alphabet letters. □

This theorem has two simple corollaries. The first corollary involves cyclic constant weight codes. We can use the fact that in a cyclic code all coordinates have the same number of ones.

Corollary 1. *Given a cyclic code of length n, with constant weight w, minimum Hamming distance $2w - 2$, and M codewords, then there exists a $GS(1, w, n, k)$, where $k = wM/n$.*

Corollary 2. *If a Steiner system $S(2, w, n)$ exists then there exists a generalized Steiner system $GS(1, w, n, k)$, where $k = (n - 1)/(w - 1)$.*

For the known Steiner systems $S(2, w, n)$ the reader is referred to [2] (we will also mention them in Sections 4 and 6). Now, we will give a few constructions for generalized Steiner systems $GS(1, w, n, k)$ (there are many more, but we choose only a few important ones). First, we give a few simple results. The first theorem needs no proof.

Theorem 3. *If a $GS(1, w, n_1, k)$ and a $GS(1, w, n_2, k)$ exist, then there exists a $GS(1, w, n_1 + n_2, k)$.*

Next, we use the dual design (exchanging blocks and points) to obtain the following theorem.

Theorem 4. *A generalized Steiner system $GS(1, w, n, k)$ exists if and only if a generalized Steiner stystem $GS(1, k, nk/w, w)$ exists.*

If $w = 2$ then $A(n,2,2) = \binom{n}{2}$ and by Theorem 1 we must have $k \leqslant n-1$. We will prove that if $k \leqslant n-1$ and nk is even then $GS(1,2,n,k)$ exists. If n is even we take k disjoint 1-factors on K_n and apply Theorem 2. If n is odd we use near-1-factorization $F = \{F_0, F_1, \ldots, F_{n-1}\}$ of K_n, such that in F_i, vertex i is isolated and $F_{n-1} = \{\{i, i+1\}$: i *even*$\}$. For each even k we take the first k near-1-factors of F and add the set $\{\{i, i+1\}$: i *even* $i < k\}$ from F_{n-1} and apply Theorem 2.

If $w = 3$ the solution becomes more complicated. If $n \equiv 3 \pmod 6$, by Theorem 1 we have $k \leqslant (n-1)/2$, and we can use k parallel classes of a resolvable Steiner triple system [2] of order n and apply Theorem 2 on this system. For other values of n we can use Corollary 1 on cyclic constant weight codes of weight 3. For information on these codes the reader is referred to [3, 8, 5, 15, 18]. For other weights we will also try to use cyclic constant weight codes, resolvable Steiner systems, and appropriate constant weight codes, and to apply Theorem 2 and Corollaries 1 and 2. For information on these codes the reader is referred to [1, 3, 6, 5, 9, 8, 14, 24].

For an array A, let $A_{i,j}$ denote the value of A in row i, column j. An $m \times n$ array A is called an *array of differences* if all the elements are from Z_q, and for any $1 \leqslant i1 \leqslant i2 \leqslant m$ and $1 \leqslant j1 < j2 \leqslant n$, we have $A_{i1,j2} - A_{i1,j1} \not\equiv A_{i2,j2} - A_{i2,j1} \pmod q$. Given an $m \times n$ array of differences A we form the following system on $Z_n \times Z_q$ over Z_{m+1}: For each row $(a_{i,1}, a_{i,2}, \ldots, a_{i,n})$ of A and $j \in Z_q$ we form the block

$$\{[(0, j + a_{i,1}), i], [(1, j + a_{i,2}), i], \ldots, [(n-1, j + a_{i,n}), i]\}$$

It is easily verified that by this construction we form a $GS(1, n, nq, m)$.

The question of constructing array of differences is interesting for itself. It is related to the question of constructing another combintorial structure, a difference array. An $n \times m$ array A is called a *difference array* if all the elements are from Z_n, and for any $1 \leqslant i1 < i2 \leqslant n$ and $1 \leqslant j1 < j2 \leqslant m$, we have $A_{i1,j2} - A_{i1,j1} \not\equiv A_{i2,j2} - A_{i2,j1} \pmod n$. It is easy to verify in a similar way to the constructions of [3] that an $n \times m$ difference array exists if and only if an $m \times n$ array of differences with elements from Z_n exists. Constructions for difference arrays are given in [5, 3, 9].

As a conclusion from all the discussion in this section we infer that many cases of $GS(t, w, n, k)$ which seem to be trivial to construct, from their written parameters, are not at all trivial to construct.

4. Generalized Steiner triple systems

In a Steiner triple system each pair is covered exactly by one triple, or in other words, each word with Hamming weight 2 is covered by exactly one codeword of weight 3. The same can be said on word of weight 3 in the Hamming code over $GF(q)$. Let $n = (q^m - 1)/(q - 1)$. Since the Hamming code of length n over $GF(q)$ is perfect and includes the zero codeword, it follows that each word of length n and Hamming weight 2, over $GF(q)$, is covered by exactly one codeword of weight 3 in the Hamming code. This is of course a generalized Steiner triple system $GS(2, 3, n, q-1)$. In this section we will give various constructions for generalized Steiner triple systems $GS(2,3,n,k)$. We start with a complete solution for alphabet with four letters, i.e., $k = 3$, since this is the most simple case. We continue with some general solutions for all alphabets, and then we will finish with the complicated complete solution for alphabet with three letters, i.e., $k = 2$. The proofs for most of the constructions are similar in some sense to traditional proofs in design theory and hence they will be omitted. From the necessary condition of Section 2 and the simple observation that $n > k$, we infer the following theorem which is also derived for $GDD(n,k,3,2)$ by Hanani [12].

Theorem 5. *A necessary condition for the existence of $GS(2,3,n,k)$ is that*

1. *If $k \equiv 0 \pmod 6$ then $n > k$.*
2. *If $k \equiv 3 \pmod 6$ then $n \equiv 1 \pmod 2$ and $n > k$.*
3. *If $k \equiv 2$ or $4 \pmod 6$ then $n \equiv 0$ or $1 \pmod 3$ and $n > k$.*
4. *If $k \equiv 1$ or $5 \pmod 6$ then $n \equiv 1$ or $3 \pmod 6$ and $n > k$.*

Construction A. If $n \equiv 1$ or $5 \pmod 6$, $n \geqslant 5$, for each $\alpha \in ZZ_3$ and each $r,s,t \in Z_n$ such that $2r \equiv s + t \pmod n$ we form the block

$$\{[r,\alpha],[s,\alpha+1],[t,\alpha+1]\}.$$

Theorem 6. *Construction* A *forms a generalized Steiner triple system $GS(2,3,n,3)$.*

Proof. Obviously, the equation $2r \equiv s + t \pmod n$ has a unique solution for any pair $\{r,s\},\{r,t\},\{s,t\}$ such that $r,s,t \in Z_n$. Hence, each pair is covered by exactly one triple and we only have to prove that the corresponding code has minimum distance 3. Two codewords which share two coordinates have minimum distance 3 since each pair is covered by exactly one triple. Two codewords which share the same three coordinates can have distance 2 only if there is a solution to the equations $2r \equiv s + t \pmod n$ and $2s \equiv r + t \pmod n$. A solution implies that $3r \equiv 3s \pmod n$ which is impossible in our case since $n \equiv 1$ or $5 \pmod 6$. $\square$

Construction B. Given a $GS(2,3,m,3)$ Q on Z_m we construct the following blocks to produce a triple system on $Z_m \times Z_3$: For each block $\{[i,\alpha],[j,\beta],[r,\gamma]\} \in Q$ and each $a \in Z_3$ we form the block

$$\{[(i,a),\alpha],[(j,a),\beta],[(r,a),\gamma]\}.$$

For each $i,j,r \in Z_m$ such that $i + j + r \equiv 0 \pmod m$ and each $\alpha \in ZZ_3$ we form the blocks

$$\{[(i,0),\alpha],[(j,1),\alpha],[(r,2),\alpha+1]\},$$

$$\{[(i,0),\alpha],[(j,1),\alpha+1],[(r+1,2),\alpha]\},$$

$$\{[(i,0),\alpha+1],[(j,1),\alpha],[(r+2,2),\alpha]\}.$$

Theorem 7. *Construction* B *forms a generalized Steiner triple system* $GS(2,3,n,3)$.

Theorems 5–7, and the generalized Steiner triple system $GS(2,3,9,3)$ given in the appendix implies

Theorem 8. *Generalized Steiner triple system* $GS(2,3,n,3)$ *exists if and only if* n *is ood and greater than* 3.

Next, we present three general constructions for every alphabet.

Construction C. Assume Q is an $GDD(m,k,3,2)$ on Z_m, and R is an $GS(2,3,n,k)$ on Z_n.

We form the following triple system on $Z_m \times Z_n$: For each block $\{[a,\alpha],[b,\beta],[c,\gamma]\} \in R$ and each $i \in Z_m$ we form the block

$$\{[(i,a),\alpha],[(i,b),\beta],[(i,c),\gamma]\}.$$

For Q, we partition the blocks of the form $\{[i,\alpha],[j,\beta],[h,\gamma]\}$ into t sets $S_0, S_1, \ldots, S_{t-1}$ such that $t \leqslant n$ and the minimum distance in S_r, $r \in Z_t$ is 3. For each block of Q, $\{[i,\alpha],[j,\beta],[h,\gamma]\} \in S_r$ and each $a,b \in Z_n$ we form the block

$$\{[(i,a),\alpha],[(j,b),\beta],[(h,a+b+r),\gamma]\}.$$

Construction D. Assume Q is an $GDD(m,k,3,2)$ on Z_m, and R is an $GS(2,3,n,k)$ on $Z_n \cup \{A\}$. We form the following triple system on $Z_m \times Z_n \cup \{A\}$.

For each block $\{[a,\alpha],[b,\beta],[c,\gamma]\} \in R$ and each $i \in Z_m$ we form the block

$$\{[(i,a),\alpha],[(i,b),\beta],[(i,c),\gamma]\}.$$

For each block $\{[a,\alpha],[b,\beta],[A,\gamma]\} \in R$ and each $i \in Z_m$ we form the block

$$\{[(i,a),\alpha],[(i,b),\beta],[A,\gamma]\}.$$

For Q, we partition the blocks of the form $\{[i,\alpha],[j,\beta],[h,\gamma]\}$ into t sets $S_0, S_1, \ldots, S_{t-1}$ such that $t \leqslant n$ and the minimum distance in S_r, $r \in Z_t$ is 3. For each block of Q, $\{[i,\alpha],[j,\beta],[h,\gamma]\} \in S_r$ and each $a,b \in Z_n$ we form the block

$$\{[(i,a),\alpha],[(j,b),\beta],[(h,a+b+r),\gamma]\}.$$

In Constructions C and D, we need to use a partition of $GDD(m,k,3,2)$ into codes with minimum Hamming distance 3. Since there cannot be more than k^2 codewords which share the same pair of coordinates we know that the maximum number of sets in the partition is k^2. In the sequel, we will need such partitions for $GDD(3,2,3,2)$ and $GDD(6,2,3,2)$. In $GDD(3,2,3,2)$ we have 4 blocks, $\{[0,1],[1,1],[2,1]\}$, $\{[0,2],[1,2],[2,1]\}$, $\{[0,2],[1,1],[2,2]\}$, and $\{[0,1],[1,2],[2,2]\}$. The partition include 4 sets, each one has one block. In $GDD(6,2,3,2)$ there are 20 blocks. It can be partitioned into two sets. The first set can have the following 17 blocks:

$\{[0,2],[2,2],[4,2]\}$, $\{[1,2],[2,1],[3,2]\}$, $\{[3,2],[4,1],[5,2]\}$, $\{[0,1],[1,2],[5,2]\}$, $\{[0,2],[3,2],[5,1]\}$, $\{[1,2],[3,1],[4,2]\}$, $\{[1,1],[2,2],[5,2]\}$, $\{[0,2],[1,2],[4,1]\}$, $\{[0,1],[2,2],[3,2]\}$, $\{[2,1],[4,2],[5,2]\}$, $\{[0,1],[4,2],[5,1]\}$, $\{[0,2],[1,1],[2,1]\}$, $\{[2,2],[3,1],[4,1]\}$, $\{[0,1],[1,1],[3,1]\}$, $\{[0,1],[2,1],[4,1]\}$, $\{[1,1],[4,1],[5,1]\}$, and $\{[2,1],[3,1],[5,1]\}$. The second set has the following 3 blocks: $\{[1,2],[2,2],[5,1]\}$, $\{[1,1],[3,2],[4,2]\}$, and $\{[0,2],[3,1],[5,2]\}$.

Construction E. Assume Q is an $S(2,v,n)$ on Z_n and R is an $GS(2,3,v,k)$ on Z_v. We form the following triple system: For each $\{x_0,x_1,\ldots,x_{v-1}\} \in Q$ such that $x_i < x_{i+1}$ and each $\{[i,\alpha],[j,\beta],[r,\gamma]\} \in R$ we form the block

$$\{[x_i,\alpha],[x_j,\beta],[x_r,\gamma]\}.$$

Theorem 9. *Constructions* C, D *and* E *form a generalized Steiner triple system* $GS(2,3,n,k)$.

As said before, the codewords of an Hamming code of length n over $GF(q)$, q a prime power, is perfect, include the zero codewords, and hence the codewords of weight 3 form a generalized Steiner triple system $GS(2,3,n,q-1)$. The Hamming codes exist for each $n=(q^m-1)/(q-1)$. Actually, we only need the first value of n, i.e., $n=(q^2-1)/(q-1)=q+1$ since $(q^m-1)/(q-1)=((q^{m-1}-1)/(q-1))q+1$ and Construction D can be applied to obtain the generalized Steiner triple systems with the other parameters. Also, the recursion can be applied starting with $GS(2,3,q+1,q-1)$ and applying constructions C and D to obtain other generalized Steiner triple systems. We also know that the necessary condition for the existence of $GDD(m,k,3,2)$ is also sufficient. Hence, when we have a suitable partition, as required by Constructions C and D, we will obtain more parameters on which Constructions C and D can be applied. As for Construction E, we know that for $k \geqslant 4$ it can be applied with $S(2,v,n)$, $v \geqslant 6$. It is well known [2] that $S(2,6,n)$ exists for $n \equiv 1$ or $6 \pmod{15}$, $n \neq 16,21,36$ with possible exceptions listed in p. 642 of [2]. For $v \geqslant 7$, $S(2,v,n)$ exists for $n \equiv 1 \pmod{v-1}$, $n(n-1) \equiv 0 \pmod{v(v-1)}$ and n sufficiently large.

Now, we turn to the interesting case of an alphabet with three letters. It was easily observed that $GS(2,3,6,2)$ cannot exists, but apart from this we will show that the necessary condition is also sufficient, i.e., $GS(2,3,n,2)$ exists for all $n \equiv 0$ or $1 \pmod 3$, $n \geqslant 4$, and $n \neq 6$. First, we need two more constructions.

Construction F. If $n \equiv 1$ or $5 \pmod 6$, $n \geqslant 5$, we form the following triple system on $Z_n \times Z_3$: For each $r,s,t \in Z_n$ such that $2r \equiv s+t \pmod n$ and each $i \in Z_3$ we form the block

$$\{[(r,i),2],[(s,i),1],[(t,i),1]\}.$$

For each $r, s \in Z_n$ we form the blocks

$$\{[(r,0),1],[(s,1),1],[(r-s+1,2),2]\},$$

$$\{[(r,0),1],[(s,1),2],[(r-s+2,2),1]\},$$

$$\{[(r,0),2],[(s,1),1],[(r-s+3,2),1]\}.$$

Let $F = \{F_0, F_1, \ldots, F_{n-1}\}$ be a near-one-factorization of K_n on the vertices Z_n, such that in F_j vertex j is isolated. For each $i \in Z_3$, $j \in Z_n$ and $\{r,s\} \in F_j$ we form the block

$$\{[(j,i),2],[(r,i+1),2],[(s,i+1),2]\}.$$

For each $j \in Z_n$ we form the block

$$\{[(j,0),2],[(j,1),2],[(j,2),2]\}.$$

For Construction G which follows we need a *good 1-factorization* as defined by Rosa [21]. A set of k edge-disjoint 1-factors of the complete graph K_{2n} is said to form a *k-good set* of 1-factors if their union is a bipartite graph. Obviously, any two edge-disjoint 1-factors of K_{2n} form a 2-good set. A 1-factorization of K_{2n} is said to be *good 1-factorization* of K_{2n} if it contains a 3-good set of 1-factors. Rosa [21] proved that a good 1-factorization of K_{2n} exists if and only if $n \geqslant 4$.

Construction G. Assume Q is a $GS(2,3,n,2)$ on Z_n, $n \equiv 1$ or $3 \pmod 6$. We construct a triple system on $Z_n \times \{0\} \cup Z_{n+1} \times \{1\}$: For each block $\{[a,\alpha],[b,\beta],[c,\gamma]\} \in Q$ we form the block

$$\{[(a,0),\alpha],[(b,0),\beta],[(c,0),\gamma]\}.$$

Let $F = \{F_0, F_1, F_2, \ldots, F_{n-1}\}$ be a good-1-factorization of K_{n+1}, where $\{F_0, F_1, F_2\}$ forms a 3-good set.

For each pair of 1-factors $\{F_a, F_b\}$, where either $a \geqslant 3$ is odd and $b = a+1$, or $(a,b) \in \{(0,1),(2,3),(0,2),(1,4),(1,2),(0,3)\}$, we order the elements of Z_n in cyclic sequences $s_0, s_1, \ldots, s_{r-1}$ such that $\{s_m, s_{m+1}\}$ (subscripts taken modulo r) is contained in F_a if m is even and in F_b if m is odd. Furthermore, let A and B be the two sides of the bipartite graph formed from the union of the edges of F_0, F_1, and F_2. We require that s_m is a vertex in A and hence s_{m+1} is a vertex in B if $\{a,b\} \subset \{0,1,2\}$.

For each odd $a \in Z_{n-1}$, $a \geqslant 5$, and each two pairs $\{s_m, s_{m+1}\}$ and $\{s_{m+1}, s_{m+2}\}$, m even, from a cyclic sequence related to the pair of 1-factors $\{F_a, F_{a+1}\}$ we form the following blocks:

$$\{[(a,0),1],[(s_m,1),1],[(s_{m+1},1),1]\},$$

$$\{[(a,0),1],[(s_{m+1},1),2],[(s_{m+2},1),2]\},$$

$$\{[(a,0),2],[(s_m,1),2],[(s_{m+1},1),2]\},$$

$$\{[(a,0),2],[(s_{m+1},1),1],[(s_{m+2},1),1]\},$$

$\{[(a+1,0),1],[(s_m,1),1],[(s_{m+1},1),2]\},$

$\{[(a+1,0),1],[(s_{m+1},1),1],[(s_{m+2},1),2]\},$

$\{[(a+1,0),2],[(s_m,1),2],[(s_{m+1},1),1]\},$

$\{[(a+1,0),2],[(s_{m+1},1),2],[(s_{m+2},1),1]\}.$

For each two pairs $\{s_m, s_{m+1}\}$ and $\{s_{m+1}, s_{m+2}\}$, m even, from a cyclic sequence related to the pair of 1-factors $\{F_3, F_4\}$ we form the following blocks.

$\{[(4,0),1],[(s_m,1),1],[(s_{m+1},1),2]\},$

$\{[(4,0),1],[(s_{m+1},1),1],[(s_{m+2},1),2]\},$

$\{[(4,0),2],[(s_m,1),2],[(s_{m+1},1),1]\},$

$\{[(4,0),2],[(s_{m+1},1),2],[(s_{m+2},1),1]\}.$

For each two pairs $\{s_m, s_{m+1}\}$ and $\{s_{m+1}, s_{m+2}\}$, m even, from a cyclic sequence related to the pair of 1-factors $\{F_0, F_1\}$ we form the following blocks:

$\{[(0,0),1],[(s_m,1),1],[(s_{m+1},1),2]\},$

$\{[(0,0),1],[(s_{m+1},1),1],[(s_{m+2},1),2]\}.$

For each two pairs $\{s_m, s_{m+1}\}$ and $\{s_{m+1}, s_{m+2}\}$, m even, from a cyclic sequence related to the pair of 1-factors $\{F_2, F_3\}$ we form the following blocks:

$\{[(0,0),2],[(s_m,1),1],[(s_{m+1},1),1]\},$

$\{[(0,0),2],[(s_{m+1},1),2],[(s_{m+2},1),2]\}.$

For each two pairs $\{s_m, s_{m+1}\}$ and $\{s_{m+1}, s_{m+2}\}$, m even, from a cyclic sequence related to the pair of 1-factors $\{F_0, F_2\}$ we form the following blocks:

$\{[(1,0),1],[(s_m,1),2],[(s_{m+1},1),1]\},$

$\{[(1,0),1],[(s_{m+1},1),2],[(s_{m+2},1),1]\}.$

For each two pairs $\{s_m, s_{m+1}\}$ and $\{s_{m+1}, s_{m+2}\}$, m even, from a cyclic sequence related to the pair of 1-factors $\{F_1, F_4\}$ we form the following blocks:

$\{[(1,0),2],[(s_m,1),1],[(s_{m+1},1),1]\},$

$\{[(1,0),2],[(s_{m+1},1),2],[(s_{m+2},1),2]\}.$

For each two pairs $\{s_m, s_{m+1}\}$ and $\{s_{m+1}, s_{m+2}\}$, m even, from a cyclic sequence related to the pair of 1-factors $\{F_1, F_2\}$ we form the following blocks:

$\{[(2,0),1],[(s_m,1),1],[(s_{m+1},1),2]\},$

$\{[(2,0),1],[(s_{m+1},1),1],[(s_{m+2},1),2]\}.$

For each two pairs $\{s_m, s_{m+1}\}$ and $\{s_{m+1}, s_{m+2}\}$, m even, from a cyclic sequence related to the pair of 1-factors $\{F_0, F_3\}$ we form the following blocks:

$$\{[(2,0),2],[(s_m,1),2],[(s_{m+1},1),2]\},$$

$$\{[(2,0),2],[(s_{m+1},1),1],[(s_{m+2},1),1]\}.$$

For each two pairs $\{s_m, s_{m+1}\}$ and $\{s_{m+1}, s_{m+2}\}$, m even, from a cyclic sequence related to the pair of 1-factors $\{F_0, F_2\}$ we form the following blocks:

$$\{[(3,0),1],[(s_m,1),1],[(s_{m+1},1),1]\},$$

$$\{[(3,0),1],[(s_{m+1},1),2],[(s_{m+2},1),2]\}.$$

For each two pairs $\{s_m, s_{m+1}\}$ and $\{s_{m+1}, s_{m+2}\}$, m even, from a cyclic sequence related to the pair of 1-factors $\{F_1, F_4\}$ we form the following blocks:

$$\{[(3,0),2],[(s_m,1),2],[(s_{m+1},1),2]\},$$

$$\{[(3,0),2],[(s_{m+1},1),1],[(s_{m+2},1),1]\}.$$

Theorem 10. *Constructions* F *and* G *form generalized Steiner triple systems.*

Theorem 11. *$GS(2,3,n,2)$ exists if and only if $n \equiv 0$ or $1 \pmod 3$, $n \geqslant 4$, $n \neq 6$.*

Proof. By Theorem 5 and since no $GS(2,3,6,2)$ exists, we only have to show that for each $n \equiv 0$ or $1 \pmod 3$, $n \geqslant 4$, $n \neq 6$ there exists a $GS(2,3,n,2)$. The proof is by induction on n. The basis is the generalized triple systems for orders 4, 7, 9, and 10, given in the appendix. Assume, we have a solution for all admissible orders less than $n, n > 10$. Now, we want to show that there is a solution of order n. We distinguish between the following cases:

Case 1: $n \equiv 1 \pmod{12}$. It is known that there exists an $S(2,4,n)$ and the system is obtained by Construction E.

Case 2: $n \equiv 7 \pmod{12}$, i.e., $n = 2m + 1$, $m \equiv 3 \pmod 6$. By the induction hypothesis we have $GS(2,3,m,2)$ and the result is obtained by Construction G.

Case 3: $n \equiv 4$ or $10 \pmod{18}$, i.e., $n = 3m + 1$, $m \equiv 1$ or $3 \pmod 6$. By using $GS(2,3,m,2)$, $GS(2,3,4,2)$, and Construction D, we obtain the required system.

Case 4: $n \equiv 16 \pmod{18}$, i.e., $n = 3m + 1$, $m \equiv 5 \pmod 6$. By using $GDD(3,2,3,2)$, $GS(2,3,m+1,2)$, and Construction D, we obtain the required system.

Case 5: $n \equiv 0$ or $12 \pmod{18}$, i.e., $n = 3m$, $m \equiv 0$ or $4 \pmod 6$. By using $GDD(3,2,3,2)$, $GS(2,3,m,2)$, and Construction C we obtain the required system.

Case 6: $n \equiv 6 \pmod{18}$, i.e., $n = 6m$, $m = 1 \pmod 3$. By using $GDD(6,2,3,2)$, $GS(2,3,m,2)$, and Construction C we obtain the required system.

Case 7: $n \equiv 3 \pmod 6$. $n = 3m$, where m is odd; if $m \equiv 1$ or $3 \pmod 6$ then $GS(2,3,n,2)$ is obtained via Construction C, and if $m \equiv 1$ or $5 \pmod 6$ then the system is obtained via Construction F.

Cases 1 and 2 implies the solution for 1(mod 6); Cases 3 and 4 implies the solution for 4(mod 6); Cases 5 and 6 implies the solution for 0(mod 6) and Case 7 implies the solution for 3(mod 6). Thus, for each $n \equiv 0$ or $1 \pmod 3$, $n \geqslant 4$, $n \neq 6$, there exists a $GS(2,3,n,2)$. □

5. Generalized Steiner quadruples systems

Generalized Steiner quadruple systems seems to be much more difficult to obtain. The knowledge on the existence of $H(n,2,4,3)$ designs [13] does not seem to help. We will discuss only generalized Steiner quadruple systems over an alphabet with 3 letters. First, we will give two recursive constructions to generate generalized Steiner quadruple system over an alphabet with 3 letters. Then we will give some ideas how to construct other systems and we will apply some of these ideas.

Construction H. Assume Q is a $GS(3,4,2n,2)$ on Z_{2n}, $F = \{F_0, F_1, \ldots, F_{2n-2}\}$ is a good 1-factorization of K_{2n}, where $\{F_0, F_1, F_2\}$ forms a 3-good set. Let A and B be the two sides of the bipartite graph formed from the union of the edges of F_0, F_1, and F_2. For each pair of 1-factors $\{F_a, F_{a+1}\}$, where $a \geqslant 3$, $a \in Z_{2n-1}$, we order the elements of Z_n in cyclic sequences $s_0, s_1, \ldots, s_{r-1}$ such that $\{s_m, s_{m+1}\}$ (subscripts taken modulo r) is contained in F_a if m is even and in F_b if m is odd. We form the following system on $Z_{2n} \times Z_2$: For each block $\{[i,\alpha],[j,\beta],[r,\gamma],[m,\delta]\} \in Q$ we form the blocks

$$\{[(i,0),\alpha],[(j,0),\beta],[(r,0),\gamma],[m,0),\delta]\},$$

$$\{[(i,1),\alpha],[(j,1),\beta],[(r,1),\gamma],[(m,1),\delta]\}.$$

For each pair $\{i,j\} \in F_a$ and each pair $\{r,m\} \in F_{a+3}$ we form the blocks

$$\{[(i,0),1],[(j,0),1],[(r,1),1],[(m,1),1]\},$$

$$\{[(i,0),2],[(j,0),2],[(r,1),2],[(m,1),2]\}.$$

For each pair $\{i,j\} \in F_a$ and each pair $\{r,m\} \in F_{a+4}$ we form the blocks

$$\{[(i,0),1],[(j,0),1],[(r,1),2],[(m,1),2]\},$$

$$\{[(i,0),2],[(j,0),2],[(r,1),1],[(m,1),1]\}.$$

For each odd $a, a \geqslant 3$, and $a \in Z_{2n-1}$, and each four pairs $\{s_m, s_{m+1}\}$ and $\{s_{m+1}, s_{m+2}\}$, m even, $\{s_r, s_{r+1}\}$ and $\{s_{r+1}, s_{r+2}\}$, r even, from the cyclic sequence related to the pair of 1-factors $\{F_a, F_{a+1}\}$ we form the following blocks:

$$\{[(s_m,0),1],[(s_{m+1},0),2],[(s_r,1),1],[(s_{r+1},1),2]\},$$

$$\{[(s_m,0),1],[(s_{m+1},0),2],[(s_{r+1},1),1],[(s_{r+2},1),2]\},$$

$$\{[(s_m,0),2],[(s_{m+1},0),1],[(s_r,1),2],[(s_{r+1},1),1]\},$$

$$\{[(s_m,0),2],[(s_{m+1},0),1],[(s_{r+1},1),2],[(s_{r+2},1),1]\},$$

$$\{[(s_{m+1},0),1],[(s_{m+2},0),2],[(s_r,1),1],[(s_{r+1},1),2]\},$$

$$\{[(s_{m+1},0),1],[(s_{m+2},0),2],[(s_{r+1},1),1],[(s_{r+2},1),2]\},$$

$$\{[(s_{m+1},0),2],[(s_{m+2},0),1],[(s_r,1),2],[(s_{r+1},1),1]\},$$

$$\{[(s_{m+1},0),2],[(s_{m+2},0),1],[(s_{r+1},1),2],[(s_{r+2},1),1]\}.$$

For each pair $\{i,j\} \in F_a$, $a \in Z_3$, and each pair $\{r,m\} \in F_a$, such that $i,r \in A, j,m \in B$, we form the blocks

$$\{[(i,0),1],[(j,0),2],[(r,1),1],[(m,1),2]\},$$

$$\{[(i,0),2],[(j,0),1],[(r,1),2],[(m,1),1]\}.$$

For each pair $\{i,j\} \in F_a$, $a \in Z_3$, and each pair $\{r,m\} \in F_{a+1}$, such that $i,r \in A$, $j,m \in B$, we form the blocks

$$\{[(i,0),1],[(j,0),2],[(r,1),2],[(m,1),1]\},$$

$$\{[(i,0),2],[(j,0),1],[(r,1),1],[(m,1),2]\}.$$

Construction I. Assume Q is a $GS(3,4,2n,2)$ on Z_{2n}, and R is an $H(m,2,4,3)$ on Z_m, with a partitioned into t sets $S_0, S_1, \ldots, S_{t-1}$, such that $t \leqslant 2n$ and the minimum distance in S_r, $r \in Z_t$ is 3. We form the following quadruple system on $Z_{2n} \times Z_m$: For each block of R, $\{[i,\alpha],[j,\beta],[h,\gamma],[p,\delta]\} \in S_r$ and each $a,b,c \in Z_{2n}$, we form the block

$$\{[(a,i),\alpha],[(b,j),\beta],[(c,h),\gamma],[(a+b+c+r,h),\delta]\}.$$

For each $a,b \in Z_m$ we form on $Z_{2n} \times \{a,b\}$ the same blocks of $Z_{2n} \times \{0,1\}$ as in construction H. Note, that the blocks of Q should be formed only once on each $Z_{2n} \times \{a\}$, $a \in Z_m$. Also, it is easy to verify that $H(m,2,4,3)$ can be always partitioned into at most 4 sets with minimum distance 3 in each set.

Theorem 12. *Constructions* H *and* I *form generalized Steiner quadruple systems.*

We have no general direct construction for generalized Steiner quadruple systems. In fact, for $n \equiv 1$ or $5 \pmod 6$ we did not find any generalized Steiner quadruple system $GS(3,4,n,2)$. But, we do have two general ideas for their construction when $n \equiv 2$ or $4 \pmod 6$.

The first idea is very simple, but quite complicated for implementation. $GS(3,4,8,2)$ given in the appendix was formed by this method. Let $n \equiv 2$ or $4 \pmod 6$, and we want to form a $GS(3,4,n,2)$ on Z_n. For each three distinct values $a, b, c \in Z_n$, we will try to form the blocks $\{[a,1],[b,1],[c,1],[d,2]\}$ and $\{[a,2],[b,2],[c,2],[d,1]\}$, for some $d \in Z_n$.

The second idea is more complicated, but it is simpler for implementation. Let $n \equiv 2$ or $4 \pmod 6$, and let $F = \{F_0, F_1, \ldots, F_{n-2}\}$ be a 1-factorization of K_n on the points of Z_n, such that for any union of two distinct 1-factors F_i and F_j there is no cycle of length 4. Assume further that Q is a Steiner quadruple system $S(3,4,n)$ on Z_n, such that for any 1-factor F_i, if $\{x_1,x_2\},\{x_3,x_4\} \in F_i$, then $\{x_1,x_2,x_3,x_4\} \notin Q$. If such 1-factorization and Steiner quadruple system exist, then we form the following quadruple system. For each two pairs $\{x_1,x_2\},\{x_3,x_4\} \in F_i$, we form the blocks

$$\{[x_1,1],[x_2,1],[x_3,2],[x_4,2]\},$$

$$\{[x_1,2],[x_2,2],[x_3,1],[x_4,1]\}.$$

For each block $\{w,x,y,z\} \in Q$ we form the blocks

$$\{[w,1],[x,1],[y,1],[z,1]\},$$

$$\{[w,2],[x,2],[y,2],[z,2]\}.$$

It is easy to verify that we have constructed a $GS(3,4,n,2)$ if the appropriate 1-factorization and Steiner quadruple system exist. These 1-factorizations are known to exist, but to make this paper more self-contained we present two simple constructions.

Let $n \equiv 2 \pmod 6$ and for $0 \leqslant i \leqslant n-2$ define

$$F_i = \{\{x,y\}: x + y \equiv 2i \pmod{n-1}\} \cup \{\{i, n-1\}\},$$

For the second construction let $m \equiv 1$ or $5 \pmod 6$. Let $G_i = \{\{x,y\}: x + y \equiv 2i \pmod m\}$ for $0 \leqslant i \leqslant m-1$ and let

$$F_i = \{\{x,y\}: \{x,y\} \in G_i\} \cup \{\{x+m, y+m\}: \{x,y\} \in G_i\} \cup \{\{i, i+m\}\}.$$

For $1 \leqslant i \leqslant m-1$ let

$F_{m-1+i} = \{\{x, x+m+i\}: x \in Z_m\}$, where $x+m+i$ is reduced modulo m to the range $\{m, m+1, \ldots, 2m-1\}$.

Using the second construction for $m = 5$ and the unique Steiner quadruple system of order 10 we have obtained an $GS(3,4,10,2)$ given in the appendix. Note that the blocks for the $GS(3,4,n,2)$, obtained from the 1-factorization of the first construction have the cycle $(0,1,\ldots,n-2)(n-1)$ as an automorphism group, while the blocks for the $GS(3,4,n,2)$, obtained from the 1-factorization of the second construction have the cycle $(0,1,\dots,r-1)(r,r+1,\dots,n-1)$, $r=n/2$ as an automorphism group. The best way to search for an appropriate Steiner quadruple system to complete the $GS(3,4,n,2)$ is to take Steiner quadruple system with the same automorphism group. A good survey on automorphism groups of Steiner quadruple systems is given in [14].

6. Generalized 2-designs

In this section we discuss the existence of generalized Steiner systems which cover pairs, i.e., $GS(2,w,n,k)$. The case of $w=3$ was discussed in Section 4. We will first give some general constructions and then discuss some specific cases. The first three constructions are generalizations of Constructions C, D, and E of Section 4.

Construction J. Assume Q is an $GS(2,w,m,k)$ on Z_m, R is an $GS(2,w,n,k)$ on Z_n, and S is $OA(2,w,n)$ over Z_n. We form the following system on $Z_m \times Z_n$: For each block $\{[a_1,\alpha_1],[a_2,\alpha_2],\dots,[a_w,\alpha_w]\} \in R$ and each $i \in Z_m$ we form the block

$$\{[(i,a_1),\alpha_1],[(i,a_2),\alpha_2],\dots,[(i,a_w),\alpha_w]\}.$$

For each block $\{[i_1,\alpha_1],[i_2,\alpha_2],\dots,[i_w,\alpha_w]\} \in Q$ and each $(j_1,j_2,\dots,j_w) \in S$ we form the block

$$\{[(i_1,j_1),\alpha_1],[(i_2,j_2),\alpha_2],\dots,[(i_w,j_w),\alpha_w]\}.$$

Construction K. Assume Q is an $GS(2,w,m,k)$ on Z_m, R is an $GS(2,w,n,k)$ on $Z_n \cup \{A\}$, and S is $OA(2,w,n)$ over Z_n. We form the following system on $Z_m \times Z_n \cup \{A\}$: For each block $\{[a_1,\alpha_1],[a_2,\alpha_2],\dots,[a_w,\alpha_w]\} \in R$ and each $i \in Z_m$ we form the block

$$\{[(i,a_1),\alpha_1],[(i,a_2),\alpha_2],\dots,[(i,a_w),\alpha_w]\}.$$

For each block $\{[a_1,\alpha_1],\dots,[a_{w-1},\alpha_{w-1}],[A,\alpha_w]\} \in R$ and each $i \in Z_m$ we form the block

$$\{[(i,a_1),\alpha_1],\dots,[(i,a_{w-1}),\alpha_{w-1}],[A,\alpha_w]\}.$$

For each block $\{[i_1,\alpha_1],[i_2,\alpha_2],\dots,[i_w,\alpha_w]\} \in Q$ and each $(j_1,j_2,\dots,j_w) \in S$ we form the block

$$\{[(i_1,j_1),\alpha_1],[(i_2,j_2),\alpha_2],\dots,[(i_w,j_w),\alpha_w]\}.$$

Construction L. Assume Q is an $S(2,v,n)$ on Z_n and R is an $GS(2,w,v,k)$ on Z_v. We form the following system on Z_n: For each $\{x_0,x_1,\dots,x_{v-1}\} \in Q$ such that $x_i < x_{i+1}$ and each $\{[i_1,\alpha_1],[i_2,\alpha_2],\dots,[i_w,\alpha_w]\} \in R$ we form the block

$$\{[x_{i_1},\alpha_1],[x_{i_2},\alpha_2],\dots,[x_{i_w},\alpha_w]\}.$$

Theorem 13. *Constructions* J, K, *and* L *result with generalized Steiner systems.*

An extensive literature on the existence of Steiner systems $S(2,v,n)$ is known [2]. $S(2,4,n)$ exists if and only if $n \equiv 1$ or $4 (\bmod 12)$. $S(2,5,n)$ exists if and only if $n \equiv 1$ or $5 (\bmod 20)$. $S(2,6,n)$ exists if and only if $n \equiv 1$ or $6 (\bmod 15)$, $n \neq 16, 21, 36$, and possible exceptions listed in [2]. $S(2,v,n)$ exists for $n \equiv 1 (\bmod v-1)$, $n(n-1) \equiv 0 (\bmod v(v-1))$, and n sufficiently large. To apply Constructions J, K, and L, we also need some specific generalized Steiner systems. We will consider the case $w = 4$ and $k = 2$. The codewords of weight 5 in the ternary Golay code of length 11, and minimum Hamming distance 5, form a generalized Steiner System $GS(3,5,11,2)$. The derived system from $GS(3,5,11,2)$ is a $GS(2,4,10,2)$. By the necessary condition given in Section 2 we have that a $GS(2,4,n,2)$ can exist only if $n \equiv 1 (\bmod 3)$. We conjecture that for $n \equiv 1 (\bmod 3)$, $n \geqslant 7$ there exists a $GS(2,4,n,2)$. However, except for these specific systems and the ones derived by Constructions J, K, and L, we do not have any other system.

The minimum Hamming distance of the code derived from $H(n,2,4,2)$ is 3, while the one derived from $GS(2,4,n,2)$ is 5. It is much more easier to obtain $GS_4(2,4,n,2)$ rather than $GS(2,4,n,2)$. For odd n a *good set of elements* is a set which contains $(n-1)/2$ nonzero residues modulo n (between 1 and $n-1$), and for each two elements x, y in the set $n - x \neq y$. Assume $n \equiv 1 (\bmod 6)$ and let $S = \{\{x_i, y_i, z_i\}: 1 \leqslant i \leqslant (n-1)/6\}$ be a set such that the union of the triples $\{x_i, y_i, z_i\}$ is a good set of elements and also the union of the sets $\{y_i - x_i, z_i - y_i, z_i - x_i\}$ is also a good set of elements. Given a good set of element $S(n)$, for each triple $\{x, y, z\} \in S(n)$ and each $i \in Z_n$ we form the following blocks to obtain a $GS_4(2,4,n,2)$:

$$\{[i,1],[i+x,2],[i+y,2],[i+z,2]\},$$

$$\{[i,2],[i+x,1],[i+y,1],[i+z,1]\}.$$

We conjecture that such a set $S(n)$ exists for all $n \equiv 1 (\bmod 6)$. Let $p \equiv 1 (\bmod 6)$ be a prime, let α be a primitive root modulo p, and let β be an element of order 3 modulo p. The set $S(p) = \{\{\alpha^i, \alpha^i\beta, \alpha^i\beta^2\}: 1 \leqslant i \leqslant (p-1)/6\}$ is a good set of elements. The proof is based on simple number theory arguments, e.g., the observation that $\beta^2 + \beta + 1 = 0$. For $n = 25$ we found the following good set of elements, $S(25) = \{\{10,23,24\},\{4,20,22\},\{8,12,18\},\{6,9,14\}\}$. More codes with minimum Hamming distance 4 can be constructed via Constructions J, K, and L.

7. Double designs

As said in the introduction, the supports of the codewords of weight 5 in the ternary Golay code of length 11 form a Steiner system $S(4,5,11)$. Also, since the Golay code is perfect with covering radius 2, the codewords of weight 5 form a generalized Steiner Systems $GS(3,5,11,2)$. The Hamming code of length $q+1$ over $GF(q)$ has a similar property. The codewords of weight 3 form a generalized Steiner system $GS(2,3,q+1,q-1)$, while their supports form the trivial Steiner system $S(3,3,q+1)$.

A code for which its codewords form a generalized Steiner system $GS(w_1, w, n, k)$, each w coordinates which are support of a codeword are support of k codewords, and their supports form a Steiner system $S(w_2, w, n)$ will be called a *double Steiner system* (or just a *double design* if we take $H(n, k, w, w_1)$ rather than $GS(w_1, w, n, k)$). Are double Steiner systems exist except for the ones derived above?

First, we ask what are the necessary conditions for the existence of double design? Assume C is a constant weight code over an alphabet with $k+1$ letters, of length n and weight w. Assume further that the codewords of C form an $H(n, k, w, w_1)$, each support is shared by k codewords, and the supports form a Steiner system $S(w_2, w, n)$. Since the codewords of C form an $H(n, k, w, w_1)$, it follows that C contains $\binom{n}{w_1}k^{w_1}/\binom{w}{w_1}$ codewords and because each support is shared by k codewords, the number of supports is $\binom{n}{w_1}k^{w_1-1}/\binom{w}{w_1}$. Now, an $S(w_2, w, n)$ have $\binom{n}{w_2}/\binom{w}{w_2}$ blocks and therefore we must have $\binom{n}{w_1}k^{w_1-1}/\binom{w}{w_1} = \binom{n}{w_2}/\binom{w}{w_2}$. Solving this equation we have that

$$\frac{(n-w_1)\cdots(n-w_2+2)(n-w_2+1)}{(w-w_1)\cdots(w-w_2+2)(w-w_2+1)} = k^{w_1-1}.$$

What are the possible solutions to this equation? We distinguish between several cases.

Case 1: $k=1$, the generalized H-design is a Steiner system and the equation does not have a meaning.

For the next four cases we assume $k>1$.

Case 2: $w_1 = 1$, the solution for the equation gives $w = n$, $w_2 = w_1$, and the H-design is a trivial one.

Case 3: $w_1 = 2$, it is easy to verify that the equation might have many kinds of solutions.

Case 4: $w_1 \geqslant 3$, the equation has solutions for $w_2 = w_1 + 1$, which implies $k^{w_1-1} = (n-w_1)/w-w_1)$.

Case 5: $w_1 = 3$, the equation has solutions for $w_2 = w_1 + 2$ and $k = 2m$, m odd. It takes the form $k^2 = (n-w_1)(n-w_1-1)/(w-w_1)(w-w_1-1)$. Now, letting $m^2 = n - w_1$ and $(m+1)/2 = w - w_1$, we obtain a solution.

We conjecture that the equation has no other solutions except for the five cases mentioned above. These five cases give the first necessary condition for the existence of double Steiner system $GS(w_1, w, n, k)$. Since for $k>1$ each support share a few codewords, it follows that the weight w is an upper bound on the minimum distance of the generalized Steiner system and hence we must have $w \geqslant 1 + 2(w - w_1)$, i.e., $2w_1 \geqslant w+1$. This condition rules out many cases which are possible by the first necessary condition. To these conditions we have to add the necessary conditions for the existence of the related Steiner system and generalized Steiner system.

It is quite difficult to find more *double Steiner systems*, especially because of the second necessary condition $2w_1 \geqslant w+1$. Less difficult is to find systems $GS_w(w_1, w, n, k)$ for which the supports form $S(w_2, w, n)$. All the systems we know have some linear properties. A *linear constant weight code*, of length n, over $GF(k+1)$ is a constant weight code C over $GF(k+1)$ with the property that $(c_1, c_2, \ldots, c_n) \in C$ implies that $(\alpha c_1, \alpha c_2, \ldots, \alpha c_n) \in C$ for any nonzero element $\alpha \in GF(k+1)$. Obviously, the codewords of any weight in a linear code form a linear constant weight code. A *quasi linear constant weight code* over ZZ_k is a constant weight code C over ZZ_k with the property that $(c_1, c_2, \ldots, c_n) \in C$ implies that $(f(\alpha + c_1), f(\alpha + c_2), \ldots, f(\alpha + c_n)) \in C$, for any element α in ZZ_k, where $f(\alpha + c) = 0$ if $c = 0$ and $f(\alpha + c) = \alpha + c$ if $c \neq 0$. All *double Steiner system* we have found are either linear or quasi linear.

From the following blocks we form a quasi linear $GS_4(2,4,8,3)$ for which the supports form $S(3,4,8)$. For each $i \in ZZ_3$ we form the blocks

$$\{[0,i],[1,i],[2,i],[3,i]\} \quad \{[0,i],[1,i+1],[4,i+1],[5,i+1]\},$$

$$\{[0,i],[1,i+2],[6,i+2],[7,i+2]\} \quad \{[0,i],[2,i+2],[4,i+2],[6,i+1]\},$$

$$\{[0,i],[2,i+1],[5,i+2],[7,i]\} \quad \{[0,i],[3,i+1],[4,i],[7,i+1]\},$$

$$\{[0,i],[3,i+2],[5,i],[6,i]\} \quad \{[4,i+2],[5,i+1],[6,i],[7,i+1]\},$$

$$\{[1,i+2],[2,i],[4,i+1],[7,i+1]\} \quad \{[1,i+2],[2,i+1],[5,i],[6,i+1]\},$$

$$\{[1,i+2],[3,i+1],[5,i+1],[7,i]\} \quad \{[1,i+2],[3,i],[4,i],[6,i]\},$$

$$\{[2,i+2],[3,i],[4,i+1],[5,i+2]\} \quad \{[2,i+2],[3,i+1],[6,i],[7,i+2]\}.$$

From the following blocks we form a quasi linear $GS_4(2,4,5,3)$ for which the supports form $S(4,4,5)$. For each $i \in ZZ_3$ we form the blocks

$$\{[0,i],[1,i],[2,i],[3,i]\} \quad \{[0,i],[1,i+1],[2,i+2],[4,i]\},$$

$$\{[0,i],[2,i+1],[3,i+2],[4,i+1]\} \quad \{[0,i],[1,i+2],[3,i+1],[4,i+2]\},$$

$$\{[1,i],[2,i+2],[3,i+1],[4,i+1]\}.$$

Finally, one can easily construct a quasilinear $GS_w(1,w,wn,k)$ for which the supports form an $S(1,w,wn)$.

8. MDS constant weight codes

Assume we are given a constant weight code of length n, and weight w over an alphabet with $k+1$ letters, with $\binom{n}{w}k$ different codewords. If $k=1$ then the minimum Hamming distance of the code is 2. If $k>1$ then the minimum Hamming distance d of the code satisfies $d \leqslant w$ since w is the maximum possible distance between two codewords which share the same coordinates. If $d = w$ the code will be called MDS Constant Weight Code (MDS-CW code). An (n,w,k) MDS-CW code is a constant weight code of weight w, minimum distance $d = w$, over an alphabet with $k+1$ letters, with $\binom{n}{w}k$ codewords. If $k=1$, MDS-CW code exists if and only if $n \geqslant 2$ and $w=2$. If $k>1$ then the codewords of minimum weight d is an MDS code over $GF(k+1)$ form an MDS-CW code. Are there any more MDS-CW codes? MDS code is equivalent to the combinatorial design of orthogonal array [16]. W.l.o.g. we can assume that the allzero row belongs to the array. With this assumption, the nonzero rows with minimum weight define an MDS-CW code as we will prove in Theorm 17. First, we will give some known theorems on orthogonal arrays, which will be extended later to MDS-CW codes.

Theorem 14. *For an $OA(t,n,q)$ if $t \leqslant q$ then*

$$n \leqslant q + t - 1 \quad \text{if } q \text{ is even}$$

$$n \leqslant q + t - 2 \quad \text{if } q \text{ is odd and } t \geqslant 3.$$

Theorem 15. *For an $OA(t,n,q)$ if $q \leqslant t$ then $n \leqslant t + 1$.*

There are three sets of parameters for trivial orthogonal arrays: $OA(1,n,q)$, $OA(n,n,q)$, and $OA(n-1,n,q)$ for all $q \geqslant 2$ and $n \geqslant 2$. Linear MDS codes exist over any Galois field $GF(q)$, q is a power of prime. The construction of the code is given in [16], while in the orthogonal array form it is given in [20]. $OA(t,q+1,q)$, with $1 \leqslant t \leqslant q+1$ exists for any power of a prime q. If q is even we also have $OA(3,2^m+2,2^m)$ and $OA(2^m-1,2^m+2,2^m)$. Blanchard [4] proved that for all t, $OA(t,n,k)$ exists for all sufficiently large n. Finally, $OA(2,n,k)$ is equivalent to a set of $n-2$ pairwise orthogonal latin squares of order k. Now, we turn our discussion to bounds and constructions for MDS-CW codes and their connection with orthogonal arrays.

Theorem 16. *If an (n,w,k) MDS-CW code exists then an $(n-1,w,k)$ MDS-CW code exists.*

Proof. Let C be an (n,w,k) MDS-CW code. By taking all blocks which do not contain the last coordinate we obtain an $(n-1,w,k)$ MDS-CW code. □

Theorem 17. *The rows of weight $w = n - t + 1$ in an orthogonal array $OA(t,n,k+1)$, which contains the all zero row, form an (n,w,k) MDS-CW code.*

Proof. We only have to prove that any $w = n - t + 1$ coordinates are supports of exactly k codewords. Assume that the orthogonal array is on the set of points Z_n. Given any set W of w coordinates, let $x \in W$, then $Z_n \setminus W \cup \{x\}$ contains t points, which must include all possible t-tuples by the orthogonal array property. By taking the k words with $t-1$ zeroes in $Z_n \setminus W$ and all the nonzero elements of the alphabet in x, we must have that the corresponding k rows have weight w or else we will see the all-zero t-tuple twice in some t coordinates. The minimum Hamming distance $d = w$ is also an immediate result from the orthogonal array property. □

Trivial MDS-CW codes are derived similarly to (or from) trivial orthogonal arrays. For $w = 1$ all possible words of weight 1 and length n over an alphabet of size $k + 1$ form an $(n, 1, k)$ MDS-CW code. For $w = 2$ the set $\{\{[i,\alpha],[j,\alpha]\}: i, j \in Z_n, \alpha \in ZZ_k\}$ forms an $(n, 2, k)$ MDS-CW code. For $w = n$ the set $\{\{[0,\alpha],[1,\alpha],\ldots,[n-1,\alpha]\}: \alpha \in ZZ_k\}$ forms an (n, n, k) MDS-CW code. Note that MDS codes of length $q + 1$, and minimum distance 3, over $GF(q)$ are the same as the Hamming codes of the same length and distance, and hence the $(q + 1, 3, q - 1)$ MDS-CW codes are the $GS(2, 3, q + 1, q - 1)$ of Section 3. Since we are interested only in the rows of minimum weight we are able to find some more MDS-CW codes.

Theorem 18. *If an (n, w, k) MDS-CW code exists then an (n, w, rk) MDS-CW codes exists for all $r > 0$.*

Proof. Let C be an (n, w, k) MDS-CW code over Z_{k+1}. We form a code C_1 over $ZZ_k \times ZZ_r \cup \{(0,0)\}$ as follows. For each codeword $\{[i_1,\alpha_1],[i_2,\alpha_2],\ldots,[i_w,\alpha_w]\} \in C$ we construct for C_1 the codewords $\{[i_1,(\alpha_1,j)],[i_2,(\alpha_2,j)],\ldots,[i_w,(\alpha_w,j)]\}$ for all $j \in ZZ_r$. □

Theorem 19. *If an (n, w, k_1) MDS-CW code and an (n, w, k_2) MDS-CW code exist then an $(n, w, k_1 + k_2)$ MDS-CW code exists.*

Proof. Assume that C_i is an (n, w, k_i) MDS-CW code over Q_i, $i = 1, 2$, such that $Q_1 \cap Q_2 = \{0\}$. One can easily verify that $C_1 \cup C_2$ is an $(n, w, k_1 + k_2)$ MDS-CW code. □

From an $OA(2, n, k)$ we can obtain an $(n, n - 1, k - 1)$ MDS-CW code. Now, we will show another type of MDS-CW code obtained from $OA(2, n, k)$. Assume M is an $OA(2, n, k)$ code over ZZ_k such that the first k symbols in the first column are ones, the next k symbols in the first column are twos, and so on. We delete the first column; in the new array we replace the first symbol in the first k rows with zeroes, the second symbol in the next k rows with zeroes and so on. All the rows after the first $(n - 1)k$ rows are deleted (note that in $OA(2, n, k)$ we have $n \leqslant k + 1$ and hence we have at least $(n - 1)k$ rows). The constructed array is an $(n - 1, n - 2, k)$ MDS-CW code. Hence we have

Theorem 20. *If there exists an $OA(2, n, k)$, then there exists an $(n - 1, n - 2, k)$ MDS-CW code.*

The MDS-CW codes which can be obtained by Theorem 20 cannot be always extended into orthogonal arrays. For example, from an $OA(2, 6, 5)$ we obtain $(5, 4, 5)$ MDS-CW code, which if can be extended would result in an $OA(2, 5, 6)$. But, it is well known [23] that there is no pair of orthogonal latin squares of order 6, and hence there is no $OA(2, 4, 6)$ and of course no $OA(2, 5, 6)$. From an $OA(2, q + 1, q)$, where q is a power of a prime, we obtain a $(q, q-1, q)$ MDS-CW code, which if can be extended would result in an $OA(2, q, q+1)$. If $q+1$ is not a power of a prime no such orthogonal array is known. Combinations of these arrays, other known MDS-CW codes obtained from orthogonal arrays, and Theorems 18 and 19 would result in other MDS-CW codes which cannot be obtained from known orthogonal arrays.

Next, we want to derive bounds on the size of the alphabet, $k+1$, of an (n, w, k) MDS-CW code for $3 \leqslant w \leqslant n-1$. If $w \leqslant n-1$, we know that each support of size w has k codewords. These codewords have distinct nonzero letters in each coordinate and in each coordinate each one of the nonzero k symbols appears. A codeword which shares exactly $w-1$ coordinates with these codewords cannot have more than one common symbol with each of these k codewords (otherwise the distance will be less than w). Therefore, we must have $k \geqslant w-1$. If $w \geqslant 3$ we consider the codewords with nonzero symbols in the first $w-1$ coordinates and the same symbol in the first coordinate. These codewords cannot have more common symbols (except for the one in the first coordinate). Therefore, we must have $n \leqslant w-1+k$. Combining these results, we have the following theorem which can be compared to Theorems 14 and 15.

Theorem 21. *Let C be an (n, w, k) MDS-CW code. If $w \geqslant 3$ then $k \geqslant n-w+1$. If $w \leqslant n-1$ then $k \geqslant w-1$.*

Theorem 22. *If an $(n, w, w-1)$ MDS-CW code exists then there exists an $OA(2, w+1, w)$.*

Proof. Consider all codewords with nonzero entries only in the first $w+1$ columns. These codewords form an $(w+1, w, w-1)$ MDS-CW code. If we add the zero word to this code we obtain a code of length $w+1$ over an alphabet with w letter, with $(w-1)(w+1)+1 = w^2$ codewords, and minimum distance w. Hence, this code is an $OA(2, w+1, w)$ and the theorem follows. □

Note that by the known parameters of orthogonal arrays, Theorems 17 and 22, we have that $OA(2, w+1, w)$ exists if and only if $(w+1, w, w-1)$ MDS-CW code exists. Finally, a very simple result is the following theorem.

Theorem 23. *For each n and w there exists a k_0 such that for each $k \geqslant k_0$ there exists an (n, w, k) MDS-CW code.*

Proof. Let m be the smallest integer such that $2^m \geqslant n-1$. As mentioned before there exists an $OA(n-w+1, n, 2^m)$ and hence there exists an $(n, w, 2^m-1)$ MDS-CW code. Similarly, there exists an $(n, w, 2^{m+1}-1)$ MDS-CW code. Since 2^m-1 and $2^{m+1}-1$ are relatively primes it follows, by using the conductor theorem of Frobenius [22, p. 376], that every integer k, such that $k > 2^{2m+1} - 2^{m+2} - 2^{m+1} + 3$ can be represented as $k = r_1(2^m-1) + r_2(2^{m+1}-1)$, where $r_1, r_2 \geqslant 0$. Therefore, by Theorems 18 and 19, for each n and w there exists a k_0 such that for each $k \geqslant k_0$ there exists an (n, w, k) MDS-CW code. $\square$

Let $KMDS(n, w)$ be the smallest integer such that for each $k \geqslant KMDS(n, w)$ there exists an (n, w, k) MDS-CW code. The upper bounds on $KMDS(n, w)$ given in Theorem 23 might be weak, while the lower bounds of Theorem 21 might be impossible to attain. An interesting question is to find the exact value of $KMDS(n, w)$. Usually, the bound of the proof of Theorem 23 can be improved by the same technique, if we will find a prime power q, $n - 1 \leqslant q < 2^{m+1}$ such that $q - 1$ and $2^m - 1$ are relatively primes. Theorem 23 can be also obtained from the results of Blanchard [4] mentioned before, but the proof of Theorem 23 is much simpler than Blanchard proofs, and the bounds are much better than the ones which can be obtained from Blanchard proofs.

9. Open problems for further research

The discussion on constant weight codes over arbitrary alphabet is far from being completed. We did not discuss bounds on sizes of codes which are not generalized Steiner systems or MDS constant weight codes. Some generalizations of Johnson bounds are quite easy to obtain [19]. Also, optimal codes which are derived from generalization of Hadamard matrices [7] are easy to obtain. The discussion given in the first eight sections raise many open problems. We gave constructions for only a few cases, but many more remain without a construction. We would really like to see progress made in the following questions.

1. Find more constructions for generalized Steiner triple systems.
2. One can easily verify that for $GS(t, w, m, k)$ with $k > 1$ and $t \neq w$, we must have $m > w$. But a better lower bound on the length of the code would be very interesting.
3. Show more values of k for which the necessary conditions for the existence of $GS(2, 3, n, k)$ is also sufficient from some $n \geqslant n_0$.
4. Find generalized Steiner quadruple system $GS(3, 4, n, 2)$ for some $n \equiv 1$ or $5 (\mathrm{mod}\, 6)$.
5. Find more constructions for generalized Steiner quadruple system.
6. Find more Steiner quadruple system which can be used in the proposed constructions of Section 5.
7. Given k and w, show that there exists an n_0 such that for all $n \geqslant n_0$, where w divides nk, $S(1, w, n, k)$ exists.
8. Find more constructions for double designs, or show when they cannot exist.
9. Find more constructions for generalized designs $GS_d(t, w, m, k)$, where $d > w - t + 1$.
10. Fine more MDS-CW codes for parameters where the relevant MDS code does not exist or not known.
11. Find better lower and upper bounds on $KMDS(n, w)$.

A. Appendix

$GS(2,3,4,2)$:

$\{[1,1],[2,1],[3,1]\}$ $\{[1,2],[2,2],[3,2]\}$ $\{[0,1],[1,2],[2,1]\}$ $\{[0,2],[1,1],[2,2]\}$

$\{[0,1],[1,1],[3,2]\}$ $\{[0,2],[1,2],[3,1]\}$ $\{[0,1],[2,2],[3,1]\}$ $\{[0,2],[2,1],[3,2]\}$

$GS(2,3,7,2)$:

$\{[0,1],[1,1],[2,1]\}$ $\{[0,1],[3,1],[4,1]\}$ $\{[0,1],[5,1],[6,1]\}$ $\{[1,1],[3,1],[5,1]\}$

$\{[1,1],[4,1],[6,1]\}$ $\{[2,1],[3,1],[6,1]\}$ $\{[2,1],[4,1],[5,1]\}$ $\{[0,2],[1,2],[3,1]\}$

$\{[0,2],[2,2],[4,1]\}$ $\{[0,2],[3,2],[5,1]\}$ $\{[0,2],[4,2],[6,1]\}$ $\{[0,2],[1,1],[5,2]\}$

$\{[0,2],[2,1],[6,2]\}$ $\{[1,2],[2,2],[5,1]\}$ $\{[1,2],[3,2],[6,1]\}$ $\{[1,2],[2,1],[4,2]\}$

$\{[1,2],[4,1],[5,2]\}$ $\{[0,1],[1,2],[6,2]\}$ $\{[0,1],[2,2],[3,2]\}$ $\{[2,2],[3,1],[4,2]\}$

$\{[2,2],[5,2],[6,1]\}$ $\{[1,1],[2,2],[6,2]\}$ $\{[1,1],[3,2],[4,2]\}$ $\{[2,1],[3,2],[5,2]\}$

$\{[3,2],[4,1],[6,2]\}$ $\{[0,1],[4,2],[5,2]\}$ $\{[4,2],[5,1],[6,2]\}$ $\{[3,1],[5,2],[6,2]\}$

$GS(2,3,9,2)$:

$\{[0,1],[1,1],[2,1]\}$ $\{[3,1],[4,1],[5,1]\}$ $\{[6,1],[7,1],[8,1]\}$ $\{[0,1],[3,1],[7,1]\}$

$\{[0,1],[4,1],[8,1]\}$ $\{[0,1],[5,1],[6,1]\}$ $\{[1,1],[3,1],[8,1]\}$ $\{[1,1],[4,1],[6,1]\}$

$\{[1,1],[5,1],[7,1]\}$ $\{[2,1],[3,1],[6,1]\}$ $\{[2,1],[4,1],[7,1]\}$ $\{[2,1],[5,1],[8,1]\}$

$\{[0,2],[1,2],[3,1]\}$ $\{[0,2],[2,2],[4,1]\}$ $\{[0,2],[2,1],[3,2]\}$ $\{[0,2],[1,1],[4,2]\}$

$\{[0,2],[5,2],[7,1]\}$ $\{[0,2],[6,2],[8,1]\}$ $\{[0,2],[6,1],[7,2]\}$ $\{[0,2],[5,1],[8,2]\}$

$\{[1,2],[2,2],[5,1]\}$ $\{[1,2],[3,2],[4,1]\}$ $\{[1,2],[2,1],[4,2]\}$ $\{[1,2],[5,2],[6,1]\}$

$\{[1,2],[6,2],[7,1]\}$ $\{[1,2],[7,2],[8,1]\}$ $\{[0,1],[1,2],[8,2]\}$ $\{[2,2],[3,2],[8,1]\}$

$\{[2,2],[4,2],[6,1]\}$ $\{[2,2],[3,1],[5,2]\}$ $\{[1,1],[2,2],[6,2]\}$ $\{[0,1],[2,2],[7,2]\}$

$\{[2,2],[7,1],[8,2]\}$ $\{[3,2],[4,2],[7,1]\}$ $\{[0,1],[3,2],[5,2]\}$ $\{[3,2],[5,1],[6,2]\}$

$\{[1,1],[3,2],[7,2]\}$ $\{[3,2],[6,1],[8,2]\}$ $\{[4,2],[5,2],[8,1]\}$ $\{[0,1],[4,2],[6,2]\}$

$\{[4,2],[5,1],[7,2]\}$ $\{[3,1],[4,2],[8,2]\}$ $\{[4,1],[5,2],[6,2]\}$ $\{[2,1],[5,2],[7,2]\}$

$\{[1,1],[5,2],[8,2]\}$ $\{[3,1],[6,2],[7,2]\}$ $\{[2,1],[6,2],[8,2]\}$ $\{[4,1],[7,2],[8,2]\}$

$GS(2,3,10,2)$:

$\{[0,1],[1,1],[2,1]\}$ $\{[3,1],[4,1],[5,1]\}$ $\{[6,1],[7,1],[8,1]\}$ $\{[0,1],[3,1],[7,1]\}$

$\{[0,1],[4,1],[8,1]\}$ $\{[0,1],[5,1],[6,1]\}$ $\{[1,1],[3,1],[8,1]\}$ $\{[1,1],[4,1],[6,1]\}$

$\{[1,1],[5,1],[7,1]\}$ $\{[2,1],[3,1],[6,1]\}$ $\{[2,1],[4,1],[7,1]\}$ $\{[2,1],[5,1],[8,1]\}$

$\{[0,2],[1,2],[2,2]\}$ $\{[3,2],[4,2],[5,2]\}$ $\{[6,2],[7,2],[8,2]\}$ $\{[5,2],[8,1],[9,1]\}$

$\{[4,2],[7,1],[9,1]\}$ $\{[3,2],[6,1],[9,1]\}$ $\{[2,2],[5,1],[9,1]\}$ $\{[1,2],[4,1],[9,1]\}$

$\{[0,2],[3,1],[9,1]\}$ $\{[2,1],[8,2],[9,1]\}$ $\{[1,1],[7,2],[9,1]\}$ $\{[0,1],[6,2],[9,1]\}$

$\{[1,2],[2,1],[6,2]\}$ $\{[0,1],[1,2],[7,2]\}$ $\{[1,2],[5,1],[8,2]\}$ $\{[0,1],[2,2],[3,2]\}$

$\{[1,1],[2,2],[4,2]\}$ $\{[2,2],[4,1],[5,2]\}$ $\{[2,2],[6,2],[7,1]\}$ $\{[2,2],[7,2],[8,1]\}$

$\{[2,2],[3,1],[8,2]\}$ $\{[0,2],[5,1],[9,2]\}$ $\{[1,2],[3,1],[9,2]\}$ $\{[1,1],[3,2],[6,2]\}$

$\{[3,2],[5,1],[7,2]\}$ $\{[3,2],[4,1],[8,2]\}$ $\{[2,2],[6,1],[9,2]\}$ $\{[4,2],[5,1],[6,2]\}$

$\{[3,1],[4,2],[7,2]\}$ $\{[4,2],[6,1],[8,2]\}$ $\{[3,1],[5,2],[6,2]\}$ $\{[2,1],[5,2],[7,2]\}$

$\{[0,1],[5,2],[8,2]\}$ $\{[2,1],[3,2],[9,2]\}$ $\{[0,1],[4,2],[9,2]\}$ $\{[1,1],[5,2],[9,2]\}$

$\{[6,2],[8,1],[9,2]\}$ $\{[4,1],[7,2],[9,2]\}$ $\{[7,1],[8,2],[9,2]\}$ $\{[0,2],[3,2],[8,1]\}$

$\{[0,2],[2,1],[4,2]\}$ $\{[0,2],[5,2],[7,1]\}$ $\{[0,2],[4,1],[6,2]\}$ $\{[0,2],[6,1],[7,2]\}$

$\{[0,2],[1,1],[8,2]\}$ $\{[1,2],[3,2],[7,1]\}$ $\{[1,2],[4,2],[8,1]\}$ $\{[1,2],[5,2],[6,1]\}$

$GS(2,3,9,3)$: For $i \in ZZ_3$ form the following blocks:

$\{[0,i],[1,i],[2,i+1]\}$ $\{[0,i],[2,i],[3,i+1]\}$ $\{[0,i],[1,i+1],[3,i]\}$

$\{[0,i],[4,i],[5,i+1]\}$ $\{[0,i],[5,i],[6,i+1]\}$ $\{[0,i],[6,i],[7,i+1]\}$

$\{[0,i],[7,i],[8,i+1]\}$ $\{[0,i],[4,i+1],[8,i]\}$ $\{[1,i],[2,i],[4,i+1]\}$

$\{[1,i],[3,i],[5,i+1]\}$ $\{[0,i+1],[1,i],[4,i]\}$ $\{[1,i],[5,i],[7,i+1]\}$

$\{[1,i],[6,i],[8,i+1]\}$ $\{[1,i],[6,i+1],[7,i]\}$ $\{[1,i],[3,i+1],[8,i]\}$

$\{[2,i],[3,i],[6,i+1]\}$ $\{[2,i],[4,i],[7,i+1]\}$ $\{[2,i],[5,i],[8,i+1]\}$

$\{[0,i+1],[2,i],[6,i]\}$ $\{[2,i],[5,i+1],[7,i]\}$ $\{[1,i+1],[2,i],[8,i]\}$

$\{[3,i],[4,i],[8,i+1]\}$ $\{[2,i+1],[3,i],[5,i]\}$ $\{[3,i],[4,i+1],[6,i]\}$

$\{[0,i+1],[3,i],[7,i]\}$ $\{[3,i],[7,i+1],[8,i]\}$ $\{[3,i+1],[4,i],[5,i]\}$

$\{[2,i+1],[4,i],[6,i]\}$ $\{[1,i+1],[4,i],[7,i]\}$ $\{[4,i],[6,i+1],[8,i]\}$

$\{[1,i+1],[5,i],[6,i]\}$ $\{[4,i+1],[5,i],[7,i]\}$ $\{[0,i+1],[5,i],[8,i]\}$

$\{[3,i+1],[6,i],[7,i]\}$ $\{[5,i+1],[6,i],[8,i]\}$ $\{[2,i+1],[7,i],[8,i]\}$

$GS(3,4,8,2)$: For $i \in ZZ_2$ form the following blocks:

$\{[0,i],[1,i],[2,i],[3,i+1]\}$ $\{[0,i],[1,i],[3,i],[4,i+1]\}$ $\{[0,i],[1,i],[2,i+1],[4,i]\}$

$\{[0,i],[1,i],[5,i],[6,i+1]\}$ $\{[0,i],[1,i],[6,i],[7,i+1]\}$ $\{[0,i],[1,i],[5,i+1],[7,i]\}$

$\{[0,i],[2,i],[3,i],[5,i+1]\}$ $\{[0,i],[2,i],[4,i],[7,i+1]\}$ $\{[0,i],[1,i+1],[2,i],[5,i]\}$

$\{[0,i],[2,i],[4,i+1],[6,i]\}$ $\{[0,i],[2,i],[6,i+1],[7,i]\}$ $\{[0,i],[3,i],[4,i],[6,i+1]\}$

$\{[0,i],[3,i],[5,i],[7,i+1]\}$ $\{[0,i],[1,i+1],[3,i],[6,i]\}$ $\{[0,i],[2,i+1],[3,i],[7,i]\}$

$\{[0,i],[3,i+1],[4,i],[5,i]\}$ $\{[0,i],[4,i],[5,i+1],[6,i]\}$ $\{[0,i],[1,i+1],[4,i],[7,i]\}$

$\{[0,i],[2,i+1],[5,i],[6,i]\}$ $\{[0,i],[4,i+1],[5,i],[7,i]\}$ $\{[0,i],[3,i+1],[6,i],[7,i]\}$

$\{[1,i],[2,i],[3,i],[6,i+1]\}$ $\{[1,i],[2,i],[4,i],[5,i+1]\}$ $\{[1,i],[2,i],[5,i],[7,i+1]\}$

$\{[0,i+1],[1,i],[2,i],[6,i]\}$ $\{[1,i],[2,i],[4,i+1],[7,i]\}$ $\{[1,i],[3,i],[4,i],[7,i+1]\}$

$\{[1,i],[2,i+1],[3,i],[5,i]\}$ $\{[1,i],[3,i],[5,i+1],[6,i]\}$ $\{[0,i+1],[1,i],[3,i],[7,i]\}$

$\{[0,i+1],[1,i],[4,i],[5,i]\}$ $\{[1,i],[3,i+1],[4,i],[6,i]\}$ $\{[1,i],[4,i],[6,i+1],[7,i]\}$

$\{[1,i],[4,i+1],[5,i],[6,i]\}$ $\{[1,i],[3,i+1],[5,i],[7,i]\}$ $\{[1,i],[2,i+1],[6,i],[7,i]\}$

$\{[0,i+1],[2,i],[3,i],[4,i]\}$ $\{[2,i],[3,i],[4,i+1],[5,i]\}$ $\{[2,i],[3,i],[6,i],[7,i+1]\}$

$\{[1,i+1],[2,i],[3,i],[7,i]\}$ $\{[2,i],[4,i],[5,i],[6,i+1]\}$ $\{[1,i+1],[2,i],[4,i],[6,i]\}$

$\{[2,i],[3,i+1],[4,i],[7,i]\}$ $\{[2,i],[3,i+1],[5,i],[6,i]\}$ $\{[0,i+1],[2,i],[5,i],[7,i]\}$

$\{[2,i],[5,i+1],[6,i],[7,i]\}$ $\{[1,i+1],[3,i],[4,i],[5,i]\}$ $\{[2,i+1],[3,i],[4,i],[6,i]\}$

$\{[3,i],[4,i],[5,i+1],[7,i]\}$ $\{[0,i+1],[3,i],[5,i],[6,i]\}$ $\{[3,i],[5,i],[6,i+1],[7,i]\}$

$\{[3,i],[4,i+1],[6,i],[7,i]\}$ $\{[4,i],[5,i],[6,i],[7,i+1]\}$ $\{[2,i+1],[4,i],[5,i],[7,i]\}$

$\{[0,i+1],[4,i],[6,i],[7,i]\}$ $\{[1,i+1],[5,i],[6,i],[7,i]\}$

$S(3,4,10,2)$. For $i \in ZZ_2$ form the following blocks:

$\{[0,i],[1,i+1],[4,i+1],[5,i]\}$ $\{[0,i],[5,i],[6,i+1],[9,i+1]\}$ $\{[0,i],[2,i+1],[3,i+1],[5,i$

$\{[0,i],[5,i],[7,i+1],[8,i+1]\}$ $\{[1,i],[4,i],[6,i+1],[9,i+1]\}$ $\{[1,i],[2,i+1],[3,i+1],[4,i$

$\{[1,i],[4,i],[7,i+1],[8,i+1]\}$ $\{[2,i+1],[3,i+1],[6,i],[9,i]\}$ $\{[6,i],[7,i+1],[8,i+1],[9,i$

$\{[2,i],[3,i],[7,i+1],[8,i+1]\}$ $\{[0,i],[1,i],[5,i+1],[6,i+1]\}$ $\{[0,i],[1,i],[2,i+1],[4,i+1$

$\{[0,i],[1,i],[7,i+1],[9,i+1]\}$ $\{[0,i],[1,i],[3,i+1],[8,i+1]\}$ $\{[2,i+1],[4,i+1],[5,i],[6,i$

$\{[5,i],[6,i],[7,i+1],[9,i+1]\}$ $\{[3,i+1],[5,i],[6,i],[8,i+1]\}$ $\{[2,i],[4,i],[7,i+1],[9,i+1$

$\{[2,i],[3,i+1],[4,i],[8,i+1]\}$ $\{[3,i+1],[7,i],[8,i+1],[9,i]\}$ $\{[0,i],[2,i],[5,i+1],[7,i+1$

$\{[0,i],[1,i+1],[2,i],[6,i+1]\}$ $\{[0,i],[2,i],[3,i+1],[4,i+1]\}$ $\{[0,i],[2,i],[8,i+1],[9,i+1$

$\{[1,i+1],[5,i],[6,i+1],[7,i]\}$ $\{[3,i+1],[4,i+1],[5,i],[7,i]\}$ $\{[5,i],[7,i],[8,i+1],[9,i+1$

$\{[1,i],[3,i+1],[4,i+1],[6,i]\}$ $\{[1,i],[6,i],[8,i+1],[9,i+1]\}$ $\{[3,i],[4,i],[8,i+1],[9,i+1$

$\{[0,i],[3,i],[5,i+1],[8,i+1]\}$ $\{[0,i],[1,i+1],[2,i+1],[3,i]\}$ $\{[0,i],[3,i],[6,i+1],[7,i+1$

$\{[0,i],[3,i],[4,i+1],[9,i+1]\}$ $\{[1,i+1],[2,i+1],[5,i],[8,i]\}$ $\{[5,i],[6,i+1],[7,i+1],[8,i$

$\{[4,i+1],[5,i],[8,i],[9,i+1]\}$ $\{[1,i],[2,i],[6,i+1],[7,i+1]\}$ $\{[1,i],[2,i],[4,i+1],[9,i+1$

$\{[4,i+1],[6,i],[7,i],[9,i+1]\}$ $\{[0,i],[4,i],[5,i+1],[9,i+1]\}$ $\{[0,i],[1,i+1],[3,i+1],[4,i$

$\{[0,i],[4,i],[6,i+1],[8,i+1]\}$ $\{[0,i],[2,i+1],[4,i],[7,i+1]\}$ $\{[1,i+1],[3,i+1],[5,i],[9,i$

$\{[5,i],[6,i+1],[8,i+1],[9,i]\}$ $\{[2,i+1],[5,i],[7,i+1],[9,i]\}$ $\{[1,i],[3,i],[6,i+1],[8,i+1$

$\{[1,i],[2,i+1],[3,i],[7,i+1]\}$ $\{[2,i+1],[6,i],[7,i+1],[8,i]\}$ $\{[0,i],[1,i+1],[6,i],[7,i+1$

$\{[0,i],[2,i+1],[6,i],[8,i+1]\}$ $\{[0,i],[3,i+1],[6,i],[9,i+1]\}$ $\{[0,i],[4,i+1],[5,i+1],[6,i$

$\{[1,i],[2,i+1],[7,i],[8,i+1]\}$ $\{[1,i],[3,i+1],[7,i],[9,i+1]\}$ $\{[1,i],[4,i+1],[5,i+1],[7,i$

$\{[2,i],[3,i+1],[8,i],[9,i+1]\}$ $\{[2,i],[4,i+1],[5,i+1],[8,i]\}$ $\{[3,i],[4,i+1],[5,i+1],[9,i$

$\{[0,i],[1,i+1],[7,i],[8,i+1]\}$ $\{[0,i],[2,i+1],[7,i],[9,i+1]\}$ $\{[0,i],[3,i+1],[5,i+1],[7,i$

$\{[0,i],[4,i+1],[6,i+1],[7,i]\}$ $\{[1,i],[2,i+1],[8,i],[9,i+1]\}$ $\{[1,i],[3,i+1],[5,i+1],[8,i$

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$\{[3,i],[4,i+1],[5,i],[6,i+1]\}$ $\{[0,i],[1,i+1],[8,i],[9,i+1]\}$ $\{[0,i],[2,i+1],[5,i+1],[8,i$

$\{[0,i],[3,i+1],[6,i+1],[8,i]\}$ $\{[0,i],[4,i+1],[7,i+1],[8,i]\}$ $\{[1,i],[2,i+1],[5,i+1],[9,i$

$\{[1,i],[3,i+1],[6,i+1],[9,i]\}$ $\{[1,i],[4,i+1],[7,i+1],[9,i]\}$ $\{[2,i],[3,i+1],[5,i],[6,i+1$

$\{[2,i],[4,i+1],[5,i],[7,i+1]\}$ $\{[3,i],[4,i+1],[6,i],[7,i+1]\}$ $\{[0,i],[1,i+1],[5,i+1],[9,i$

$\{[0,i],[2,i+1],[6,i+1],[9,i]\}$ $\{[0,i],[3,i+1],[7,i+1],[9,i]\}$ $\{[0,i],[4,i+1],[8,i+1],[9,i$

$\{[1,i],[2,i+1],[5,i],[6,i+1]\}$ $\{[1,i],[3,i+1],[5,i],[7,i+1]\}$ $\{[1,i],[4,i+1],[5,i],[8,i+1$

$\{[2,i],[3,i+1],[6,i],[7,i+1]\}$ $\{[2,i],[4,i+1],[6,i],[8,i+1]\}$ $\{[3,i],[4,i+1],[7,i],[8,i+1$

$\{[0,i],[1,i],[2,i],[5,i]\}$ $\{[1,i],[3,i],[4,i],[5,i]\}$ $\{[1,i],[2,i],[3,i],[6,i]\}$

$\{[0,i],[2,i],[4,i],[6,i]\}$ $\{[2,i],[3,i],[4,i],[7,i]\}$ $\{[0,i],[1,i],[3,i],[7,i]\}$

$\{[0,i],[3,i],[4,i],[8,i]\}$ $\{[1,i],[2,i],[4,i],[8,i]\}$ $\{[0,i],[1,i],[4,i],[9,i]\}$

$\{[0,i],[2,i],[3,i],[9,i]\}$ $\{[4,i],[7,i],[8,i],[9,i]\}$ $\{[4,i],[5,i],[6,i],[8,i]\}$

$\{[0,i],[5,i],[8,i],[9,i]\}$ $\{[0,i],[6,i],[7,i],[9,i]\}$ $\{[1,i],[5,i],[6,i],[9,i]\}$

$\{[1,i],[5,i],[7,i],[8,i]\}$ $\{[2,i],[5,i],[6,i],[7,i]\}$ $\{[2,i],[6,i],[8,i],[9,i]\}$

$\{[3,i],[6,i],[7,i],[8,i]\}$ $\{[3,i],[5,i],[7,i],[9,i]\}$ $\{[3,i],[4,i],[6,i],[9,i]\}$

$\{[0,i],[4,i],[5,i],[7,i]\}$ $\{[0,i],[1,i],[6,i],[8,i]\}$ $\{[1,i],[2,i],[7,i],[9,i]\}$

$\{[2,i],[3,i],[5,i],[8,i]\}$ $\{[2,i],[4,i],[5,i],[9,i]\}$ $\{[0,i],[3,i],[5,i],[6,i]\}$

$\{[1,i],[4,i],[6,i],[7,i]\}$ $\{[0,i],[2,i],[7,i],[8,i]\}$ $\{[1,i],[3,i],[8,i],[9,i]\}$

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