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On the chromatic number, colorings, and codes of the Johnson graph

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Abstract

We consider the Johnson graph $J(n,w)$, $0 \leqslant w \leqslant n$. The graph has $\binom{n}{w}$ vertices representing the $\binom{n}{w}$ w-subsets of an n-set. Two vertices are connected by an edge if the intersection between their w-subsets is a $(w-1)$-set. Let $\theta(n,w)$ be the chromatic number of this graph. It is well known that $\theta(n,w) \leqslant n$. We give some constructions which yield $\theta(n,w) < n$ for some cases of n and w. The colorings associated with the chromatic number and other colorings of the graph lead to improvements on the lower bounds on the sizes of some constant weight codes.

1. Introduction

The Johnson graph $J(n,w)$ is defined as follows. The vertex set, V_w^n, consists of all w-subsets of a fixed n-set (or all binary n-tuples with constant weight w). Two such w-subsets (n-tuples) are adjacent if and only if their intersection has size $w-1$ (there are $w-1$ coordinates in which both n-tuples have ONEs). This graph is very interesting since it is a distance-regular graph and because of the codes which it produces. With respect to $J(n,w)$ we will consider two fundamental questions in graph theory, the chromatic number of the graph and its largest independent set. These problems can be translated into coding theory and block design. A maximum independent set is the largest code of length n, constant weight w, and minimum Hamming distance (distance in short) 4 [4, 15]. It is also the largest packing of $(w-1)$-subsets by w-subsets. The chromatic number of the graph is the minimum number of disjoint constant weight codes of length n, weight w, and distance 4, for which the union is the set of all n-tuples with weight w [4, 15]. It is also the minimum number of disjoint packings of $(w-1)$-subsets by w-subsets, for which the union is the set of all w-subsets of the n-set.

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These partitions of packings are used to produce other independent sets in a method called the *partitioning construction* [4, 15]. All these questions are of considerable interest and we will try to give some answers now.

Let (n,d,w) denote a code of length n, constant weight w, and distance 4, and let $A(n,d,w)$ denote the maximum size of an (n,d,w) code. For bounds on $A(n,d,w)$ the reader is referred to [4, 10]. Let $\theta(n,w)$ denote the chromatic number of $J(n,w)$. Graham and Sloane [10] proved that $\theta(n,w) \leqslant n$ for all $0 \leqslant w \leqslant n$. Trivially we have $\theta(n,w) = \theta(n,n-w)$, $\theta(n,0) = 1$, and $\theta(n,1) = n$ [4, 15] . Also, it is easily observed that for even n, $\theta(n,2) = n-1$ and for odd n, $\theta(n,2) = n$. For $w = 3$, it is known [11, 12, 14], that for $n > 7$, $n \equiv 1$ or 3 (mod 6), $\theta(n,3) = n-2$, and for $n > 7$, $n \equiv 0$ or 2 (mod 6), $\theta(n,3) = n-1$, and $\theta(7,3) = 6$, $\theta(n,3) = n$ for $n \leqslant 6$. For $n \equiv 4$ (mod 6) we have $\theta(n,3) = n$, and it is conjectured that for $n > 5$, $n \equiv 5$ (mod 6), $\theta(n,3) = n-1$, but this is proved [5, 6] only for some infinite cases. For $w > 3$ the evaluation of $\theta(n,w)$ becomes more difficult. van Pul and Etzion [15] proved that if $\theta(n,w) < n$ for all even w, then $\theta(2^i n,w) < 2^i n$ for all even w, and $i \geqslant 0$. This result can be applied on $n = 4,6$, and 10 [15, 4]. The result proved in [15] is in fact that if for a given w_0, $\theta(n,w) < n$ for all even w, $2 \leqslant w \leqslant w_0$, then $\theta(2n,w) < 2n$ for all even w, $2 \leqslant w \leqslant w_0$. For $w = 4$, clearly $\theta(2n,4) < 2n$ implies $\theta(2^i n,4) < 2^i n$, $i \geqslant 1$. This result can be applied on $n = 2,3,5$, and 7 [4]. Etzion [7] proved that if $\theta(2n,4) \leqslant 2n-2$, $n \equiv 2$ or 4 (*mod* 6), then $\theta(4n,4) \leqslant 4n-2$. This result was applied only to obtain $\theta(2^i,4) \leqslant 2^i - 2$, for $i \geqslant 3$.

Two types of constructions can be given for attaining new upper bounds on $\theta(n,w)$, direct constructions and recursive constructions. In Section 3, we will present a direct approach, which unfortunately we could not apply without a computer search. We will show cases for which $\theta(n,w) \leqslant n-1$, when $(w-1)w$ is relatively prime to $n-1$. In Section 4 we will give recursive constructions, a simple doubling construction and a more complicated quintupling one which can be applied only on $w = 4$. In Section 5 we will discuss applications of the previous results. In Section 6 we will introduce a specific interesting coloring of $J(11,4)$. In Section 7 we will improve the partitioning construction for constructing independent sets in $J(n,w)$ by using appropriate colorings. But, we start in Section 2 with the definitions for the designs and methods used in this paper.

2. The used designs and the partitioning construction

Since we use the partitioning construction to obtain new codes (larger independent sets), and since we will improve this method we will first introduce the concept of partitioning. The representation is taken from [4].

A *partition* $\Pi(n,w) = (X_1,\ldots,X_m)$ is a collection of disjoint sets or *classes* $X_1,\ldots,X_m$, each of which is a code of length n, distance 4 and constant weight w, and whose union contains all $\binom{n}{w}$ vectors of weight w. The vector $\pi(n,w) = (|X_1|,\ldots,|X_m|)$ with integer components is the *index vector* of the partitions $\Pi(n,w)$, and

$$\pi(n,w)\cdot\pi(n,w)=\sum_{i=1}^{m}|X_i|^2$$

is its *norm*. We always assume $|X_1|\geqslant\cdots\geqslant|X_m|$. When there are several different partitions available for a given n and w we often denote them by $\Pi_1(n,w),\Pi_2(n,w),\cdots$ and their index vectors by $\pi_1(n,w),\pi_2(n,w),\ldots$.

The *direct product* $\Pi(n_1,w_1)\times\Pi(n_2,w_2)$ of two partitions $\Pi(n_1,w_1)=(X_1,\ldots,X_{m_1})$, $\Pi(n_2,w_2)=(Y_1,\ldots,Y_{m_2})$ consists of the vectors

$$\{(u,v)\colon u\in X_i,\quad v\in Y_i,\quad 1\leqslant i\leqslant m\},$$

where $m=\min\{m_1,m_2\}$. This set (which is only part of the final code) clearly has length n_1+n_2, distance 4, weight w_1+w_2, and contains

$$\pi(n_1,w_1)\cdot\pi(n_2,w_2)=\sum_{i=1}^{m}|X_i||Y_i|$$

words.

The construction: To obtain a code of length n, distance 4 and weight w by the partitioning construction we write $n=n_1+n_2$, choose $\varepsilon=0$ or 1, and take the union of the direct products

$$\begin{aligned}&\Pi(n_1,\varepsilon)\times\Pi(n_2,w-\varepsilon),\\&\Pi(n_1,\varepsilon+2)\times\Pi(n_2,w-\varepsilon-2),\\&\Pi(n_1,\varepsilon+4)\times\Pi(n_2,w-\varepsilon-4),\\&\qquad\vdots\end{aligned}$$

It is apparent that this code does have the required properties, and contains

$$\pi(n_1,\varepsilon)\cdot\pi(n_2,w-\varepsilon)+\pi(n_1,\varepsilon+2)\cdot\pi(n_2,w-\varepsilon-2)\\+\pi(n_1,\varepsilon+4)\cdot\pi(n_2,w-\varepsilon-4)+\cdots$$

codewords. For examples of how the construction is applied to obtain specific codes the reader is referred to [4, 15].

We next discuss the choice for a good partition. We say that one partition $\Pi_1(n_1,w_1)$ *dominates* another $\Pi_2(n_1,w_1)$ if

$$\pi_1(n_1,w_1)\cdot\pi(n_2,w_2)\geqslant\pi_2(n_1,w_1)\cdot\pi(n_2,w_2)$$

holds for all choices of n_2, w_2, and all possible index vectors $\pi(n_2,w_2)$. If a partition is dominated it need never be used in the construction. As was proved by [4], $\pi_1(n_1,w_1)=(a_1,\cdots,a_{m_1})$ dominates $\pi_2(n_1,w_1)=(b_1,\cdots,b_{m_2})$ if and only if

$$\sum_{i=1}^{j}a_i\geqslant\sum_{i=1}^{j}b_i\quad\text{for all } j=1,\ldots,\max\{m_1,m_2\}.$$

A partition $\Pi(n,w)$ is *optimal* if it dominates all other partitions $\Pi'(n,w)$ with the same n and w.

We will also use in our paper some concepts of block design, and hence we give some necessary definitions.

A *Steiner system* $S(t,k,n)$ is a collection of k-subsets (called *blocks*) of an n-set (whose elements are called *points*) such that each t-subset of the n-set is contained in exactly one block. A *packing quadruple system* (*PQ*) of order n is a pair (Q,q), where $Q = \{0,1,\dots,n-1\}$ is a set of points and q is a collection of 4-element subsets of Q such that every 3-element subset of Q is a subset of at most one block of q. If every 3-element subset of Q is a subset of exactly one block of q the packing is a Steiner quadruple system $S(3,4,n)$. A *packing triple system* (*PT*) of order n is a pair (Q,q), where $Q = \{0,1,\dots,n-1\}$ is a set of points and q is a collection of 3-element subsets of Q such that every 2-element subset of Q is a subset of at most one block of q. If every 2-element subset of Q is a subset of exactly one block of q the packing is a Steiner triple system $S(2,3,n)$. A *near-1-factorization* of the complete graph K_n, n odd, is a coloring of the graph edges with n colors. Each color consists of $(n-1)/2$ edges and one vertex is isolated. The set of edges of each color is called a *near-1-factor*. If the vertex set of the graph is Z_n, then the near-1-factorization $F = \{F_0, F_1, \dots, F_{n-1}\}$ is a partition of pairs into n disjoint packings. It is well known that near-1-factorization exists for every odd n.

A very interesting concept in block design is the *large set*. The large set is a partition of the space into disjoint optimal designs. A near-1-factorization is a large set of near-1-factors. For $n \equiv 1$ or 3 (mod 6), $n > 7$, there exists a large set of Steiner triple systems of order n, with $n-2$ $S(2,3,n)$ [11, 12, 14]. For $n \equiv 0$ or 2 (mod 6), $n > 7$, there exists a large set of PTs of order n, with $n-1$ PTs which can be derived from the large set of Steiner triple systems. Large sets of packing quadruple systems are not known, except for trivial ones for $n = 4$ and $n = 5$, and the best results in this area are given in [9].

3. A direct construction with computer search

In this section we will find partitions of all n-tuples with weight w into $n-1$ codes with minimum Hamming distance 4. This will be done by using computer search. To make the search more effective, we must limit it, and this can be done by looking for codes and partitions with a simple combinatorial and algebraic structure. We will try to find partitions in which each code is obtained from the first code by a simple permutation on the coordinates. To simplify the search even more, we will try that the first code in the partition will have some nice automorphism, or "almost" a nice automorphism.

Let C be an $(n+1,d,w)$ code. Each word $c \in C$ is represented by a w-tuple $c = \langle x_1,x_2,\dots,x_w\rangle$, where $x_i \in \{\infty\} \cup Z_n$. For each binary word, c, of length $n+1$ with constant weight w we define a shift operator $Sh_i(c)$ by

$$Sh_i(\langle x_1,x_2,\dots,x_w\rangle) = \langle x_1 + i(\mathrm{mod}\ n),\dots,x_w + i(\mathrm{mod}\ n)\rangle,$$

where $\infty + i = \infty$; for a code C, we define $Sh_i(C) = \{Sh_i(c) : c \in C\}$. The partition will have the structure $\Pi(n+1,w) = \{C_0,C_1,\dots,C_{n-1}\}$ where $C_i = Sh_i(C_0)$, $1 \leqslant i \leqslant n-1$. For n relatively prime to $(w-1)w$ we define the layered graph $G^{n+1}(V,E)$, with s layers, $s = \binom{n+1}{w}/n$, where $V = \bigcup_{i=1}^{s} V_i$, is the set of all w-subsets of $\{\infty\} \cup Z_n$, and $E = \{(u,v)\,|\,u,v \in V,$ and $|u \cap v| = w-1\}$. All the layers V_i, $1 \leqslant i \leqslant s$, are disjoint, and are of equal size n. Each layer V_i consists of a single orbit of the operator $Sh_k(\cdot)$, on some w-subset of $\{\infty\} \cup Z_n$. Clearly an independent set in the graph $G^{n+1}(V,E)$ corresponds to an $(n+1,4,w)$ code. If, in addition to this, we find an independent set C of size s in the graph that contains exactly one vertex from each layer V_i, $1 \leqslant i \leqslant s$, then the sets $C_i = Sh_i(C)$, $0 \leqslant i < n$, are also independent sets in the graph, and $\{C_0,C_1,\dots,C_{n-1}\}$ is a partition of the vectors of length $n+1$ and weight w.

Let $M(n)$ be the multiplicative group of the residues, between 1 and $n-1$, modulo n, which are relatively primes to n. For $\beta \in M(n)$, we define $T_\beta(\langle x_1,\dots,x_w\rangle) = \langle \beta x_1,\dots,\beta x_w\rangle$, where the multiplication is done modulo n, and $T_\beta(V_i) = \{T_\beta(v) : v \in V_i\}$. The graph $\tilde{G}_\beta^{n+1} = G^{n+1}(T_\beta(V),E)$, where E is defined as before, is isomorphic to $G^{n+1}(V,E)$. Let $\mathscr{T}_\beta = \{T_{\beta^i} : i \geqslant 0\}$; $\mathscr{T}_\beta$ is a cyclic subgroup of the automorphism group of the graph $G^{n+1}(V,E)$. Note that the order of $\mathscr{T}_\beta$ is equal to the order of β in $M(n)$. For each $v \in V$, let the orbit of v under T_β be $[v]_\beta = \{T(v) : T \in \mathscr{T}_\beta\}$.

We now turn to the description of our search program. The program receives three parameters: an integer n, weight w, and $\beta \in M(n)$. The program builds the graph $G^{n+1}(V,E)$, and using a backtracking algorithm it tries to build an independent set C of size s, that contains exactly one vertex from each layer V_i. Let $[v]_\beta^*$ denote the largest subset of $[v]_\beta$ for which, for each $u_1,u_2 \in [v]_\beta^*$, we have $Sh_k(u_1) \neq u_2$ for all $1 \leqslant k < n$. In each step the program chooses a vertex $v \in V_i$ where V_i is a layer that still does not contain a vertex from the independent set, C, and checks if $C \cup [v]_\beta^*$ is still an independent set. If it is, then $[v]_\beta^*$ is joined to C to form the new C. If for all $v \in V_i$ that we take, $[v]_\beta = [v]_\beta^*$ then $\bigcup_{c\in C} T_\beta(c) = C$, i.e., T_β is an automorphism of C. Clearly the complexity of the search decreases as the order of β in $M(n)$ increases, but β with too high order might not result in a code of size s. If you choose $\beta = 1$ then the search is the trivial search. The best choice (complexity wise) is to choose β as a generator of $M(n)$.

Using this search program we found several partitions. These partitions are listed in the Appendix, together with β and the order of β we used for the first code. Only for $n = 12$ and $w = 6$, T_β is not an automorphism of C.

4. Recursive constructions

In this section we present two recursive constructions to obtain $\theta(n,w) \leqslant n-1$. The first one is a simple generalization of the result obtained in [15].

Theorem 1. *If for a given* w_0, $\theta(n,w) < n$ *for all* $2 \leqslant w \leqslant w_0$, *then* $\theta(2n,w) < 2n$ *for all* $2 \leqslant w \leqslant w_0$.

Proof. For even w the result was proven by [15] as mentioned in the Introduction. For odd w note that if we take the union of $\Pi(n,1)\times\Pi(n,w-1)$ and $\Pi(n,w-1)\times\Pi(n,1)$ as the first code, then by using all distinct direct products of the partitions to obtain $(2n,4,w)$ codes we need no more than another $2n-2$ codes to cover all vectors of length $2n$ and weight w. This is as in "Construction B" for combining partitions (see [4, 15]). □

Let A_i, $0 \leqslant i \leqslant 4$, denote a code of length 5 whose codewords have weight 4 in the Lee metric, and $a_0a_1a_2a_3a_4$ is a codeword in A_i if and only if $\sum_{j=0}^{4} ja_j \equiv 4i \pmod 5$. Let $\{PQ_0,\ldots,PQ_{n-1}\}$ be a partition of quadruples on $Z_n \cup \{\alpha\}$. Let $\{PT_0,\ldots,PT_{n-1}\}$ be a partition of triples on $Z_n \cup \{\alpha\}$, and $\{F_0,\ldots,F_{n-1}\}$ be a near-1-factorization of K_n on Z_n, such that in F_i, $i \in Z_n$, vertex i is isolated. From these sets we form the following sets S_{ij}, $i \in Z_5$, $j \in Z_n$, on the points $Z_5 \times Z_n \cup \{\beta\}$. For any given word $a_0a_1a_2a_3a_4 \in A_i$ we form in S_{ij} one or two of the following block types:

(A) $\{(r,x),(s,y),(t,z),(q,x+y+z+j)\}$,
if $a_r = a_s = a_t = a_q = 1$ and all $x,y,z \in Z_n$.

(B) $\{(r,x),(r,y),(s,v),(s,w)\}$,
if $a_r = a_s = 2$, and all $\{x,y\} \in F_m$, $m \in Z_n$, and $\{v,w\} \in F_{m+j}$.

(C) $\{(i,x),(i,y),(i,z),(i,w)\}$,
for any $\{x,y,z,w\} \in PQ_j$.

(D) $\{(i,x),(i,y),(i,z),\beta\}$,
for any $\{x,y,z,\alpha\} \in PQ_j$.

(E) $\{(r,x),(r,y),(r,z),(s,m+j)\}$,
if $a_r = 3$, $a_s = 1$, and $\{x,y,z\} \in PT_m$ for $m \in Z_n$.

(F) $\{(r,x),(r,y),(s,m+j),\beta\}$,
if $a_r = 3$, $a_s = 1$, and all $\{x,y,\alpha\} \in PT_m$ for $m \in Z_n$.

(G) $\{(r,x),(r,y),(s,l),(t,m+l+j)\}$,
if $a_r = 2$, $a_s = a_t = 1$, and $(x,y) \in F_m$ for $m,l \in Z_n$.

(H) $\{(i,m),(s,l),(t,m+l+j),\beta\}$,
if $a_i = 2$, $a_s = a_t = 1$ and for all $m,l \in Z_n$.

We leave the proof of the following theorem to the reader.

Theorem 2. $\{S_{ij} : i \in Z_5, j \in Z_n\}$ *is a partition of all quadruples from a* $(5n+1)$*-set.*

Corollary 1. *If* $\theta(n+1,4) \leqslant n$ *for* $n \equiv 1$ *or* $5 \pmod 6$, $n > 5$, *then* $\theta(5n+1,4) \leqslant 5n$.

To generalize this construction in order to obtain $\theta(kn+1,4) \leqslant kn$ we need a set of codes with weight 4 in the Lee metric, which satisfy certain properties. Unfortunately, we are not optimistic about the existence of such codes, and hence we will not go into details about this generalization.

5. Applications of the constructions

By using the partition of $\Pi(12,5)$, given in the Appendix and obtained via the construction of Section 3, the partitions in [4] and the partitioning construction, four new bounds on the $A(n,4,w)$ were obtained.

$A(24,4,7) \geqslant 15656$.

$A(24,4,9) \geqslant 59387$.

$A(24,4,11) \geqslant 116937$.

$A(25,4,8) \geqslant 46832$.

Another motivation to find partitions of quadruples with codes of the same size is the suggested construction of Stanton [13] for producing Steiner quintuple system $S(4,5,2n+1)$. For this construction a coloring of $J(n+1,4)$ with n colors, related to n codes with the same size, is needed. Unfortunately, none of the partitions that we have found was good enough to produce a new Steiner quintuple system.

Finally, we would like to mention that from the known results on $\theta(n,4)$ given in the Introduction, from $\Pi(18,4)$ given in the Appendix, and from Corollary 1, we can get many infinite sequences of values for n, for which $\theta(n,4) \leqslant n-1$. On the other hand, Theorem 1 can be applied only on values of n where $n \equiv 0 \pmod 6$. The reason for this is that for odd n, $\theta(n,2)=n$, for $n \equiv 4 \pmod 6$, $\theta(n,3)=n$, and for $n \equiv 2 \pmod 6$, $2n \equiv 4 \pmod 6$ and $\theta(2n,3)=n$. By using $\theta(12,5)=11$, with its coloring given in the Appendix, and the partitions for $n=12$ given in [4] we obtain that for $2 \leqslant w \leqslant 10$ we have $\theta(3\cdot 2^i, w) \leqslant 3\cdot 2^i - 1$ for $i \geqslant 2$. Other results that can be obtained are very specific, e.g., for $2 \leqslant w \leqslant 6$ we have $\theta(16,w) \leqslant 15$.

6. An interesting partition of length 11 and weight 4

The following code is a constant weight code of length 11, weight 4, and minimum Hamming distance 4,

100 100 100 01	001 110 000 01	100 001 010 10	100 010 110 00
010 010 010 01	001 000 110 01	010 011 000 10	010 001 011 00
001 001 001 01	000 100 010 11	010 000 110 10	001 100 101 00
100 011 000 01	000 010 001 11	001 101 000 10	010 110 100 00
100 000 011 01	000 001 100 11	001 000 011 10	001 011 010 00
010 101 000 01	010 100 001 10	100 110 000 10	100 101 001 00
010 000 101 01	001 010 100 10	100 000 101 10	000 101 110 00

000 110 011 00
000 011 101 00
011 010 001 00
101 001 100 00
110 100 010 00
110 000 000 11

By applying the eight permutations $(1,2,3,6,4,5,8,9,7,10,11)$, $(1,2,3,5,6,4,9.7,8,10,11)$, (9,7,8,1,2,3,4,5,6,10,11), (7,8,9,1,2,3,6,4,5,10,11), (8,9,7,1,2,3,5,6,4,10,11), (4,5,6,8,9,7,1,2,3,10,11), (6,4,5,9,7,8,1,2,3,10,11), (5,6,4,7,8,9,1,2,3,10,11), on the first 33 codewords we obtain 8 codes with 33 codewords for which we add the vectors 10100000011, 01100000011, 00011000011, 00010100011, 00001100011, 00000011011, 00000010111, and 000000001111, respectively. Now, we have 9 disjoint codes of length 11, weight 4, minimum distance 4, and size 34. The 24 missing vectors of length 11 and weight 4 are the 24 vectors which contain the triples 11100000000, 00011100000, and 000000011100. These vectors can be easily partitioned to 8 disjoint codes of size 3. This partition has norm 10476.

By exchanging some codewords between the codes we obtain a partition with index vector (34,34,34,34,34,34,34,34,34,12,3,3,3,3) and its norm is equal to 10584. Its coloring is:

```
897CA325616445D136424556E589783971221453566 4B67897289314978917823DBEC
A563142312971289456739784523 16A235188642389715319A212374645236123 1889
7641235A75612789343129693178645921243 7238A152899757846456296 4A8575674
38964125312379453197826823975648 1127A3789563124A23185316294397786 8954
6485156437945A9678178894976556A744596286451897CABED312
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where the words are ordered lexicographically and the list given is of their color's number [4]. This partition is of special interest for two reasons:

1. It leads to some new lower bounds on $A(n,4,w)$ as we will see in the next section.
2. Although $A(11,4,4) = 35$ [1], usually the best lower bounds on $A(n,4,4)$ for $n \equiv 5 \pmod 6$ are obtained by using the partitioning construction (other better bounds are given in [2]). A code, with the same size of the one obtained by the partitioning construction, is also derived by taking all codewords which start with 0 from a code which attains $A(n+1,4,4)$ [3]. For $n = 11$ and $w = 4$ the size of the derived code is 34. The maximum number of disjoint codes with this size is $n-2$. So, in some sense this partition is very close to a large set of quadruples. Unfortunately, the codes of size 34 cannot be extended to codes which attain $A(12,4,4) = 51$. Also, we could not generalize this partition to obtain other partitions of length $n \equiv 5 \pmod 6$ with $n-2$ codes like this.

As we will see in the next section, although the second partition dominates the first one, the first one is more useful and it will be used in a modification of the partitioning construction.

7. Improving on the partitioning method

In this Section we will give some new bounds on $A(n,4,w)$ which improve on the ones that appeared in Brouwer et al. [4]. We use a method of improving on partitioning. This method was applied on three parameters and we will present it with these specific parameters.

Assume the partitioning method is applied on $n = n_1 + n_2$. Assume further that n_2 is odd and the direct products of $\Pi(n_1,w) \times \Pi(n_2,0)$ and $\Pi(n_1,w-2) \times \Pi(n_2,2)$ are taken. Usually, for $\Pi(n_2,2)$ we use the optimal partition with n_2 codes of size $(n_2-1)/2$. But without using the last coordinate, for the first n_2-2 codes, we have a partition with n_2-2 codes of size $(n_2-1)/2$ and n_2-1 codes of size 1 with a codeword which have a ONE in the last coordinate (usually we do not use these n_2-1 codes in the offered method). Now, we can take the direct product of a code which attains of length n_1, weight $w-1$, and minimum Hamming distance 4, by the word of length n_2, weight 1, with ONE in the last coordinate and join it to the code produced by the partitioning construction. We must make sure that:
1. The union of the codes of weight w (taken for $\Pi(n_1,w) \times \Pi(n_2,0)$) and $w-1$ have minimum Hamming distance 3.
2. The union of each of the codes with weight $w-2$, if taken with direct products with the codes of size 1, and the code of weight $w-1$ have minimum Hamming distance 3.

A similar idea can be applied if we use $\Pi(n_1,w-3) \times \Pi(n_2,3)$ and we take $\Pi(n_2,3)$ with codes for which some pair is not covered, and n_2-2 codes of size 1 with ONES in the uncovered coordinates (which again we usually do not use). A partition like this can be obtained for $n = 3k+2$, $k \geq 1$, k odd. We were able to find a partition of order $3k+2$ on the points Z_{3k+2} with $3k$ optimal packings, each one does not cover the pair $\{3k, 3k+1\}$, and the remaining triples are $\{\{i, k+i, 2k+i\} : 0 \leqslant i \leqslant k-1\} \cup \{\{i, 3k, 3k+1\} : 0 \leqslant i \leqslant 3k-1\}$. The proof takes ideas given in [5, 6, 8]. $\{\{i, k+i, 2k+i\} : 0 \leqslant i \leqslant k-1\}$ forms the $(3k+1)$th code. As an example, for $n = 11$, the code

111000000 00	000110100 00	000101000 10	001001000 01
100000101 00	100001010 00	001000100 10	000000110 01
001100001 00	010100010 00	000000011 10	010000001 01
010001100 00	001010010 00	010010000 10	100010000 01
000011001 00			

and the eight cyclic permutations on the first 9 coordinates result in 9 disjoint codes of size 17. The remaining vectors are 10010010000, 01001001000, 00100100100, which form the 10th code of the partition, and nine codes of size 1 with 2 ONES in the last two coordinates.

For weight 4, we can use a similar idea if some triples are not covered. The partition for $n = 11$, $w = 4$, and index vector (34,34,34,34,34,34,34,34,34,3,3,3,3,3,3,3,3,3) is good for this purpose since the vectors 11100000000, 00011100000, 00000011100 are not covered in the codes of size 34.

Now, we will present the six new bounds for $n \leqslant 28$ obtained by this method. The partitions of pairs, triples, and quadruples discussed in this method will be called special.

$A(21,4,7) \geqslant 6156$ with the following direct products. $\Pi(10,0) \times \Pi(11,7)$, $\Pi(10,2) \times \Pi(11,5)$, $\Pi(10,4) \times \Pi(11,3)$ (we take the special $\Pi(11,3)$), $\Pi(10,6) \times \Pi(11,1)$ (we take $\Pi(10,6)$ which attains $\theta(10,6) \leqslant 9$), and the direct product of a code which attains $A(10,4,5)$ by the vector of length 11 with 2 ONES in the last two coordinates.

$A(21,4,8) \geqslant 10753$ with the following direct products. $\Pi(10,1) \times \Pi(11,7)$, $\Pi(10,3) \times \Pi(11,5)$, $\Pi(10,5) \times \Pi(11,3)$ (take the special $\Pi(11,3)$), $\Pi(10,7) \times \Pi(11,1)$ (from $\Pi(10,7)$ we use only 9 codes of size 13), and the direct product of a code which attains $A(10,4,6)$ by the vector of length 11 with 2 ONES in the last two coordinates.

$A(21,4,9) \geqslant 16897$ with the following direct products. $\Pi(10,2) \times \Pi(11,7)$ (we take the special $\Pi(11,7)$), $\Pi(10,4) \times \Pi(11,5)$, $\Pi(10,6) \times \Pi(11,3)$ (we take the special $\Pi(11,3)$), $\Pi(10,8) \times \Pi(11,1)$, the direct product of a code which attains $A(10,4,7)$ by the vector of length 11 with 2 ONES in the last two coordinates, the direct product of the vector 0000000001 by the vectors 00011111111, 11100011111, 11111100011, and the direct product of the vector 0000000000 by the vectors 01101111111, 10110111111, 11011011111, 11111101101, 11111110110.

$A(22,4,7) \geqslant 8252$ with the following direct products. $\Pi(11,1) \times \Pi(11,6)$, $\Pi(11,3) \times \Pi(11,4)$, $\Pi(11,5) \times \Pi(11,2)$ (we take the special $\Pi(11,2)$). We take a code which attains $A(12,4,7) = 80$, delete its last coordinate to get a code A with weight 6, and a code B with weight 7. Now we take the direct product of B by 00000000000, and the direct product of A by the vector of length 11 with ONE in the last coordinate. Since not all the vectors of weight 5 are covered by A we can use an appropriate permutation such that we can add the codeword which is formed by a direct product of the last and only unused codeword of the partition $\Pi(11,5)$ by the word 10000000001.

$A(22,4,8) \geqslant 16430$ with the following direct products. $\Pi(11,7) \times \Pi(11,1)$ (for $\Pi(11,7)$ we use only 9 codes of size 34 from the special $\Pi(11,7)$), $\Pi(11,5) \times \Pi(11,3)$ (we take the special $\Pi(11,3)$), the direct product of a code which attains $A(11,4,6)$ by the vector of length 11 with 2 ONES in the last two coordinates, and from $\Pi(11,8) \times \Pi(11,0)$ take the direct product of the three uncovered vectors of weight 8 in $\Pi(11,7)$ by the vector 00000000000. Similarly we take vectors from the direct products $\Pi(11,3) \times \Pi(11,5)$, $\Pi(11,2) \times \Pi(11,6)$, $\Pi(11,1) \times \Pi(11,7)$, and $\Pi(11,0) \times \Pi(11,8)$.

$A(25,4,10) \geqslant 140340$ with the direct products as in Brouwer et al. [4], where in $\Pi(12,8) \times \Pi(13,2)$ we take the special $\Pi(13,2)$, we add the direct product of a code which attains $A(12,4,9)$ by the vector of length 13 with a ONE in the last coordinate, and in $\Pi(12,10) \times \Pi(13,0)$ we take the only code of length 12, weight 10, and size 6, for which the union with the code which attains $A(12,4,9)$ is a code with minimum distance 3.

Appendix

Let $\beta \in GF(p)$, p prime, and let $r = \mathrm{o}(\beta)$. For each partition we list a set of words, $\{c_0, \ldots, c_{s-1}\}$ of weight w. $\Pi(p+1, w) = \{C_0, C_1, \ldots, C_{p-1}\}$ is obtained as follows.

$$C = \{\beta^j c_i : 0 \leqslant i < s, \text{ and } 0 \leqslant j < r,\ \beta^{j_1} c_i \neq Sh_k(\beta^{j_2} c_i),\ 0 \leqslant j_1 < j_2 < r,$$
$$1 \leqslant k < p\},$$
$$C_i = Sh_i(C) \text{ for } 0 \leqslant i < p.$$

$\Pi(12,4)$

$\beta = 4,\ \mathrm{o}(\beta) = 5,$

$\langle \infty, 0, 1, 10 \rangle$	$\langle \infty, 3, 4, 6 \rangle$	$\langle \infty, 5, 7, 8 \rangle$
$\langle 2, 3, 4, 5 \rangle$	$\langle 4, 5, 6, 8 \rangle$	$\langle 5, 6, 7, 10 \rangle$
$\langle 0, 1, 2, 6 \rangle$	$\langle 1, 2, 3, 9 \rangle$	$\langle 0, 2, 3, 10 \rangle$

$\Pi(12,5)$

$\beta = 4,\ \mathrm{o}(\beta) = 5,$

$\langle \infty, 0, 1, 2, 3 \rangle$	$\langle \infty, 3, 4, 5, 7 \rangle$	$\langle \infty, 5, 6, 7, 10 \rangle$
$\langle \infty, 1, 6, 7, 8 \rangle$	$\langle \infty, 0, 1, 7, 10 \rangle$	$\langle \infty, 1, 2, 4, 5 \rangle$
$\langle 0, 1, 8, 9, 10 \rangle$	$\langle 4, 5, 6, 7, 9 \rangle$	$\langle 0, 1, 2, 5, 10 \rangle$
$\langle 3, 7, 8, 9, 10 \rangle$	$\langle 2, 3, 4, 5, 10 \rangle$	$\langle 3, 5, 6, 7, 8 \rangle$
$\langle 0, 2, 3, 9, 10 \rangle$	$\langle 0, 1, 2, 6, 8 \rangle$	
$\langle 2, 6, 7, 8, 10 \rangle$	$\langle 1, 3, 4, 5, 9 \rangle$	

$\Pi(12,6)$

$\beta = 4,\ \mathrm{o}(\beta) = 5,$

$\langle \infty, 1, 2, 3, 4, 5 \rangle$	$\langle \infty, 1, 7, 8, 9, 10 \rangle$	$\langle \infty, 0, 1, 2, 5, 10 \rangle$
$\langle \infty, 0, 4, 5, 6, 7 \rangle$	$\langle \infty, 0, 5, 8, 9, 10 \rangle$	$\langle \infty, 3, 5, 6, 7, 8 \rangle$
$\langle \infty, 5, 6, 7, 9, 10 \rangle$	$\langle \infty, 2, 6, 7, 8, 10 \rangle$	$\langle \infty, 0, 4, 6, 9, 10 \rangle$
$\langle \infty, 1, 2, 3, 7, 10 \rangle$	$\langle 0, 6, 7, 8, 9, 10 \rangle$	$\langle 0, 1, 2, 4, 9, 10 \rangle$
$\langle 1, 5, 6, 7, 8, 9 \rangle$	$\langle 0, 1, 2, 3, 4, 8 \rangle$	$\langle 0, 2, 3, 4, 5, 6 \rangle$
$\langle 1, 2, 3, 4, 6, 7 \rangle$	$\langle 3, 4, 5, 6, 8, 10 \rangle$	$\langle 2, 4, 5, 6, 7, 9 \rangle$
$\langle 0, 1, 2, 4, 5, 7 \rangle$	$\langle 0, 1, 3, 4, 5, 9 \rangle$	

$\Pi(14,4)$

$\beta = 8,\ o(\beta) = 4,$

$\langle\infty,0,1,12\rangle$	$\langle\infty,2,3,5\rangle$	$\langle\infty,3,4,7\rangle$	$\langle\infty,1,2,6\rangle$
$\langle\infty,0,6,7\rangle$	$\langle\infty,0,2,11\rangle$	$\langle\infty,6,8,12\rangle$	$\langle 5,6,7,8\rangle$
$\langle 8,9,10,12\rangle$	$\langle 0,1,2,5\rangle$	$\langle 0,7,8,9\rangle$	$\langle 4,10,11,12\rangle$
$\langle 1,2,11,12\rangle$	$\langle 3,4,6,8\rangle$	$\langle 4,5,7,11\rangle$	$\langle 0,6,10,11\rangle$
$\langle 4,7,8,10\rangle$	$\langle 4,6,7,9\rangle$	$\langle 4,5,8,9\rangle$	$\langle 0,3,6,12\rangle$
$\langle 1,5,10,11\rangle$	$\langle 3,6,7,10\rangle$	$\langle 2,3,10,11\rangle$	$\langle 2,6,7,11\rangle$
$\langle 1,3,10,12\rangle$	$\langle 1,5,8,12\rangle$		

$\Pi(18,4)$

$\beta = 8,\ o(\beta) = 8,$

$\langle\infty,0,1,16\rangle$	$\langle\infty,2,3,5\rangle$	$\langle\infty,3,4,7\rangle$	$\langle\infty,4,15,16\rangle$
$\langle\infty,4,5,11\rangle$	$\langle\infty,0,3,14\rangle$	$\langle 7,8,9,10\rangle$	$\langle 4,5,6,8\rangle$
$\langle 10,11,12,15\rangle$	$\langle 1,2,3,7\rangle$	$\langle 9,10,11,16\rangle$	$\langle 0,6,15,16\rangle$
$\langle 1,2,15,16\rangle$	$\langle 0,11,12,14\rangle$	$\langle 0,4,14,15\rangle$	$\langle 4,5,7,14\rangle$
$\langle 1,2,4,12\rangle$	$\langle 5,10,11,13\rangle$	$\langle 3,4,6,16\rangle$	$\langle 6,8,9,11\rangle$
$\langle 6,7,10,11\rangle$	$\langle 0,1,4,7\rangle$	$\langle 1,8,14,15\rangle$	$\langle 5,8,9,12\rangle$
$\langle 5,6,11,12\rangle$	$\langle 2,7,13,14\rangle$	$\langle 3,8,9,14\rangle$	$\langle 2,8,9,15\rangle$

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