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Partitions of Triples into Optimal Packings

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It is well known that for $n \equiv 1$ or 3 (mod 6), $n > 7$, it is possible to partition all triples of order n into $n-2$ pairwise disjoint optimal packings (Steiner triple systems). For $n \equiv 0$ or 2 (mod 6), $n > 7$, it is possible to partition all triples of order n into $n-1$ pairwise disjoint optimal packings. We prove that for $n \equiv 4$ (mod 6) it is possible to partition all triples of order n into $n-1$ pairwise disjoint optimal packings and one packing of size $(n-1)/3$. For $n \equiv 5$ (mod 6) we show that it is possible to partition all triples of order n into $n-2$ pairwise disjoint optimal packings, which is the maximum possible number of pairwise disjoint optimal packing of order n. The remaining triples are partitioned into two packings for which the first is at least of size $n-2$ and the second is at most of size $(n-2)/3$. We also give recursive and non-recursive constructions of partitions for triples of order $n \equiv 5$ (mod 6) with $n-1$ pairwise disjoint packings and recursive constructions for optimal partitions for triples of order $n \equiv 5$ (mod 6).

1. INTRODUCTION

Partitions of triples into disjoint packings are important in the design of constant weight codes with minimum Hamming distance 4 [1, 7].

A *packing triple system* (PT) of order n, (PT(n)) is a pair (Q, q), where $Q = \{0, 1, ..., n-1\}$ is a set of n *points* and q is a collection of three-element subsets of Q called *blocks* such that every two-element subset of Q is a subset of at most one block of q (in other words, every pair is *covered* by at most one block). A PT is *optimal* if there is no PT of the same order with a larger size.

Spencer [11] proved that optimal PT(n) has $\lfloor n/3 \lfloor (n-1)/2 \rfloor \rfloor$ triples for $n \not\equiv 5$ (mod 6) and $\lfloor n/3 \lfloor (n-1)/2 \rfloor \rfloor - 1$ triples for $n \equiv 5$ (mod 6).

A *Steiner triple system* (STS) of order n (ST(n)) is a (PT(n)) such that every two-element subset of Q is a subset of exactly one block of q. It is well known that an ST(n) exists if and only if $n \equiv 1$ or 3 (mod 6). It is clear

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that $ST(n)$ is an optimal PT. Two PTs (Q, q_1) and (Q, q_2) are *disjoint* if $q_1 \cap q_2 = \emptyset$. It is known that for $n \equiv 1$ or 3 (mod 6), $n \geqslant 9$, there exists a set of $n-2$ pairwise disjoint STSs [5, 6, 12]. This set is called a *large set*. By taking from each $ST(n)$ the blocks which do not contain the last point, we obtain a set of $n-2$ disjoint optimal PTs of order $n-1$. Thus, the problem is solved for $n \equiv 0$, 1, 2, or 3 (mod 6), $n \geqslant 8$. In a previous paper [3] we obtain some results on partitions for triples of order $n \equiv 4$ or 5 (mod 6). The two main results were as follows. We proved the existence of optimal partitions for every order $n \equiv 4$ or 16 (mod 18). These partitions have $n-1$ optimal PTs and one PT of size $(n-1)/3$. It was also proved that for $n \equiv 5$ or 17 (mod 18), $n > 5$, there exists a partition of order n with $n-2$ optimal PTs, which is the maximum possible number of disjoint optimal PTs. The remaining triples can be partitioned into two PTs, for which the first is at least of size $n-2$ and the second is at most of size $(n-2)/3$.

In this paper we continue handle the cases $n \equiv 4$ or 5 (mod 6). The key idea is the existence of a special set with $3k$ PTs of order $3k$, k odd, called a base set. We believe this set, which is defined in Section 2, exists for all odd k, but we were able to prove its existence only for $k \equiv 1$ or 3 (mod 6) and for some values of $k \equiv 5$ (mod 6). In Section 2 we present three constructions for base sets. In Section 3 we present optimal partitions of orders $n \equiv 10$ (mod 18) and thus complete the proof of the existence of optimal partitions for all orders $n \equiv 4$ (mod 6) with $n-1$ optimal PTs and one PT of size $(n-1)/3$. In Section 4 we present partitions of orders $n \equiv 11$ (mod 18) with $n-2$ optimal PTs, one PT at least of size $n-2$, and one PT at most of size $(n-2)/3$. Thus completing the proof of the existence of such partitions for all $n \equiv 5$ (mod 6), $n > 5$. In Section 5 we examine the recursive constructions of Section 4 and a previous construction from [3] to prove the existence of partitions of orders $n \equiv 5$ (mod 6) with $n-1$ PTs. We also prove that if there exists a partition of order $n \equiv 5$ (mod 6) with $n-1$ PTs, for which the first $n-2$ are optimal then our results make it possible to find more partitions of this form. In Section 6 we give a nonrecursive construction of partitions of order $n \equiv 5$ (mod 6), with $n-1$ PTs.

2. Constructions of Base Sets

In this section we prove that base sets, of orders $n = 3k$, exist for all $k \equiv 1$ or 3 (mod 6) and for some $k \equiv 5$ (mod 6). But, first we define another design which has some similarity to a base set and is used in constructions of partition for triples [3, 5, 8, 10, 13].

Let $ST(q+2)$ be any STS of order $q+2$. It is clear that we can order the elements Z_q in cyclic sequences $a_0, a_1, \ldots, a_{r-1}$ such that $a_i \neq a_j$ for $i \neq j$, and for each i, $0 \leqslant i \leqslant r-1$, $\{a_i, a_{i+1}, q\} \in \mathrm{ST}(q+2)$ or $\{a_i, a_{i+1}, q+1\} \in \mathrm{ST}(q+2)$, where subscripts taken modulo r. Note, that each element of Z_q appears in exactly one sequence. Let $\mathrm{index}(j, m, \mathrm{ST}(q+2)) = 1$ if $j = a_i$ and $m = a_{i+1}$ in some cyclic sequence, and $\mathrm{index}(j, m, \mathrm{ST}(q+2)) = 0$ for all other ordered pairs.

Property A. A set of q pairwise disjoint STSs, $\mathrm{ST}_i(q+2)$, $0 \leqslant i \leqslant q-1$, has this property if for $j \neq m$, $0 \leqslant j$, $m \leqslant q-1$, $\mathrm{index}(j, m, \mathrm{ST}_d(q+2)) = 1$, for some d, $0 \leqslant d \leqslant q-1$, then for each $s \neq d$, $\mathrm{index}(j, m, \mathrm{ST}_s(q+2)) = 0$.

Let $D(A)$ denote the set of orders for which a set of q SQSs of order $q+2$, with Property A exists. The set of $n-2$ pairwise disjoint STSs of order n constructed by Schreiber [8] and Wilson [13] has property A. Hence, we have the following result.

LEMMA 1 [8, 13]. *If q is an integer such that the order of -2 modulo each of its prime factors is congruent to 2 modulo 4 then $q+2 \in D(A)$.*

The condition of Lemma 1 is satisfied for all primes of the form $8t-1$ and "roughly" $\frac{1}{6}$ of those of the form $8t+1$ [9]. Lu [5] found that also 19, $21 \in D(A)$, and Schreiber [10] found that $15 \in D(A)$ by using the 13 pairwise disjoint STSs of Denniston [2]. We also have the following result.

LEMMA 2 [10]. *If $a+2 \in D(A)$ and $b+2 \in D(A)$ then $ab+2 \in D(A)$.*

As said in the Introduction the key to most of the constructions in this paper is a set of $3r$ PTs of order $3r$ which covers all the triples. Let Q_{ij}, $i \in Z_r$, $j \in Z_3$, be a set of $3r$ PTs of order $3r$ on the points $Z_r \times Z_3$. This set is called a *base set* if it has the following properties:

(1) Each set has $3r(r-1)/2+1$ triples.

(2) The union of all the PTs is the set of all triples of order $3r$.

(3) For each i, $i \in Z_r$, Q_{i0}, Q_{i1}, and Q_{i2} have exactly one triple in common which is $\{(i,0), (i,1), (i,2)\}$. In addition, in these three PTs all the pairs which include an element from $\{(i,0), (i,1), (i,2)\}$ are covered.

(4) Any other two PTs are disjoint.

(5) For a PT, Q_{ij} there are $3r-3$ elements in the uncovered pairs, each element appears in exactly two uncovered pairs, and we can order these elements in cyclic sequences $a_0, a_1, ..., a_{k-1}$ such that $\{a_m, a_{m+1}\}$ (subscripts taken modulo k) is an uncovered pair, and $a_l \neq a_m$ for $l \neq m$. Let $\mathrm{index}(b, c, Q_{ij}) = 0$ if and only if there is no m such that $b = a_m$ and $c = a_{m+1}$ in some cyclic sequence.

(6) For each m, $m \in Z_r$, $\mathrm{index}((m,x), (m,y), Q_{ij}) = 0$ for each $i \in Z_r$ and $j \in Z_3$.

(7) Each other ordered pair appears exactly once in some cyclic sequence defined by (5) for some Q_{kl}.

(8) $\text{index}(b, c, Q_{ij}) \in \{0, 1, 2\}$, $\text{index}(a_m, a_{m+1}, Q_{ij}) \neq \text{index}(a_{m+1}, a_{m+2}, Q_{ij})$, and if $\text{index}(b, c, Q_{ij}) = x$, $x \in \{1, 2\}$, then $\text{index}(c, b, Q_{kl}) \neq x$, for all $k \in Z_r$, $l \in Z_3$, $(i, j) \neq (k, l)$.

We now prove the existence of a base set for every $r \equiv 1$ or $3 \pmod 6$, and for every $r = qk$, $q + 2 \in D(A)$ and $k \equiv 1 \pmod 6$. First, we present a base set of order 9. Let

$$Q = \{\{0, 3, 6\}, \{1, 4, 7\}, \{2, 5, 8\}, \{0, 4, 5\}, \{0, 7, 8\}, \{1, 3, 5\},$$
$$\{1, 6, 8\}, \{2, 3, 4\}, \{2, 6, 7\}, \{0, 1, 2\}\}.$$

By applying the eight permutations, (0, 1, 2, 5, 3, 4, 7, 8, 6), (0, 1, 2, 4, 5, 3, 8, 6, 7), (8, 6, 7, 0, 1, 2, 3, 4, 5), (6, 7, 8, 0, 1, 2, 5, 3, 4), (7, 8, 6, 0, 1, 2, 4, 5, 3), (3, 4, 5, 7, 8, 6, 0, 1, 2), (5, 3, 4, 8, 6, 7, 0, 1, 2), and (4, 5, 3, 6, 7, 8, 0, 1, 2), on Q, and replacing 0 with (0, 0), 1 with (0, 1), 2 with (0, 2), 3 with (1, 0), 4 with (1, 1), 5 with (1, 2), 6 with (2, 0), 7 with (2, 1), 8 with (2, 2), and calling the PTs Q_{00}, Q_{01}, Q_{02}, Q_{10}, Q_{11}, Q_{12}, Q_{20}, Q_{21}, and Q_{22}, respectively, we obtain the desired base set of order nine. The interested reader can check that indeed the nine PTs, Q_{ij}, $0 \leqslant i, j \leqslant 2$, form a base set of order nine.

Using a base set of order 3^r, $r > 1$, and the base set of order nine, one can construct a base set of order 3^{r+1}, using a special case of Construction C which follows. A base set of order 3^r, $r > 1$ is enough to prove the results of Sections 3 and 4. But, we will use a general construction for base sets by using STSs constructed by Lindner and Rosa [4]. This will enable us to obtain some general results given in Section 5. For this construction we need the following definitions. Two STSs, (Q, q_1) and (Q, q_2), are said to be *almost disjoint* if q_1 and q_2 have exactly one triple in common. A large set of mutually almost disjoint (MAD) STSs of order n, is a set of STSs of order n, such that each two are almost disjoint, and the union of all the STSs is the set of all triples of order n. A *Steiner quadruple system* (SQS) of order n, is a pair (Q, q), where $Q = \{0, 1, ..., n-1\}$ is a set of n points and q is a collection of four-element subsets of Q, such that every three-element subset of Q is a subset of exactly one quadruple of q. SQS of order n exists if and only if $n \equiv 2$ or $4 \pmod 6$. If (Q, q) is a SQS and x is any element of Q, then the set of all triples $\{a, b, c\}$ such that $\{x, a, b, c\} \in q$ is an STS denoted by $q(x)$. Let T_i, $0 \leqslant i \leqslant n-2$, be the STS obtained from $q(i)$ by replacing $n-1$ with i. Lindner and Rosa proved that the set of T_i's is a large set of MAD STSs. The following property of this large set of MAD STSs is easy to verify and is used for the construction of our base sets.

LEMMA 3. (i) *If* $\{a, b, c\} \in T_i \cap T_j$ *then* $\{i, j\} \subset \{a, b, c\}$.
(ii) $\{i, j, l\} \in T_i \cap T_j$ *if and only if* $\{i, j, l\} \in T_l$.

Now, we give three constructions for base sets.

Construction A. Let $k = n-1 \equiv 1$ or $3 \pmod 6$, and let $\{T_i\}$, $0 \leqslant i \leqslant k-1$, be the set of k MAD STSs defined by Lindner and Rosa. Let $F_0 = \{\{1, 2\}\}$, $F_1 = \{\{0, 2\}\}$, and $F_2 = \{\{0, 1\}\}$. Let Q_{ij}, $0 \leqslant i, j \leqslant 2$, be the base set of order nine on the points $Z_3 \times Z_3$. For each $\{i, a, b\} \in T_i$ we define two functions h and g, such that $\{h(i), h(a), h(b)\} = \{0, 1, 2\}$, $h(l) < h(m)$ iff $l < m$. Also we have $g(i, a, b) = 0$ if $i < a$ and $i < b$, $g(i, a, b) = 2$ if $i > a$ and $i > b$, and $g(i, a, b) = 1$ otherwise. From these sets we construct a base set of order $3k$ with the PTs S_{ij}, $i \in Z_k$, $j \in Z_3$, on the points $Z_k \times Z_3$.

Each set S_{ij} has the following blocks:

$$\{(a, x), (b, y), (c, x+y+j)\}, \qquad \{a, b, c\} \in T_i,\ i \notin \{a, b, c\}, \qquad x, y \in Z_3. \tag{A.1}$$

There are $9(k(k-1)/6 - (k-1)/2)$ blocks of this type.

$$\begin{aligned} &\{(i, x), (a, y), (b, z)\},\\ &\{i, a, b\} \in T_i,\ \{(h(i), x), (h(a), y), (h(b), z)\} \in Q_{g(i,a,b)j}, \end{aligned} \tag{A.2}$$

$$\begin{aligned} &\{(i, x), (a, y), (a, z)\},\\ &\{i, a, b\} \in T_i,\ \{(h(i), x), (h(a), y), (h(a), z)\} \in Q_{g(i,a,b)j}. \end{aligned} \tag{A.3}$$

There are $9(k-1)/2$ blocks of these two types. The last block is

$$\{(i, 0), (i, 1), (i, 2)\}. \tag{A.4}$$

THEOREM 1. *Construction A defines a base set of order* $3k$, $k \equiv 1$ *or* 3 $\pmod 6$.

Proof. To show that Construction A generates a base set of order $3k$, we have to show that the set $\{S_{ij}\}$ has properties (1) through (8) which define a base set. Property (1) holds since each set S_{ij} is a PT and has

$$9\left(\frac{k(k-1)}{6} - \frac{k-1}{2}\right) + 9\,\frac{k-1}{2} + 1 = \frac{3k(k-1)}{2} + 1 \text{ blocks.}$$

Now, note that by the definition of h, g, and Lemma 3, the triples of types (A.2), (A.3), and (A.4), define the base set of order nine on the points

$\{i, j, l\} \times Z_3$ for $\{i, j, l\} \in T_i \cap T_j \cap T_l$. We call this property the sub-base set property.

By the definition of type (A.4), S_{i0}, S_{i1}, and S_{i2} have the triple $\{(i, 0), (i, 1), (i, 2)\}$ and, since $Q_{g(i,a,b)j}$ is a PT in the base set of order nine, all the pairs which include an element from $\{(g(i, a, b), 0), (g(i, a, b), 1), (g(i, a, b), 2)\}$ are covered in Q_{ij}. The same is true for the elements from $\{(i, 0), (i, 1), (i, 2)\}$ in S_{ij} by this fact, the sub-base set property, and the fact that for each $a \in Z_k$, $a \neq i$, the pair $\{i, a\}$, is covered by $\{i, a, b\} \in T_i$.

Except for triples of type (A.4) we claim that the intersection between two PTs is empty. To see this, note that two PTs can only intersect in blocks of the same type. The disjointness of blocks of type (A.1) follows from the fact that $T_i \cap T_j$ is one block $\{a, b, c\}$ and $\{i, j\} \subset \{a, b, c\}$. The disjointness of the blocks of types (A.2) and (A.3) follows from the sub-base set property.

We now set $\text{index}((a, x), (b, y), S_{ij}) = v$, for $\{i, a, b\} \in T_i$, iff $\text{index}((h(a), x), (h(b), y), Q_{g(i,a,b)j}) = v$. Properties (5) through (8) of the function *index* follow from the existence of the base set of order nine, Lemma 3, and the definition of the blocks of types (A.2), (A.3), and (A.4). Finally, a simple counting argument shows that each triple of $Z_k \times Z_3$ is covered by at least one PT. Q.E.D.

Construction B. Let $\text{ST}_i(q+2)$, $i \in Z_q$, be a large set of STSs on the points Z_{q+2}, having Property A. We arrange this set in a way that $\{i, q, q+1\} \in \text{ST}_i(q+2)$. Let $F_0 = \{\{1, 2\}\}$, $F_1 = \{\{0, 2\}\}$, and $F_2 = \{\{0, 1\}\}$. From these sets we generate a base set of order $3q$, with the PTs S_{ij}, $i \in Z_q$, $j \in Z_3$, on the points $Z_q \times Z_3$.

Each set S_{ij}, has the blocks:

$$\{(a, x), (b, y), (c, x+y+j)\}, \quad a, b, c \in Z_q, \ \{a, b, c\} \in \text{ST}_i(q+2), \ x, y \in Z_3 \tag{B.1}$$

There are $9((q-1)(q-2)/6)$ blocks of this type.

$$\{(a, x), (a, y), (b, t+j)\}, \qquad t \in Z_3, \ \{x, y\} \in F_t, \ \text{index}(a, b, \text{ST}_i(q+2)) = 1. \tag{B.2}$$

There are $3(q-1)$ blocks of this type. The last block is

$$\{(i, 0), (i, 1), (i, 2)\}. \tag{B.3}$$

THEOREM 2. *Construction B defines a base set of order* $3q$, *where* $q+2 \in D(A)$.

Proof. To show that Construction B generates a base set of order $3q$, we have to show that the set $\{S_{ij}\}$ has properties (1) through (8) which define a base set. Property (1) holds since each set S_{ij} is a PT and has $9((q-1)(q-2)/6)+3(q-1)+1=3q(q-1)/2+1$ blocks. By the definition of type (B.3), S_{i0}, S_{i1}, and S_{i2} have the triple $\{(i,0),(i,1),(i,2)\}$ and, since each pair $\{i,a\}$, $a\in Z_q$, is covered in $\mathrm{ST}_i(q+2)$ by the triple $\{i,a,b\}$, $b\in Z_q$, then, by the definition of the blocks of types (B.1) and (B.3), all the pairs which include elements from $\{(i,0),(i,1),(i,2)\}$ are covered in S_{ij}.

Except for triples of type (B.3) we claim that the intersection between two PTs is empty. To see this note that two PTs can only intersect in blocks of the same type. The disjointness of blocks of type (B.1) follows from the fact that the STSs $\mathrm{ST}_l(q+2)$ and $\mathrm{ST}_m(q+2)$, $l,m\in Z_q$, $l\neq m$, are disjoint. The disjointness of blocks of type (B.2) follows from the facts that the sets F_t, $t\in Z_3$, are disjoint and, if $\mathrm{index}(a,b,\mathrm{ST}_l(q+2))=1$, then $\mathrm{index}(a,b,\mathrm{ST}_m(q+2))=0$, $l\neq m$.

We now set $\mathrm{index}((a,t),(b,t+j),S_{ij})=1$, $t\in Z_3$, iff $\{a,b,q\}\in\mathrm{ST}_i(q+2)$. We also set $\mathrm{index}((a,t),(b,t+j),S_{ij})=2$, $t\in Z_3$, iff $\{a,b,q+1\}\in \mathrm{ST}_i(q+2)$. Properties (5) through (8) of the function *index* follow from Property A and the definition of the blocks of type (B.2). Finally, a simple counting argument shows that each triple of $Z_q\times Z_3$ is covered by at least one PT. Q.E.D.

Construction C. Let $F=\{F_0,F_1,\dots,F_{k-1}\}$ be a near-one factorization of K_k, such that in F_i vertex i is isolated. Let Q_{il}, $i\in Z_q$, $l\in Z_3$, be a base set of order $3q$ on the points $Z_q\times Z_3$. Let R_{jl}, $j\in Z_k$, $l\in Z_3$, be a base set of order $3k$ on the points $Z_k\times Z_3$. From these sets we generate a base set of order $3qk$ with the PTs S_{ijl}, $i\in Z_q$, $j\in Z_k$, $l\in Z_3$, on the points $Z_q\times Z_k\times Z_3$ (it can be mapped into a base set on the points $Z_{qk}\times Z_3$).

Each set S_{ijl}, $i\in Z_q$, $j\in Z_k$, $l\in Z_3$, has the blocks:

$$\{(a,x,b),(c,y,d),(e,x+y+j,f)\},$$
$$x,y\in Z_k,\ \{(a,b),(c,d),(e,f)\}\in Q_{il}-\{(i,0),(i,1),(i,2)\} \tag{C.1}$$

There are $(3q(q-1)/2)k^2$ blocks of this type.

$$\{(i,a,b),(i,c,d),(i,e,f)\},\ \{(a,b),(c,d),(e,f)\}\in R_{jl}. \tag{C.2}$$

There are $3k(k-1)/2+1$ blocks of this type.

$$\{(a,x,b),(a,y,b),(c,t+j,d)\},$$
$$t\in Z_k,\ \{x,y\}\in F_t,\ \mathrm{index}((a,b),(c,d),Q_{il})>0. \tag{C.3}$$

There are $(3q-3)\binom{k}{2}$ blocks of this type.

THEOREM 3. *Construction C defines a base set of order* $3qk$, *where* q *and* k *are odd, and there exist base sets of orders* $3q$ *and* $3k$, *respectively.*

Proof. To show that Construction C generates a base set of order $3qk$, we have to show that the set $\{S_{ijl}\}$ has properties (1) through (8) which define a base set. Property (1) holds since each set S_{ijl} is a PT and has $(3q(q-1)/2)k^2 + 3k(k-1)/2 + 1 + (3q-3)\binom{k}{2} = 3qk(qk-1)/2 + 1$ blocks. Since R_{jl}, $j \in Z_k$, $l \in Z_3$, is a base set of order $3k$ it follows by the definition of type (C.2) that S_{ij0}, S_{ij1}, and S_{ij2} have the triple $\{(i, j, 0), (i, j, 1), (i, j, 2)\}$ and since each pair $\{(i, a), (m, b)\}$, $m \in Z_q$, $m \neq i$, $a, b \in Z_3$, is covered in Q_{il}, and each pair with element from $\{(j, 0), (j, 1), (j, 2)\}$ is covered in R_{jl}, then by the definition of the blocks of types (C.1) and (C.2) it follows that all the pairs which include an element from $\{(i, j, 0), (i, j, 1), (i, j, 2)\}$ are covered in S_{ijl}.

Except for triples of type (C.2) we claim that the intersection between two PTs is empty. To see this, note that two PTs can only intersect in blocks of the same type. The disjointness of the blocks of type (C.1) follows from the almost disjointness of Q_{i1}, Q_{i2}, and Q_{i3} and the disjointness of Q_{il} and Q_{rs} for $i \neq r$. The disjointness of the other blocks of type (C.2) follows from the fact that R_{il}, $i \in Z_k$, $l \in Z_3$, is a base set of order $3k$. The disjointness of blocks of type (C.3) follows from the facts that the sets F_t, $t \in Z_k$, are disjoint and if index$((a, b), (c, d), Q_{il}) > 0$ then index$((a, b), (c, d), Q_{rs}) = 0$, $(i, l) \neq (r, s)$.

We now set index$((i, a, b), (i, c, d), S_{ijl}) = x$, iff index$((a, b), (c, d), R_{jl}) = x$, $x \in Z_3$, and index$((a, t, b), (c, t + j, d), S_{ijl}) = x$, $t \in Z_k$, iff index$((a, b), (c, d), Q_{il}) = x$, $x \in Z_3$. Properties (5) through (8) of the function *index* follow from the facts (i) the R_{il} are the PTs of a base set of order $3k$ and the definition of the blocks of type (C.2) and (ii) the Q_{il} are the PTs of a base of order $3q$ and the definition of the blocks of type (C.3).

Finally, a simple counting argument shows that each triple of $Z_q \times Z_k \times Z_3$ is covered by at least one PT. Q.E.D.

From Theorems 1 through 3 we infer

THEOREM 4. *There exists a base set of order* $3qk$, *where* $q + 2 \in D(A)$ *and* $k \equiv 1 \pmod 6$.

3. PARTITIONS OF ORDERS $n \equiv 4 \pmod 6$

In this section we prove the existence of optimal partitions for all orders $n \equiv 4 \pmod 6$. An optimal partition has $n - 1$ optimal PTs and one PT of size $(n-1)/3$.

Construction D. Let Q_{ij}, $i \in Z_r$, $j \in Z_3$, be the PTs of a base set of order $3r$ on the points $Z_r \times Z_3$. From this base set we can generate an optimal partition of order $3r+1$, with the $3r+1$ PTs S_{ij}, $i \in Z_r$, $j \in Z_3$, and S_A, on the points $Z_r \times Z_3 \cup \{A\}$.

Define S_{ij} by

$$S_{ij} = Q_{ij} - \{(i, 0), (i, 1), (i, 2)\} \cup \{\{x, y, A\}: \text{index}(x, y, Q_{ij}) = 1\} \\ \cup \{\{(i, j), (i, j+1), A\}\}.$$

It is easy to verify that each pair appears at most once as a subset of a triple from S_{ij} and therefore S_{ij} is a PT. Each set S_{ij} has

$$\frac{3r(r-1)}{2} + \frac{3r-3}{2} + 1 = \frac{3r^2-1}{2}$$

blocks and hence it is an optimal PT.

S_A has the k blocks of the form $\{(i, 0), (i, 1), (i, 2)\}$, $i \in Z_r$.

It is not difficult to see that the $3r+1$ PTs are pairwise disjoint and together include all the triples from the set $Z_r \times Z_3 \cup \{A\}$. Hence, by using Theorem 1 we have the following theorem.

THEOREM 5. *For $n = 3r+1$, $r \equiv 1$ or $3 \pmod 6$ there exists a partition for triples with n pairwise disjoint PTs of order n, for which the first $n-1$ are optimal.*

Etzion [3] proved the existence of optimal partitions of orders $3k+1$, $k \equiv 1$ or $5 \pmod 6$. Hence, by using Theorem 5 we have

THEOREM 6. *For $n \equiv 4 \pmod 6$, there exists a partition with n pairwise disjoint PTs of order n, for which the first $n-1$ are optimal.*

4. PARTITIONS OF ORDERS $n = 3rk + 2$, k ODD

The main result of this section is that for every $n \equiv 5 \pmod 6$ there exists a partition with $n-2$ pairwise disjoint optimal PTs, which is the maximum possible number of pairwise disjoint optimal PTs for these orders.

Construction E. Let $F = \{F_0, F_1, ..., F_{k-1}\}$ be a near-one-factorization of K_k, such that in F_i vertex i is isolated. Let Q_{il}, $i \in Z_r$, $l \in Z_3$, be a base set of order $3r$ on the points $Z_r \times Z_3$. Let R_{il}, $i \in Z_k$, $l \in Z_3$, R_A, and R_B, be a partition of order $3k+2$, on the points $Z_k \times Z_3 \cup \{A, B\}$, for which the first $3k$ PTs are optimal. Each of these $3k$ PTs has size $(3k^2+3k-2)/2$.

From these sets we can generate a partition of order $3rk+2$, with the PTs S_{ijl}, $i \in Z_r$, $j \in Z_k$, $l \in Z_3$, S_A, and S_B, on the points $Z_r \times Z_k \times Z_3 \cup \{A, B\}$.

Each set S_{ijl}, has the blocks:

$$\{(a, x, b), (c, y, d), (e, x+y+j, f)\},$$
$$x, y \in Z_k,\ \{(a, b), (c, d), (e, f)\} \in Q_{il} - \{(i, 0), (i, 1), (i, 2)\}. \tag{E.1}$$

There are $(3r(r-1)/2)k^2$ blocks of this type.

$$\{(i, a, b), (i, c, d), (i, e, f)\},\ \{(a, b), (c, d), (e, f)\} \in R_{jl}$$
$$\{(i, a, b), (i, c, d), A\},\ \{(a, b), (c, d), A\} \in R_{jl}$$
$$\{(i, a, b), (i, c, d), B\},\ \{(a, b), (c, d), B\} \in R_{jl}$$
$$\{(i, a, b), A, B\},\ \{(a, b), A, B\} \in R_{jl}. \tag{E.2}$$

There are $(3k^2+3k-1)/2$ blocks of this type.

$$\{(a, x, b), (a, y, b), (c, t+j, d)\},$$
$$t \in Z_k,\ \{x, y\} \in F_t,\ \text{index}((a, b), (c, d), Q_{il}) > 0. \tag{E.3}$$

There are $(3r-3)\binom{k}{2}$ blocks of this type.

$$\begin{aligned} &\{(a, t, b), (c, t+j, d), A\},\ t \in Z_k,\ \text{index}((a, b), (c, d), Q_{il}) = 1 \\ &\{(a, t, b), (c, t+j, d), B\},\ t \in Z_k,\ \text{index}((a, b), (c, d), Q_{il}) = 2. \end{aligned} \tag{E.4}$$

There are $(3r-3)k$ blocks of this type.

It is easy to verify that each set S_{ijl}, $i \in Z_r$, $j \in Z_k$, $l \in Z_3$, is a PT and has

$$\frac{3r(r-1)}{2}k^2 + \frac{3k^2+3k-1}{2} + (3r-3)\binom{k}{2} + 2(3r-3)k$$
$$= \frac{3(rk)^2+3rk-2}{2}$$

blocks. Hence, each PT is optimal. Note, that, since not all the pairs can be covered by $R_A \cup R_B$, we can assume that $\{j, A, B\} \notin R_A \cup R_B$, $j \in Z_k$.

The set, S_x, $x \in \{A, B\}$, has the blocks:

$$\{(i, a, b), (i, c, d), (i, e, f)\}, \quad i \in Z_r,\ \{(a, b), (c, d), (e, f)\} \in R_x$$
$$\{(i, a, b), (i, c, d), A\}, \quad i \in Z_r,\ \{(a, b), (c, d), A\} \in R_x$$
$$\{(i, a, b), (i, c, d), B\}, \quad i \in Z_r,\ \{(a, b), (c, d), B\} \in R_x.$$

If R_A has n_0 blocks and R_B has n_1 blocks, then S_A and S_B have $r \cdot n_0$ and $r \cdot n_1$ blocks, respectively.

Similarly to the proofs of Theorems 1 through 3 it can be verified that the $3rk+2$ PTs are pairwise disjoint and together include all the triples from the set $Z_r \times Z_k \times Z_3 \cup \{A, B\}$. For $n = 3k+2$, $k \equiv 1$ or $5 \pmod 6$, $k > 1$, it was proved [3] that there exists a partition of order n with $n-2$ pairwise disjoint optimal PTs of order n, and two PTs for which the first is at least of size $n-2$. Hence, we have the following theorem.

THEOREM 7. *For $n = 3k+2$, $k > 1$ odd, there exists a partition for triples with $n-2$ pairwise disjoint optimal PTs of order n, and two PTs for which the first is at least of size $n-2$.*

5. OPTIMAL PARTITIONS OF ORDER $n \equiv 5 \pmod 6$

The question whether a partition of order $n \equiv 5 \pmod 6$ with $n-1$ PTs for which the first $n-2$ are optimal remains open. We believe they exist for every $n \geqslant 11$ and we will refer to them as optimal partitions of order $n \equiv 5 \pmod 6$. An easier problem is the existence of partitions of order $n \equiv 5 \pmod 6$ with $n-1$ PTs. Brouwer *et al.* [1] proved that a partition of order 11 with 10 PTs exists. In [3] we proved the following result.

THEOREM 8. *Let $n \equiv 3 \pmod 6$.*

(1) *If there exists an optimal partition of order $n+2$ then there exists an optimal partition of order $qn+2$, $q+2 \in D(A)$.*

(2) *If there exists a partition of order $n+2$ with $n+1$ PTs then there exists a partition of order $qn+2$, $q+2 \in D(A)$, with $qn+1$ PTs.*

Construction E will work if, instead of the $3k$ optimal PTs of order $3k+2$, R_A, and R_B, we take a partition of order $3k+2$ with $3k+1$ PTs, R_{ij}, $i \in Z_k$, $j \in Z_3$, and R_A. We can make sure that the pair $\{A, B\}$ is not covered by the triples of R_A, since any PT of order $3k+2$ has at least four uncovered pairs [11]. Using Construction E and Theorems 1 through 4, we can extend Theorem 8 as follows.

THEOREM 9. *Let $n \equiv 3 \pmod 6$ and let k be an integer such that $k \equiv 1$ or $3 \pmod 6$ or $k = qr$, $q+2 \in D(A)$ and $r \equiv 1 \pmod 6$.*

(1) *If there exists an optimal partition of order $n+2$ then there exists an optimal partition of order $kn+2$.*

(2) *If there exists a partition of order $n+2$ with $n+1$ PTs then there exists a partition of order $kn+2$ with $kn+1$ PTs.*

In [3] we obtained a partition of order 11 with nine optimal PTs, one PT of size 10 and one PT of size 2. For order 17 we obtained a partition with 15 optimal PTs, one PT of size 17 and one PT of size 3. These two partitions are given in the Appendix. Hence, by using Construction E, we have the following theorem.

THEOREM 10. *Let k be an integer such that $k \equiv 1$ or 3 (mod 6) or $k = qr$, $q+2 \in D(A)$ and $r \equiv 1$ (mod 6). Then*

(1) *there exists a partition of order $9k+2$ with $9k$ optimal PTs, one PT of size $10k$, and one PT of size $2k$;*

(2) *there exists a partition of order $15k+2$ with $15k$ optimal PTs, one PT of size $17k$, and one PT of size $3k$.*

6. Partitions of Order $n \equiv 5$ (mod 6) with $n-1$ PTs

In Section 5 we gave some theorems for the recursive constructions of partitions of orders $n \equiv 5$ (mod 6) with $n-1$ PTs. In this section we give a non-recursive construction for partitions of orders $n \equiv 5$ (mod 6) with $n-1$ PTs. For this construction we need a special partition for triples of order $k \equiv 5$ (mod 6) into k pairwise disjoint PTs. For $k \equiv 1$ or 5 (mod 6) we say that $k \in D(B)$ if there exist k PTs $Q_i(k)$, $i \in Z_k$, on the points Z_k such that the following properties hold:

(1) Each set has $(k-1)(k-2)/6$ triples.

(2) The union of all the PTs is the set of all triples of order $3k$.

(3) Each two PTs are disjoint.

(4) All the pairs $\{i, a\}$, $a \in Z_k$, $a \neq i$, are covered in $Q_i(k)$.

(5) For a PT, $Q_i(k)$ there are $k-1$ elements in the uncovered pairs, each element appears in exactly two uncovered pairs, and we can order these elements in cyclic sequences $a_0, a_1, ..., a_{r-1}$ such that $\{a_m, a_{m+1}\}$ (subscripts taken modulo r) is an uncovered pair, and $a_l \neq a_m$ for $l \neq m$. Let $\text{index}(b, c, Q_i(k)) = 0$ if and only if there is no m such that $b = a_m$ and $c = a_{m+1}$ in some cyclic sequence.

(6) $\text{index}(b, c, Q_i(k)) \in \{0, 1, 2, 3\}$.

(7) If $\text{index}(b, c, Q_i(k)) > 0$ then $\text{index}(b, c, Q_j(k)) = 0$ for $j \neq i$.

(8) If $\text{index}(b, c, Q_i(k)) = 1$ then there exists $j \in Z_k$, $j \neq i$, such that $\text{index}(c, b, Q_j(k)) \in \{2, 3\}$.

(9) If $\text{index}(a_m, a_{m+1}, Q_i(k)) = 1$ then $\text{index}(a_{m+1}, a_{m+2}, Q_i(k)) \in \{1, 2\}$.

(10) If $\text{index}(a_m, a_{m+1}, Q_i(k)) = 3$ then $\text{index}(a_{m+1}, a_{m+2}, Q_i(k)) \in \{1, 2\}$.

(11) If $\text{index}(a_m, a_{m+1}, Q_i(k)) = 2$ then $\text{index}(a_{m+1}, a_{m+2}, Q_i(k)) = 3$.

Some values which belong to $D(B)$ can be detected similarly to the values which belong to $D(A)$.

LEMMA 4. *If p is a prime such that the order of -2 modulo p is congruent to 0 modulo 4 then $p \in D(B)$.*

Proof. Let $p \equiv 1$ or $5 \pmod 6$ be a prime such that the order of -2 modulo p is congruent to 0 modulo 4. Define $Q_i(p) = \{\{a, b, c\} : a+b+c \equiv 3i \pmod p\}$, $i \in Z_p$. It is clear that properties (1) through (4) hold. It is also clear that each PT is a cyclic shift of the other PTs and that the same is true for the uncovered pairs. The uncovered pairs in $Q_0(p)$ have the form $\{x, y\}$, where $2x + y \equiv 0 \pmod p$. Hence the cyclic sequences defined in (5) can be defined in a way that $2a_m + a_{m+1} \equiv 0 \pmod p$ or $a_{m+1} \equiv -2a_m \pmod p$ for $Q_0(p)$. Therefore, the elements of $Z_p - \{0\}$ are ordered in cycles $a_0 a_1 \cdots a_{r-1}$ such that $a_{m+1} \equiv -2a_m \pmod p$. Now, we set

(1) $\text{index}(a_m, a_{m+1}, Q_0(p)) = 1$ iff $0 \leqslant m \leqslant r/2 - 1$,

(2) $\text{index}(a_m, a_{m+1}, Q_0(p)) = 2$ iff $r/2 \leqslant m$ and m is even,

(3) $\text{index}(a_m, a_{m+1}, Q_0(p)) = 3$ iff $r/2 < m$ and m is odd,

(4) $\text{index}(b+j, c+j, Q_j(p)) = x$ iff $\text{index}(b, c, Q_0(p)) = x$.

It is easy to verify that by this definition of the function *index* properties (5) through (11) hold. Q.E.D.

Schreiber [10] observed that the condition of Lemma 4 holds for all primes of the form $8t+5$ and some primes of the form $8t+1$. The first primes for which it holds are 5, 13, 17, 29, 37, 41, 53, 61, 97, and 101.

To obtain more values which belong to $D(B)$, we use a result which is similar to Lemma 2. The proof is left for the reader.

LEMMA 5. *If $q \in D(B)$ and $k \in D(B)$ then $qk \in D(B)$.*

Construction F. Let $k \equiv 5 \pmod 6$, where $k \in D(B)$ and let $\{Q_i(k)\}$, $i \in Z_k$, be the corresponding partition. From this partition we generate a partition of order $2k+1$, with the $2k$ PTs S_{ij}, $i \in Z_k$, $j \in Z_2$, on the points $Z_k \times Z_2 \cup \{A\}$. Each set S_{ij} has the blocks

$$\{(a, x), (b, y), (c, x+y+j)\}, \qquad \{a, b, c\} \in Q_i(k),\ x, y \in Z_2. \quad \text{(F.1)}$$

There are $4((k-1)(k-2)/6)$ blocks of this type.

$$\begin{aligned} \{(a, j), (b, j+1), A\}, &\quad \text{index}(a, b, Q_i(r)) = 1 \\ \{(a, j), (b, j), A\}, &\quad \text{index}(a, b, Q_i(r)) = 2 \\ \{(a, j+1), (b, j+1), A\}, &\quad \text{index}(a, b, Q_i(r)) = 3. \end{aligned} \tag{F.2}$$

There are $k-1$ blocks of this type.

$$\begin{aligned} \{(a, 0), (a, 1), (b, j)\}, &\quad \text{index}(a, b, Q_i(r)) \in \{1, 3\} \\ \{(a, 0), (a, 1), (b, j+1)\}, &\quad \text{index}(a, b, Q_i(r)) = 2. \end{aligned} \tag{F.3}$$

There are $k-1$ blocks of this type. For $j=0$ we have also the block

$$\{(i, 0), (i, 1), A\}. \tag{F.4}$$

Each set S_{ij}, $i \in Z_k$, $j=0$, has

$$4\left(\frac{(k-1)(k-2)}{6}\right) + 2(k-1) + 1 = \frac{(k-1)(2k+2)}{3} + 1$$

blocks, and each set S_{ij}, $i \in Z_k$, $j=1$, has $(k-1)(2k+2)/3$ blocks.

THEOREM 11. *Construction F generates a partition for triples of order* $2k+1$, $k \equiv 5 \pmod 6$, $k \in D(B)$, *with* $2k$ *PTs.*

Proof. It is easy to verify that each pair appears at most once as a subset of a block in S_{ij}. Hence, S_{ij} is a PT of order $2k+1$.

It is easy to verify that two PTs can intersect only in blocks of the same type. The disjointness of blocks of type (F.1) follows from the disjointness of the k PTs $Q_i(k)$, $i \in Z_k$. The disjointness of the blocks of types (F.2) and (F.3) follows directly from the definition of the function *index*. The disjointness of the blocks of type (F.4) is trivial.

Now, since

$$k\left(\frac{(k-1)(2k+2)}{3} + 1\right) + k\left(\frac{(k-1)(2k+2)}{3}\right) = \frac{(2k+1)\,2k(2k-1)}{6}$$

it follows that construction F generates a partition of order $2k+1$. Q.E.D.

7. CONCLUSION

We proved that optimal partitions for triples of orders $n \equiv 4 \pmod 6$ exist for every n. We proved that partitions for triples of orders $n \equiv 5 \pmod 6$ with $n-2$ optimal PTs, which is the maximum possible number of pairwise disjoint optimal PTs that exist for every n. We proved that the existence of one optimal partition of order $n \equiv 5 \pmod 6$ implies the existence of many other optimal partitions. We gave constructions of partitions for orders $n \equiv 5 \pmod 6$ with $n-1$ PTs. There are still a few important open questions in this context. We give the main ones in the form of the following two conjectures:

The Weak Conjecture. For every $n \equiv 5 \pmod 6$, $n > 5$, there exists a partition for triples of order n with $n-1$ PTs.

The Strong Conjecture. For every $n \equiv 5 \pmod 6$, $n > 5$, there exists a partition for triples of order n with $n-1$ PTs, for which the first $n-2$ PTs are optimal.

Appendix

This appendix gives the partitions of triples into disjoint packings. The notation is as in Brouwer *et al.* [1]. We ordered all the triples in lexicographic order and the disjoint packings are numbered by 1, 2, 3, 4, 5, 6, 7, 8, 9, *A*, *B*, and so on. Then we list the triples in lexicographic order giving the number of packing to whom they belong for $n = 11$ and $n = 17$, respectively.

```
2619473214B67842328598678 5A5234319674B9A88956169345432 17A345
37186991A67279145651432835145 66A9829371124A78A87125676825439
198A235617754193A74822365A894958236787 2936541
```

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AB68H1E7943712F98A254F682E315A9B436FD917A93CF42E6BAC254713BE
AC28ABA48D153DF7CBD7365879G4C7FB6DH9FBCB519E2645E18DCE284769
823AG55D781C67EGAB1CDC612AF37546F29EDFC39587AGC534BBCD66E389
2DA78F5B8C2DED7923B1486E5713AFE1FD4A698B584D624GFCCDBE77F349
A3EGB8916C19D3EFE8DA34C2597EF6824B1F261E5B7A9C6B295DE7935GC1
DDE7CF881645AB4FEGC9A2GD32AE4F1F98EB45D36A84F17935C213572F6C
8BADC4DG31A6BEF85A467CD2EEG38D1992F756BC518FGDAB3HE943BF5121
A29FC56E47B975128A46D324A68317D9CBEDEC5EFG432B7ACF1986B5784D
E3FFG549E2AA3B1867CD62D91GEBC49F7A54C16232B93A1D67F58CA18623
9B57E4356B79428EADCFE57FD36F81G5D43C89BD12AB97C68965EF411246
5AF3BB4AC2978DE731EA2GFCD5A1C8B65D27343CFA4B2E78169DBE29734A
C68F54627C8A539FBED1
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