119497558976453, 153639718684011, 185427249719400, 210150883015320, 223708998029760]. Syndromes $\in \overline{(0, \alpha)}$.

Our computer programs written in the computer algebra system Maple have taken 391 sec. on a SUN 3/60 station to produce the coset weight enumerators of the (63, 51, 5) BCH code and to determine the parameter $\bar{r}$.

We have observed the remarkable fact that there are only seven distinct proper coset weight enumerators. So in this case, we have the parameters $e = 2$, $\rho = 3$, $t = 5$, $\gamma = 7$. Finally, by using Theorem 4 and the algorithm mentioned in Remark 4, we have obtained that $\bar{r} = 16$.

C. The Binary (255, 239, 5) BCH Code

Here the G-orbits partition design contains 39 elements and so the associated matrix M have dimension 39×39.

Our programs has taken 28539 sec. on a SUN 3/280 to produce this matrix M, the coset weight enumerators and the regularity number $\bar{r}$ for this BCH code of length 255.

Here also we have observed the remarkable fact that there are only seven distinct proper coset weight enumerators. The first parameters are thus $e = 2$, $\rho = 3$, $t = 5$, $\gamma = 7$. Finally our calculations have given $\bar{r} = 38$.

A theoretical explanation of our observation that $\gamma = 7$ in all the considered cases would be very interesting. In the two last cases, the partition design with minimum cardinality is our G-orbits partition whereas this is not the case for the code of length 15. This is also a remarkable fact that may be related to the determination of the full automorphism group of these codes.

References

[1] L. A. Bassalygo, G. V. Zaitsev, and V. A. Zinoviev, "Uniformly packed codes," *Probl. Peredach. Inform.*, vol. 10, no. 1, pp. 9–14, 1974.

[2] L. A. Bassalygo and V. A. Zinoviev, "Remark on uniformly packed codes," *Probl. Peredach. Inform.*, vol. 13, no. 3, pp. 22–45, 1977.

[3] P. Camion, "Difference sets in elementary abelian group," *Les Presses de l'Université de Montréal*, Montréal, 1979.

[4] P. Camion, B. Courteau, and P. Delsarte, "On r-partition designs in Hamming spaces," tech. rep. 626, INRIA, 1987.

[5] P. Camion, B. Courteau, and P. Delsarte, "On r-partition designs in Hamming spaces," *Applicable Algebra in Eng., Commun. and Comput.*, vol. 2, no. 2, pp. 147–162, 1992.

[6] P. Camion, B. Courteau, G. Fournier and S. V. Kanetkar, "Weight distribution of translates of linear codes and generalized Pless identities," *J. Inform. Optim. Sci.*, vol. 8, pp. 1–23, 1987.

[7] I. M. Chakravarti and K. V. Suryanarayana, "Partial difference sets and partially balanced weighting designs for calibration and tournaments," *J. Combinat. Theory (A)*, vol. 13, pp. 426–431, 1972.

[8] P. Delsarte, "An algebraic approach to the association schemes of coding theory," Philips Res. Repts Suppl. 10, 1973, Ph.D. thesis, Université Catholique de Louvain, 1973.

[9] ——, "Four fundamental parameters of a code and their combinatorial significance," *Inform. Contr.*, vol. 23, pp. 407–438, 1973.

[10] J. M. Goethals and H. C. A. van Tilborg, "Uniformly packed codes," Philips Res. Rep., vol. 30, pp. 9–36, 1975.

[11] D. C. Gorenstein, W. W. Peterson, and N. Zierler, "Two-error correcting Bose–Chaudhuri codes are quasi-perfect," *Inform. Contr.*, vol. 3, pp. 291–294, 1960.

[12] T. Kasami, "Weight distribution of Bose–Chaudhuri–Hocquenghem codes," in *Combinatorial Mathematics and Its Applications*, H. F. Mattson, Jr., Ed. Chapel Hill, NC: Univ. North Carolina Press, 1967, pp. 335–357.

[13] A. Montpetit, "Codes dans les graphes réguliers," Ph.D. thesis, Université de Sherbrooke, 1987.

[14] A. Montpetit, "Codes, et partitions cohérentes dans les graphes réguliers," *Ann. sc. math. Québec*, vol. 14, no. 2, pp. 183–191, 1990.

[15] N. V. Semakov, V. A. Zinoviev, and G. V. Zaitsev, "Uniformly packed codes," *Probl. Peredach. Inform.*, vol. 7, no. 1, pp. 38–50, 1971.

[16] H. C. A. van Tilborg, "Uniformly packed codes," Ph.D. thesis, Tech. Univ. Eindhoven, 1976.

Optimal Codes for Correcting Single Errors and Detecting Adjacent Errors

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Abstract—**Optimal codes that correct single errors and detect double errors within nibbles of power of two length are presented. For each n, a code of length n, with the largest possible dimension which corrects single errors and detects double adjacent errors is presented. The problem of constructing optimal codes, which correct single errors and detect double adjacent errors within nibbles of length l is discussed.**

Index Terms—**Cyclic Gray code, double adjacent errors detection within nibbles, extended Hamming code, single error correction.**

I. Introduction

In certain memory systems, e.g., some spacecraft memories subject to soft upsets, the most common error is a single error and the next most common error is two errors in bits which are stored physically adjacent in the memory [6]. Blaum and Bruck [1] showed that there exists a (12, 8) code that corrects single errors and detects any double adjacent errors within three 4-bit nibbles. These codes are closely related to codes that correct and detect burst (adjacent) errors and byte (nibble) errors [2]–[4]. For the extensive literature on these subjects the reader is referred to Rao and Fujiwara [5]. Generally, the best known linear codes which correct single errors and detect double errors are the $(2^m, 2^m - m - 1)$ extended Hamming code and the codes obtained from it by shortening. In this correspondence, we construct codes which correct single errors and detect double adjacent errors. In Section II we present, for each $r > k \geq 1$, an optimal $(n, n - r)$ code which corrects single errors and detects all double errors within nibbles of length 2^k. In Section III, we present for each r the longest $(n, n - r)$ code that corrects single errors and detects double adjacent errors. In Section IV, we show how to construct optimal codes which detect the double adjacent errors within nibbles whose lengths are not power of 2.

II. Optimal Codes with Nibbles of a Power of 2 Length

Let *$SD(m)$ code* be a code which corrects single errors and detects double adjacent errors within nibbles of length m. For a matrix A, let A^R denote the reverse of A, i.e., the columns of A taken from the last to the first. Given r and k with $r > k \geq 1$, we want to find the largest n such that an $(n, n - r)$ SD(2^k) code exists. Since the parity check matrix of the code does not include the all-zero column it follows that $n \leq 2^r - 2^k$. We claim that for each $r > k \geq 1$, there exists an $(2^r - 2^k, 2^r - 2^k - r)$ SD(2^k) code. Moreover, we claim that there exists such a code which detects all double errors within the nibbles. We now show a recursive reconstruction for generating the parity check matrices of such codes. Assume we have a parity check matrix $H(r, k) = [H_1, H_2, \cdots$

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$H_{2^{r-k}-1}]$, where H_i has length 2^k (corresponds to a nibble), of a $(2^r - 2^k, 2^r - 2^k - r)$ SD(2^k) code which detects all double errors within the nibbles. Assume further that the 2^k vectors which do not appear as columns in $H(r, k)$ are $v_0, v_1, \cdots, v_{2^k-1}$ and for each $i, j, 0 \le i, j \le 2^k - 1$, $v_i + v_j = v_m$ for some $0 \le m \le 2^k - 1$. The recursive construction has two steps.

Step 1) Since in $\begin{bmatrix} 0 \cdots 0 \\ H(r,k) \end{bmatrix}$ addition of two columns within a nibble results in $\begin{pmatrix} 0 \\ v_i \end{pmatrix}$ for some i, it follows that the same is true for $\begin{bmatrix} 1 \cdots 1 \\ H(r,k) \end{bmatrix}$. Since $v_i + v_j = v_m$, it follows that the addition of two columns from $\begin{bmatrix} 1 & \cdots & 1 \\ v_1 & & v_{2^k-1} \end{bmatrix}$ results in $\begin{pmatrix} 0 \\ v_i \end{pmatrix}$ for some i. Hence,

$$\begin{bmatrix} 0 \cdots 0 & 1 \cdots 1 & 1 & 1 & \cdots & 1 \\ H(r,k) & H(r,k) & v_0 & v_1 & & v_{2^k-1} \end{bmatrix}$$

is a parity check matrix for an SD(2^k) code that detects all double errors within the nibbles, i.e., $H(r+1, k)$. The vectors of length $r + 1$ which do not appear in $H(r + 1, k)$ are $u_0, u_1, \cdots, u_{2^k-1}$, where $u_i = \begin{pmatrix} 0 \\ v_i \end{pmatrix}$. Since $v_i + v_j = v_m$, it follows that $u_i + u_j = u_m$.

Step 2) Since in $\begin{bmatrix} 0 \cdots 0 \\ H_i \end{bmatrix}$ addition of two columns results in $\begin{pmatrix} 0 \\ v_j \end{pmatrix}$ for some j, it follows that the same is true for $\begin{bmatrix} 1 \cdots 1 \\ H_i \end{bmatrix}$, and addition of one column from $\begin{bmatrix} 0 \cdots 0 \\ H_i \end{bmatrix}$ and one column from $\begin{bmatrix} 1 \cdots 1 \\ H_i \end{bmatrix}$ results in $\begin{pmatrix} 1 \\ v_j \end{pmatrix}$ for some j. Let $u_i = \begin{pmatrix} 0 \\ v_i \end{pmatrix}$ and $u_{i+2^k} = \begin{pmatrix} 1 \\ v_i \end{pmatrix}$, $0 \le i \le 2^k - 1$. Since $v_i + v_j = v_m$, it follows that $\begin{pmatrix} 0 \\ v_i \end{pmatrix} + \begin{pmatrix} 0 \\ v_j \end{pmatrix} = \begin{pmatrix} 1 \\ v_i \end{pmatrix} + \begin{pmatrix} 1 \\ v_j \end{pmatrix} = \begin{pmatrix} 0 \\ v_m \end{pmatrix}$ and $\begin{pmatrix} 0 \\ v_i \end{pmatrix} + \begin{pmatrix} 1 \\ v_j \end{pmatrix} + \begin{pmatrix} 1 \\ v_m \end{pmatrix}$. Hence,

$$\begin{bmatrix} 0 \cdots 0 & 1 \cdots 1 & 0 \cdots 0 & 1 \cdots 1 & \cdots & 0 \cdots 0 & 1 \cdots 1 \\ H_1 & H_1 & H_2 & H_2 & & H_{2^{r-k}-1} & H_{2^{r-k}-1} \end{bmatrix}$$

is a parity check matrix for an SD(2^{k+1}) code which detects all double errors within the nibbles, i.e., $H(r+1, k+1)$. The vectors of length $r + 1$ that do not appear in $H(r + 1, k + 1)$ are $u_0, u_1, \cdots, u_{2^{k+1}-1}$.

Assume we want to construct a $(2^r - 2^k, 2^r - 2^k - r)$ SD(2^k) code. We start with $H(2, 1) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $v_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, and $v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Then, we have to apply Step 1), $r - k - 1$ times, and Step 2), $k - 1$ times, to obtain the parity check matrix $H(r, k)$ of our code. Note, that by applying Steps 1) and 2) in different orders we obtain different parity check matrices for equivalent codes (different only by permutations of rows, nibbles, and columns within the nibbles). Therefore, we have the following theorem.

Theorem 1: For any given r and k such that $r > k \ge 1$, there errors within the nibbles.

Example 1: Assume we want to construct a (12,8) SD(4) code ($H(4, 2)$). We start with $H(2, 1) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $v_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Applying Step 1), we obtain

$$H(3,1) = \begin{bmatrix} 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}, v_0 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, v_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}.$$

Now, we apply Step 2) and obtain

$$H(4,2) = \begin{bmatrix} 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \end{bmatrix},$$

$$v_0 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad v_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}, v_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, v_3 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}.$$

If we first apply Step 2) and then Step 1), we obtain

$$H(4,2) = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \end{bmatrix},$$

$$v_0 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, v_1 \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}, v_2 \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, v_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}.$$

Note that if we need SD(2^k) code of length $l \cdot 2^k$, we have to find the smallest r such that $l \cdot 2^k \le 2^r - 2^k$, generate a $(2^r - 2^k, 2^r - 2^k - r)$ SD(2^k) code and shorten it to form an $(l \cdot 2^k, l \cdot 2^k - r)$ SD(2^k) code. There is only one case in which the extended Hamming code or a code obtained by shortening it will obtain a code with the same parameters. If we need an SD(2^k) code of length 2^m, then the $(2^m, 2^m - m - 1)$ extended Hamming code has the same parameters as the code obtained by shortening our code. We will use the extended Hamming code in this case since it is capable to correct all double errors.

III. Optimal SD(n) Codes of Length n

By using the $(2^m, 2^m - m - 1)$ extended Hamming code, and by shortening it up to $2^{m-1} - 1$ times, we obtain one of the 2^{m-1} codes, $(2^m - j, 2^m - m - j - 1)$ SD($2^m - j$) code, $0 \le j \le 2^{m-1} - 1$, which corrects single errors and detects all double errors. For detecting adjacent errors, usually, we can do slightly better. We first ask, for a given r, what is the largest n such that an $(n, n - r)$ SD(n) code exists. An upper bound on n is given by the following lemma.

Lemma 1: For a given $r > 3$ the largest n such that an $(n, n - r)$ SD(n) code exists satisfies $n \le 2^r - r - 2$.

Proof: Assume that in the parity check matrix H of the $(n, n - r)$ SD(n) code, $n > 2^r - r - 2$, the vectors of length r that do not appear as columns are $v_1, \cdots, v_k$. Let s be the dimension of the space spanned by these vectors. Since in H any two adjacent columns sum to some v_i, we have that $n \le 2^s$ (each column of H is a linear combination of the first column of H and some of the v_i's). For $r > 3, 2^{r-1} < 2^r - r - 2$, and hence $n > 2^r - r - 2$ implies $s = r$. Since the all-zero vector is one of the v_i's, it follows that $k \ge r + 1$. We now have to prove that $k \ne r + 1$. Assume that $k = r + 1$ then the v_i's excluding the all-zero vector form a base B of Z_2^r. Let $h_i, 0 \le i \le n - 1$, be column i of from B which sum to h_i. Clearly, d_i is even (odd), if and only if

d_{i+1} is odd (even). On the other hand there are 2^{r-1} linear combinations of odd number of vectors from B and 2^{r-1} linear combinations of even number of vectors from B. But since r combinations of a single vector (odd number of vectors) from B do not appear in H, and only one combination of even number of vectors (the all-zero vector), we have a contradiction. Hence, for $r > 3$ we have $n \leq 2^r - r - 2$. □

The construction of an optimal code which attains the upper bound of Lemma 1 is based on the existence of cyclic Gray codes. A *cyclic Gray code* of order m is a code $c_0, c_1, \cdots, c_{2^m-1}$ such that each c_i is a word of length m, $\cup_{i=0}^{2^m-1} c_i = Z_2^m$, c_i and c_{i+1}, $0 \leq i \leq 2^m - 1$, differ in exactly one position, and also c_0 and c_{2^m-1} differ in exactly one position, It is well known that for every $m \geq 1$ there exists a cyclic Gray code. For our purpose we need a specific type of cyclic Gray code.

Lemma 2: For every $m > 1$ there exists a cyclic Gray code in which

a) the vectors adjacent to the all-zero vector are the vectors with unique one at the first position and the vector with unique one at the second position;

b) one of the vectors adjacent to the all-one vector is the vector with a unique zero at the first position.

$$\begin{bmatrix} 1&1&0&0&0&1&1&0&1&1&1&1&0&0&1&1&0&1&1&0&0&1&1&0&0 \\ 1&0&1&1&0&0&0&1&1&1&1&1&1&0&0&0&1&1&1&1&0&0&0&0&1 \\ 1&1&1&1&1&1&0&0&0&1&1&0&0&0&0&0&0&0&1&1&1&1&1&1&1 \\ 0&0&0&1&1&1&1&1&1&1&1&1&1&1&1&0&0&0&0&0&0&0&1&1&1 \\ 0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1 \end{bmatrix},$$

Proof: We give a recursive construction which generates a cyclic Gray code in each step $m > 1$. For $m = 2$ we take the cyclic Gray code, $\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}$.

Given the code generated in step $m > 1$, $c_0, c_1, \cdots, c_{2^m-1}$, we construct the code of step $m + 1$. Let $d_i = \begin{pmatrix}c_i\\0\end{pmatrix}$ and $d_{2^{m+1}-i-1} = \begin{pmatrix}c_i\\1\end{pmatrix}$ for $0 \leq i \leq 2^m - 1$. $d_0, d_1, \cdots, d_{2^{m+1}-1}$ is the code of step $m + 1$. By a simple induction we can prove that the code constructed in each step $m > 1$ is a cyclic Gray code which satisfies a) and b) in Lemma 2. □

Lemma 3: For a given $r > 3$ there exists an $(n, n - r)$ SD(n) code with $n = 2^r - r - 2$.

Proof: Let C_k the $k \times (2^k - 1)$ matrix obtained from the cyclic Gray code of order k defined in Lemma 2 by deleting the all-zero vector, fixing the all-one vector as the first codeword and the vector with a unique zero in the first position as the last codeword (if necessary we reverse the order of the codewords). We define two types of matrices with size $r \times (2^k - 1)$.

For even k,

$$M_k = \begin{bmatrix} C_k \\ 1 \cdots 1 \\ 0 \cdots 0 \\ \cdots \\ \cdots \\ 0 \cdots 0 \end{bmatrix}.$$

For odd k,

$$M_k = \begin{bmatrix} C_k^R \\ 1 \cdots 1 \\ 0 \cdots 0 \\ \cdots \\ \cdots \\ 0 \cdots 0 \end{bmatrix}.$$

It is easy to verify that the matrix $H = [M_2 M_3 \cdots M_{r-1}]$ is a parity check matrix of an $(n, n - r)$ SD(n) code with $n = 2^r - r - 2$. □

Note, that the only vectors which do not appear in the parity check matrix H of Lemma 3 are the all-zero vector, the r vectors of weight 1, and the vector of weight 2 with ones in the first and second positions. Lemmas 1 and 3 imply the following theorem.

Theorem 2: For a given $r > 3$ the largest n for which an $(n, n - r)$ SD(n) code exists is $2^r - r - 2$.

For $r = 1$ and $r = 2$ the optimal codes have dimension 0. For $r = 3$ we have a (4, 1) SD(4) code which is the extended Hamming code.

Example 2: For $r = 4$ we obtain a (25, 20) SD(25) code with the parity check matrix

where the first 3 columns belong to M_2, the next 7 columns belong to M_3, and the last 15 columns belong to M_4.

Now, we can also answer the question, what is the largest dimension k of a code of length n, which corrects single errors and detects all adjacent errors. For $2^{r-1} - r - 1 < n \leq 2^r - r - 2$, $r > 3$, $k = n - r$. If $2^{r-1} - r - 1 < n \leq 2^{r-1}$, the best code is the extended Hamming code and the codes obtained by shortening it, since they are capable to detect all double errors. For $2^{r-1} < n \leq 2^r - r - 2$, the code constructed in this section, and the codes obtained by shortening it, will be taken to do the job.

IV. Other Optimal Codes

In this section, we discuss the solution for nibbles which are not of a power of 2 length. If we have nibbles of length l and we want to construct an $(n, n - r)$ SD(l) code with the largest possible n, we take the largest n which is a multiple of l and not greater than $2^r - r - 2$, and the first n column of the $(2^r - r - 2, 2^r - 2r - 2)$ SD$(2^r - r - 2)$ code, defined in Lemma 3, form a code which corrects single errors and detects double adjacent errors within nibbles of length l. Next, we prove that usually this code is also optimal.

Assume that in the parity check matrix H of the $(n, n - r)$ SD(l) code, the vectors of length r that do not appear as columns are $v_1, \cdots, v_k$. Let s be the dimension of the space spanned by these vectors.

First, we show that $s = r$. If $s \leq r - 1$ we can take s vectors that form a base of the v_i's as the rows of a generator matrix for an (r, s) code C. All the v_i's are codewords of this code, and if two vectors u and w sum to some v_i, it implies that u and w are in the same coset of C. Therefore, in each nibble of H we have columns that are words from the same coset of C. The number of words in a coset of C is 2^s and since the length of the nibble is not a power of 2, it follows that at least one word from each coset does not appear as a column of H and must belong to the v_i's, a contradiction. Hence, $s = r$ and $k \geq r$.

If $k \geq r + 2$ then certainly our construction produces an optimal code and since the all-zero vector is one of the v_i's it follows that $k \geq r + 1$. So, the only remaining case is $k = r + 1$. In this case, if $2^r - r - 1 = n_1 l_1$ and either $n_1 < r - 1$ or l_1 is even, there is no $(2^r - r - 1, 2^r - 2r - 1)$ SD(l_1) code. The proof follows similarly to the one of Lemma 1. The v_i's excluding the all-zero vector form a base B of Z_2^r. Let $h_i, 0 \leq i \leq n - 1$, be columns i of H. For each $i, 0 \leq i \leq n - 1$, let d_i be the number of vectors from B which sum to h_i. There are 2^{r-1} linear combinations of odd number of vectors from B and 2^{r-1} linear combinations of even number of vectors for B. In H, exactly r combinations of a single vector from B and one combination of zero vectors from B, do not appear. Hence, H can exist only if l_1 is odd and $n_1 \geq r - 1$.

If $2^r - r - 1 = n_1 l_1$ for $n_1 \geq r - 1$ and odd l_1, we conjecture that there exists a $(2^r - r - 1, 2^r - 2r - 1)$ SD(l_1) code. We give the parity check matrices (written in octal) for the first three cases:

1) $2^6 - 6 - 1 = 19 \cdot 3$. The following matrix is a parity check matrix of a (57, 51) SD(3) code:

$$\begin{bmatrix} 0&7&7&1&4&0&1&7&0&7&3&0&1&7&0&1&7&7&7 \\ 0&0&7&7&1&0&0&0&7&0&7&3&7&1&3&7&7&4&7 \\ 4&0&0&7&7&3&7&1&0&7&0&7&0&3&1&7&7&6&4 \\ 1&4&0&0&7&7&3&7&1&0&7&0&3&0&7&7&6&7&1 \\ 7&1&4&0&0&7&0&3&3&1&0&7&7&7&0&7&0&7&7 \\ 7&7&1&4&0&1&7&0&7&3&1&1&0&0&7&4&3&7&7 \end{bmatrix}.$$

2)–3) $2^7 - 7 - 1 = 8 \cdot 15 = 40 \cdot 3$. The following matrix is a parity check matrix of a (120,113) SD(15) code which is of course a parity check matrix of a (120,113) SD(3) code:

$$\begin{bmatrix} 06370 & 07607 & 17036 & 74074 & 70360 & 74170 & 70340 & 76077 \\ 37076 & 16314 & 31463 & 07147 & 14631 & 46314 & 14607 & 74361 \\ 77600 & 00776 & 03770 & 61760 & 37700 & 17740 & 40077 & 77003 \\ 03743 & 77740 & 00177 & 00377 & 76000 & 00777 & 01763 & 17630 \\ 00017 & 37777 & 77777 & 00000 & 77777 & 77777 & 00000 & 00000 \\ 60000 & 00000 & 77777 & 00000 & 00003 & 77777 & 77777 & 37777 \\ 00000 & 74000 & 00000 & 37776 & 77777 & 77776 & 77776 & 70000 \end{bmatrix}.$$

The first two unknown cases are for $2^8 - 8 - 1 = 13 \cdot 19$, for which we can have a (247,239) SD(13) code and a (247,239) SD(19) code.

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References

[1] M. Blaum and J. Bruck, "A note on a systematic (12,8) code for correcting single errors and detecting adjacent errors," preprint.
[2] C. L. Chen, "Error-correcting codes with byte error detection capability," *IEEE Trans. Comput.*, vol. C-32, pp. 615–621, July 1983.
[3] C. L. Chen, "Byte oriented error-correcting code for semiconductor memory systems," *IEEE Trans. Comput.*, vol. C-35, pp. 646–648, July 1986.
[4] A. A. Davydov and L. M. Tombak, "An alternative to the Hamming code in the class of SEC-DED codes in semiconductor memory," *IEEE Trans. Inform. Theory,* vol. 37, pp. 897–902, May 1991.
[5] T. R. N. Rao and E. Fujiwara, *Error-Control Coding for Computer Systems.* London, UK: Prentice Hall Int., 1989.
[6] J. W. Schwartz and J. K. Wolf, "A systematic (12,8) code for correcting single errors and detecting adjacent errors," *IEEE Trans. Comput.,* vol. 39, pp. 1403–1404, Nov. 1990.

The Free Distance of Fixed Convolutional Rate 2/4 Codes Meets the Costello Bound

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Abstract—**The long standing question whether the free distance of fixed rate convolutional codes is as good as the Costello bound was almost solved by Zigangirov and Massey. They proved that this is indeed the case for codes with long branch length and rates $2/c$, $c \geq 5$. It is shown that there exist fixed convolutional codes of rate 2/4 whose free distance d_{free} meets the Costello bound originally derived for time varying convolutional codes.**

Index Terms—**Convolutional codes, Costello bound, free distance, fixed convolutional codes.**

I. Introduction

The free distance, d_{free}, is an important criterion for the error correction capability of convolutional codes. Consequently, for many years researchers tried to find and improve lower bounds on the value of d_{free} for fixed (non time-varying) convolutional codes. For time-varying codes such a lower bound was obtained by Costello [1], i.e., the well-known Costello bound. Costello's result says that there exist codes with $d_{\text{free}} \geq \nu \rho_c(R)$, where $\rho_c(R) = -R/\log(2^{1-R} - 1)$ is the Costello parameter for a rate R convolutional code, and where ν is the constraint length of the code.[1] Unfortunately, the fraction of fixed codes among the set of time-varying codes is very small, and hence Costello's result gives us little information about the free distance of the fixed codes. Neuman gave a bound for fixed codes but his result is weaker than the Costello bound [2].

Recently it was shown that for large branch length b it is possible to find rate $R = b/c$ fixed codes which have a d_{free} that approaches the Costello bound [3]. This result is further developed in [4] and [5] in such a way that the previous result also holds for small branch length and $c \geq 5$. In particular it was shown that there exist fixed codes for which $d_{\text{free}}/\nu \geq \max(\rho_c(R), 3H^{-1}(1 - R)) - o(1)$, where $H(x) = -x \log(x) - (1 - x) \log(1 - x)$. Un-

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[1] In this correspondence, all logarithms are with respect to base 2.

