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Optimal Partitions for Triples

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The best known method to obtain constant weight codes with distance 4 is the partitioning method. To apply this method one has to partition sets of n-tuples into disjoint constant codes of weight w and distance 4, such that the number of codes will be minimal and the codes will be as large as possible. In this paper we consider the case of $w=3$. For $n \equiv 0, 1, 2, 3 \pmod 6$ the optimal partition is derived from disjoint Steiner triple systems. We give optimal partitions for all order $n=3k+1$, $k \equiv 1$ or $5 \pmod 6$. For orders $n=3k+2$, $k \equiv 1$ or $5 \pmod 6$, $k>1$, or $k \equiv 3 \pmod{12}$ we present constructions which get the maximal number, $3k$, of disjoint optimal codes. We give a construction which gets a partition of order $n=qk+i$, $i=1, 2$, $q \equiv 1$ or $5 \pmod 6$, $q>1$, $k \equiv 3 \pmod 6$, with qk disjoint optimal codes, if a partition of order $k+i$ with k optimal codes exists, and a set of q pairwise disjoint Steiner triple systems, or order $q+2$, with some special property exists. For these q's, we also prove that for $n=9q+2$ there exists a partition with $9q+1$ codes.

1. INTRODUCTION

Let $A(n, d, w)$ denote the maximum number of codewords in a binary code of length n, minimum distance d, and constant weight code w. $A(n, d, w)$ is a fundamental combinatorial quantity, which is also used in the construction of codes for asymmetric channels, DC-free codes, and spherical codes [2]. The best known method to design constant weight codes with distance 4 is the partitioning method [2] which was introduced in [11]. To apply this method we have to partition sets of n-tuples into disjoint constant weight codes of weight w and distance 4. In this paper we discuss partitions of n-tuples with weight 3. For this case, Spencer [16] proved that

$$A(n, 4, 3) = \left[\frac{n}{3}\left[\frac{n-1}{2}\right]\right] \qquad \text{for} \quad n \not\equiv 5 \pmod 6,$$

$$A(n, 4, 3) = \left[\frac{n}{3}\left[\frac{n-1}{2}\right]\right] - 1 \qquad \text{for} \quad n \equiv 5 \pmod 6.$$

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A *packing triple system* (PT) of order n (PT(n)) is a pair (Q, q), where $Q = \{0, 1, ..., n-1\}$ is a set of n *points* and q is a collection of 3-element subsets of Q called *blocks* such that every 2-element of Q is a subset of at most one block of q (in other words every pair is *covered* by at most one block). A PT is *optimal* if there is no PT of the same order with a larger size. A *Steiner triple system* (STS) of order n(ST(n)) is a (PT(n)) such that every 2-element subset of Q is a subset of exactly one block of q. It is well known that an ST(n) exists if and only if $n \equiv 1$ or 3 (mod 6). It is clear that $A(n, 4, 3)$ is equal to the number of blocks in an ST(n), i.e., ST(n) is an optimal PT. Two PTs (Q, q_1) and (Q, q_2) are *disjoint* if $q_1 \cap q_2 = \emptyset$. It is known that for $n \equiv 1$ or 3 (mod 6), $n \geqslant 9$, there exists a set of $n-2$ pairwise disjoint STSs [6, 7, 18]. By taking from each ST(n) the blocks which do not contain point $n-1$, we obtain a set of $n-2$ disjoint PTs which attain $A\ (n-1, 4, 3)$. Thus, the problem is solved for $n \equiv 0, 1, 2$, or 3 (mod 6), $n \geqslant 8$. In this paper we handle the cases $n \equiv 4$ or 5 (mod 6).

In Section 2 we present some definitions and some of the basic ideas of our first construction. In Section 3 we present optimal partitions of orders $n = 3k+1$, $k \equiv 1$ or 5 (mod 6), $k > 5$. Each partition contains $3k$ PTs of optimal size $(3k^2-1)/2$ and one PT of size k. In Section 4 we present partitions of order $n = 3k+2$, $k \equiv 1$ or 5 (mod 6), $k > 5$. Each partition contains $3k$ PTs of optimal size $(3k^2+3k-2)/2$, one PT of size $3k$ and one PT of size k. In Section 5 we generalize the construction of Rosa [12]. For order $n = 3k+2$, $k \equiv 3$ (mod 12), we obtain $3k$ optimal PTs, one PT of size $5m+2$ and one PT of size $3m+2$, where $k = 2m+1$. In Section 6 we present a recursive construction similar to the one of Lu [6]. From a partition with k optimal PTs of order $k+i$, $i = 1, 2$, $k \equiv 3$ (mod 6), and q disjoint STSs of order $q+2$ with a special property, we obtain a partition with qk optimal PTs of order $qk+i$. We also prove that for $n = 9q+2$ there exists a partition with $9q+1$ PTs. In Section 7 we present optimal partitions with the maximum number of optimal PTs for orders $n = 10$, 11, 16, and 17, and a possible way to attack the orders which are not covered by the other constructions. In the conclusion we end this paper with some interesting questions and conjectures.

2. On the Structure of the First Construction

In our first construction we will use some known partitions for pairs and triples into disjoint constant weight codes with distance 4. Partition of pairs on n points is equivalent to a one-factorization of the complete graph K_n when n is even and to a near-one-factorization of K_n when n is odd. We only need the case where n is odd. In each near-one-factor (one set of pairs) of K_n, with $V(K_n) = Z_n$, one vertex is isolated and it is easy to choose a

near-one-factorization $F=\{F_0, F_1, ..., F_{n-1}\}$ such that in F_i vertex i is isolated. For more information on one-factorizations the reader is referred to [8].

We construct partitions for triples of order $3k+1$ and $3k+2$, where k is odd. We consider the points as taken from the set $Z_k \times Z_3 \cup \{3k\}$ for $n=3k+1$, and $Z_k \times Z_3 \cup \{3k, 3k+1\}$ for $n=3k+2$. Throughout Sections 2, 3, and 4, we assume that for a point (i, j) addition in the first coordinate is taken modulo k and in the second modulo 3. We consider 13 types of configurations for the triples. In a (i, j, m) configuration i points are from $Z_k \times \{0\}$, j from $Z_k \times \{1\}$, and m from $Z_k \times \{2\}$. There are 10 configurations of this form. In a $(3k)$ configuration point $3k$ is in the block. Similarly, we have $(3k+1)$ and $(3k, 3k+1)$ configurations when the order of the PT is $3k+2$. The configurations in our PTs have some similarity to the configurations which were used by Teirlinck [17]. If $n=3k+1$ we use k PTs with blocks from configurations $(1, 1, 1)$, $(3, 0, 0)$, $(0, 3, 0)$, $(0, 0, 3)$, and $(3k)$. We also use k PTs with blocks from configurations $(1, 2, 0)$, $(0, 1, 2)$, $(2, 0, 1)$, and $(3k)$ and, similarly, k PTs with blocks from configurations $(2, 1, 0)$, $(0, 2, 1)$, $(1, 0, 2)$, and $(3k)$. The pairs which are not covered by the blocks of the first three configurations in the last $2k$ PTs are of the form $\{(x, j), (y, j+1)\}$. We have to determine x and y in such a way that for each of these $2k$ PTs we will have $(3k-1)/2$ blocks with configuration $(3k)$. This is done by using k sequences with elements from $Z_k \times Z_3$ such that the following three properties are satisfied:

(P.1) Each sequence is a permutation of $Z_k \times Z_3$.

(P.2) Each element (x, i) is followed by $(y, i+1)$.

(P.3) Each ordered pair (x, i), $(y, i+1)$ appears exactly once as consecutive elements in one of the sequences.

These three properties are sufficient for $n=3k+1$. For $n=3k+2$ we need one more property which our sequences have and which will be discussed later. Sequence i, $0 \leqslant i \leqslant k-1$, has elements $A_j^{(i)}$, $0 \leqslant j \leqslant 3k-1$, which are defined as follows: $A_0^{(i)}=(i, 0)$, $A_1^{(i)}=(2i, 1)$, $A_2^{(i)} \equiv (4i, 2)$, and $A_j^{(i)}=A_{j-3}^{(i)}+(1, 0)$, $3 \leqslant j \leqslant 3k-1$.

LEMMA 1. *If* $k \equiv 1$ *or* 5 (mod 6) *then the Sequences* $A^{(i)}$, $0 \leqslant i \leqslant k-1$, *satisfy properties* (P.1) *through* (P.3).

Proof. The initialization of $A^{(i)}$ and the fact that $A_j^{(i)}=A_{j-3}^{(i)}+(1, 0)$ implies that $A^{(i)}$ is a permutation of $Z_k \times Z_3$. The cyclic sequence $A^{(i)}$ is the same as the following cyclic sequence $B^{(i)}$, $B_0^{(i)}=(0, 0)$, $B_1^{(i)}=(i, 1)$, $B_2^{(i)}=(3i, 2)$, $B_j^{(i)}=B_{j-3}^{(i)}+(1, 0)$. It is clear that since $k \equiv 1$ or 5 (mod 6) we have that $\{B_1^{(i)}\}=Z_k \times \{1\}$ and $\{B_2^{(i)}\}=Z_k \times \{2\}$, and together with the facts that $B_0^{(i)}=(0, 0)$ for all i, and $B_j^{(i)}=B_{j-3}^{(i)}+(1, 0)$, it follows that each

one of the ordered pairs $(x, 0)$, $(y, 1)$ and $(x, 2)$, $(y, 0)$ appears exactly once as consecutive elements in one of the $B^{(i)}$s. By definition, the pair $(x, 1)$, $(x+2i, 2)$, $0 \leqslant x \leqslant k-1$, appears in $B^{(i)}$, and since k is odd $\{2i \bmod k): 0 \leqslant i \leqslant k-1\} = \{j \pmod k: 0 \leqslant j \leqslant k-1\}$. Hence, (P.1) through (P.3) are satisfies. Q.E.D.

Now, we make sure that all the uncovered pairs in PTS i and $k+i$ of the last $2k$ PTs are exactly all the pairs of consecutive elements in $A^{(i)}$. This is done in Sections 3 and 4.

3. Partitions of Orders $n = 3k+1$, $k \equiv 1$ or $5 \pmod 6$

Let $F = \{F_0, F_1, ..., F_{k-1}\}$ be a near-one-factorization of K_k, such that in F_i vertex i is isolated. Let $ST_i(k+2)$, $0 \leqslant i \leqslant k-1$, $k \equiv 1$ or $5 \pmod 6$, $k > 5$, be STS number i from a set of k pairwise disjoint STSs of order $k+2$.

Each set S_i, $0 \leqslant i \leqslant k-1$, has the blocks

$$\{(j, 0), (m, 1), (m+j+i, 2)\}, \qquad 0 \leqslant j, m \leqslant k-1$$

$$\{(j, t), (m, t), (l, t)\}, \ t = 0, 1, 2, \ \{j, m, l\} \in ST_i(k+2), \ 0 \leqslant j, m, l \leqslant k-1$$

$$\{(j, t), (m, t), 3k\}, \ t = 0, 1, 2, \ \{j, m, k\} \in ST_i(k+2), \ 0 \leqslant j, m \leqslant k-1.$$

Each set S_i, $0 \leqslant i \leqslant k-1$, has

$$k^2 + 3 \cdot \frac{1}{k}\binom{k}{3} + 3 \cdot \frac{k-1}{2} = \frac{3k^2-1}{2} \text{ blocks.}$$

Each set S_{k+i}, $0 \leqslant i \leqslant k-1$, has the blocks

$$\{(j, t), (m, t+1), (l, t+1)\}, \qquad t = 0, 1, 2, \ 0 \leqslant j, m, l \leqslant k-1, \ \{m, l\} \in F_r,$$

where (j, t), $(r, t+1)$ are consecutive elements in $A^{(i)}$:

$$\{(j, t), (m, t+1), 3k\}, \qquad A_r^{(i)} = (j, t), \qquad A_{r+1}^{(i)} = (m, t+1), \qquad r \text{ odd.}$$

Each set S_{2k+i}, $0 \leqslant i \leqslant k-1$, has the blocks

$$\{(m, t), (l, t), (j, t+1)\}, \qquad \{m, l\} \in F_r,$$

where (r, t), $(j, t+1)$ are consecutive elements in $A^{(i)}$:

$$\{(j, t), (m, t+1), 3k\}, \quad A_r^{(i)} = (j, t), \quad A_{r+1}^{(i)} = (m, t+1), \quad r \text{ even}, r \neq 0.$$

Each set S_i, $k \leqslant i \leqslant 3k-1$, has

$$3 \cdot \binom{k}{2} + \frac{3k-1}{2} = \frac{3k^2-1}{2} \text{ blocks.}$$

The last set, S_{3k}, has the k blocks of the form $\{(j, 0), (2j, 1), 3k\}$, $0 \leqslant j \leqslant k-1$. All the $3k+1$ sets defined are PTs and together include all the triples on the set $Z_k \times Z_3 \cup \{3k\}$. Also it is easy to verify that each two PTs are disjoint. Hence, we have the following theorem.

THEOREM 1. *The S_i's, $0 \leqslant i \leqslant 3k$, define a set of $3k+1$ pairwise disjoint PTs of order $3k+1$, $k \equiv 1$ or 5 (mod 6), for which the first $3k$ are optimal.*

4. PARTITIONS OF ORDERS $n = 3k+2$, $k \equiv 1$ OR 5 (mod 6)

In this case we construct the PTs in two phases. In Phase 1, we construct $3k$ optimal PTs similarly to the construction for $n = 3k+1$ and then, in Phase 2, we exchange some of the triples which are not included in the first $3k$ PTs with some triples which are in the PTs. This is done to obtain $3k$ optimal PTs, one PT of size $3k$, and one PT of size k. The definitions of F and $A^{(i)}$ remain as in Section 3. Let $ST_i(k+2)$, $0 \leqslant i \leqslant k-1$, $k \equiv 1$ or 5 (mod 6), $k > 5$, be the STS, with the block $\{i, k, k+1\}$, from a set of k pairwise disjoint STSs of order $k+2$. Phase 1 is very similar to the construction of Section 3.

Phase 1

S_0 has the blocks

$$\{(j, 0), (m, 1), (m+j, 2)\},$$

$$\{(j, t), (m, t), (l, t)\}, \qquad t = 0, 1, \qquad \{j, m, l\} \in ST_1(k+2),$$

$$\{(j, 2), (m, 2), (l, 2)\}, \qquad \{j, m, l\} \in ST_0(k+2),$$

$$\{(j, t), (m, t), 3k\}, \qquad t = 0, 1, \qquad \{j, m, k\} \in ST_1(k+2),$$

$$\{(j, 2), (m, 2), 3k\}, \qquad \{j, m, k\} \in ST_0(k+2),$$

$$\{(j, t), (m, t), 3k+1\}, \qquad t = 0, 1, \qquad \{j, m, k+1\} \in ST_1(k+2),$$

$$\{(j, 2), (m, 2), 3k+1\}, \qquad \{j, m, k+1\} \in ST_0(k+2),$$

$$\{(1, 1), 3k, 3k+1\},$$

where $0 \leqslant j, m, l \leqslant k-1$, and the subscripts are taken modulo k.

Each set S_i, $1 \leqslant i \leqslant k-2$, has the blocks

$$\{(j, 0), (m, 1), (m+j+3i, 2)\},$$

$$\{(j, t), (m, t), (l, t)\}, \qquad t=0, 1, \qquad \{j, m, l\} \in ST_{i+2}(k+2),$$

$$\{(j, 2), (m, 2), (l, 2)\}, \qquad \{j, m, l\} \in ST_{4(i+1)}(k+2),$$

$$\{(j, t), (m, t), 3k\}, \qquad t=0, 1, \qquad \{j, m, k\} \in ST_{i+2}(k+2),$$

$$\{(j, 2), (m, 2), 3k\}, \qquad \{j, m, k\} \in ST_{4(i+1)}(k+2),$$

$$\{(j, t), (m, t), 3k+1\}, \qquad t=0, 1, \qquad \{j, m, k+1\} \in ST_{i+2}(k+2),$$

$$\{(j, 2), (m, 2), 3k+1\}, \qquad \{j, m, k+1\} \in ST_{4(i+1)}(k+2),$$

$$\{(i+2, 1), 3k, 3k+1\},$$

where $0 \leqslant j, m, l \leqslant k-1$, and the subscripts are taken modulo k.

S_{k-1} has the blocks

$$\{(j, 0), (m, 1), (m+j-3, 2)\},$$

$$\{(j, t), (m, t), (l, t)\}, \qquad t=0, 1, \qquad \{j, m, l\} \in ST_2(k+2),$$

$$\{(j, 2), (m, 2), (l, 2)\}, \qquad \{j, m, l\} \in ST_4(k+2),$$

$$\{(j, t), (m, t), 3k\}, \qquad t=0, 1, \qquad \{j, m, k\} \in ST_2(k+2),$$

$$\{(j, 2), (m, 2), 3k\}, \qquad \{j, m, k\} \in ST_4(k+2),$$

$$\{(j, t), (m, t), 3k+1\}, \qquad t=0, 1, \qquad \{j, m, k+1\} \in ST_2(k+2),$$

$$\{(j, 2), (m, 2), 3k+1\}, \qquad \{j, m, k+1\} \in ST_4(k+2),$$

$$\{(2, 1), 3k, 3k+1\},$$

where $0 \leqslant j, m, l \leqslant k-1$, and the subscripts are taken modulo k. Each set S_i, $0 \leqslant i \leqslant k-1$, has

$$k^2 + 3 \cdot \frac{1}{k}\binom{k}{3} + 2 \cdot 3 \cdot \frac{k-1}{2} + 1 = \frac{3k^2+3k-2}{2} \text{ blocks.}$$

Each set S_{k+i}, $0 \leqslant i \leqslant k-1$, has the blocks

$$\{(j, t), (m, t+1), (l, t+1)\}, \qquad t=0, 1, 2, \ 0 \leqslant j, m, l \leqslant k-1, \ \{m, l\} \in F_r,$$

where (j, t), $(r, t+1)$ are consecutive elements in $A^{(i)}$:

$$\{(j, t), (m, t+1), 3k\}, \quad A_r^{(i)} = (j, t), \quad A_{r+1}^{(i)} = (m, t+1), \quad r \text{ odd.}$$

$$\{(j, t), (m, t+1), 3k+1\}, \quad A_r^{(i)} = (j, t), \quad A_{r+1}^{(i)} = (m, t+1), \quad r \neq 0 \text{ even.}$$

Each set S_{2k+i}, $0 \leqslant i \leqslant k-1$, has the blocks

$$\{(m, t), (l, t), (j, t+1)\}, \qquad \{m, l\} \in F_r$$

where (r, t), $(j, t+1)$ are consecutive elements in $A^{(i)}$:

$$\{(j, t), (m, t+1), 3k\}, A_r^{(i)} = (j, t), A_{r+1}^{(i)} = (m, t+1), r \neq 0 \text{ even.}$$

$$\{(j, t), (m, t+1), 3k+1\}, A_r^{(i)} = (j, t), A_{r+1}^{(i)} = (m, t+1), r \neq 1 \text{ odd, or } r = 0.$$

Each set S_i, $k \leqslant i \leqslant 3k-1$, has

$$3 \cdot \binom{k}{2} + 2 \cdot \frac{3k-1}{2} = \frac{3k^2+3k-2}{2} \text{ blocks.}$$

It is not difficult to see that all the $3k$ sets are $3k$ pairwise disjoint *PT*s. The $4k$ triples of the form $\{(j, 0), (2j, 1), 3k\}$, $\{(2j, 1), (4j, 2), 3k+1\}$, $\{(j, 0), 3k, 3k+1\}$, $\{(j, 2), 3k, 3k+1\}$, $0 \leqslant j \leqslant k-1$ do not appear in one of these $3k$ PTs.

Before describing Phase 2 we discuss some properties of the sequences $A^{(i)}$ and the partition which was generated in Phase 1. The first lemma is an immediate consequence of definition.

LEMMA 2. *The pairs* $A_3^{(i)}$, $A_2^{(i)}$ *have the form* $(i+1, 0)$, $(4i, 2)$, *and the pairs* $A_0^{(i)}$, $A_{3k-1}^{(i)}$ *have the form* $(i, 0)$, $(-1+4i, 2)$, $0 \leqslant i \leqslant k-1$.

It is clear that both sets $\{4i\}$ and $(-1+4i\}$, $0 \leqslant i \leqslant k-1$, are equal to $\{0, 1, ..., k-1\}$. Let $t_i = \{(i+2, 0), (2, 1), (4 \cdot (i+1), 2)\}$, $0 \leqslant i \leqslant k-1$. On can readily verify that

LEMMA 3. *The set* S_i, $0 \leqslant i \leqslant k-1$, *contains the triple* t_i.

The only triples from the set $\{t_i\}_{i=0}^{k-1}$ which cover a pair from the set $\{(j, 0), (2j, 1)\} \cup \{(2j, 1), (4j, 2)\}$, $0 \leqslant j \leqslant k-1$, are $\{(1, 0), (2, 1), (0, 2)\}$ and $\{(2, 0), (2, 1), (4, 2)\}$. The first triple is from S_{k-1} and the second is from S_0. In S_{k-1} we also have the triple $\{(2, 0), (5, 1), (4, 2)\}$ and in S_0 we have also have the triple $\{(1, 0), (k-1, 1), (0, 2)\}$. Finally, note that the pairs $\{(i, 0), 3k\}$ and $\{(4i, 2), 3k+1\}$ are not covered by S_{k+i} and S_{2k+i}, respectively. The pairs $\{(1, 0), 3k\}$ and $\{(0, 2), 3k\}$ are not covered by S_0. The pairs $\{(i+2, 0), 3k\}$ and $\{(4(i+1), 2), 3k\}$ are not convered by S_i, $1 \leqslant i \leqslant k-2$. The pairs $\{(2, 0), 3k\}$ and $\{(4, 2), 3k\}$ are not covered by S_{k-1}. Phase 2 makes use of all these properties of the $3k$ PTs and the sequences $A^{(i)}$.

Phase 2

Let S_i, $0 \leqslant i \leqslant 3k-1$, be the $3k$ PTs from Phase 1:

From S_{k+i}, $0 \leqslant i \leqslant k-1$, remove the triple $\{(i, 0), (-1+4i, 2), 3k+1\}$ and join the triple $\{(i, 0), 3k, 3k+1\}$.

From S_{2k+i}, $0 \leqslant i \leqslant k-1$, remove the triple $\{(i+1, 0), (4i, 2), 3k\}$ and join the triple $\{(4i, 2), 3k, 3k+1\}$.

From S_0 remove the triple $\{(1, 0), (k-1, 1), (0, 2)\}$ and join the triple $\{(1, 0), (0, 2), 3k\}$. From S_i, $1 \leqslant i \leqslant k-2$, remove the triple $\{(i+2, 0), (2, 1), (4(i+1), 2)\}$ and join the triple $\{(i+2, 0), (4(i+1), 2), 3k\}$.

From S_{k-1} remove the triple $\{(2, 0), (5, 1), (4, 2)\}$ and join the triple $\{(2, 0), (4, 2), 3k\}$.

The set S_{3k} has the blocks

$$\{(j, 0), (2j, 1), 3k\}, \qquad 0 \leqslant j \leqslant k-1$$

$$\{(2j, 1), (4j, 2), 3k+1\}, \qquad 0 \leqslant j \leqslant k-1$$

$$\{(j, 0), (2, 1), (4(j-1), 2)\}, \qquad j=0 \text{ or } 3 \leqslant j \leqslant k-1$$

$$\{(2, 0), (5, 1), (4, 2)\}$$

$$\{(1, 0), (k-1, 1), (0, 2)\}.$$

The set S_{3k+1} has the blocks

$$\{(j, 0), (-1+4j, 2), 3k+1\}, \qquad 0 \leqslant j \leqslant k-1.$$

All the $3k+2$ sets defined are PTs and together include all the triples of the set $Z_k \times Z_3 \cup \{3k, 3k+1\}$. Also, it is easy to verify that each two PTs are disjoint. Hence, we have the following theorem.

THEOREM 2. *The S_i's, $0 \leqslant i \leqslant 3k+1$, define a set of $3k+2$ pairwise disjoint PTs of order $3k+2$, $k \equiv 1$ or 5 (mod 6), for which the first $3k$ are optimal, and the last two are of sizes $3k$ and k.*

5. PARTITIONS OF ORDERS $n = 3k+2$, $k \equiv 3$ (mod 12)

For $n = 3k+2$, $k \equiv 3$ (mod 12) we need some definitions on quadruples. A *packing quadruple system* (PQ) of order n is a pair (Q, q), where $Q = \{0, 1, ..., n-1\}$ is a set of n points and q is a collection of 4-element subsets of Q called blocks such that every 3-element subset of Q is a subset of at most one block of q. A PQ is optimal if there is no PQ of the same order with a larger size. A *Steiner quadruple system* (SQS) of order n is a PQ of order n such that every 3-element subset of Q is a subset of exactly one block of q. It is well known [5] that an SQS of order n exists if and

only if $n \equiv 2$ or 4 (mod 6). It is clear that $A(n, 4, 4)$ is equal to the number of blocks in an optimal PQ of order n.

From a partition of order $n = 3k+2$, k odd, we can obtain a partition of order $2n+1$ by generalizing the construction of Rosa [12]. He proved that $d(2v+1) = v+1+d(v)$, where $d(v)$ is the number of disjoint Steiner triple systems of order v. In his proof, for the case $d(v) = v-2$, he used the following combinatorial structures:

(1) A good one-factorization F of K_{v+1}.

(2) A Latin square N^* of order $v+1$ associated with F.

(3) A Latin square L of order v with two specified rows.

(4) A one-factorization H of K_{v+1}.

(5) An SQS of order $v+1$.

(6) $v-2$ pairwise disjoint STSs of order v.

The good one-factorization F, the Latin squares N^* and L, and the one-factorization H exist for all odd v, and hence can be taken also when $v \equiv 5$ (mod 6). The SQS of order $v+1$ was taken to obtain $v+1$ pairwise disjoint PTs of order $v+1$. The blocks of PT number i are constructed by taking, from the quadruples which contain point i, the other three points. If we consider the PT i as taken from a set with v points (remove point i) then the PT is an ST(v). Instead of the SQS we take an optimal PQ of order $v+1$, $v+1 \equiv 0$ (mod 6). Based on the construction of Mills [9], Brouwer [1] proved that $A(v+1, 4, 4) = ((v+1)/4)\, A(v, 4, 3)$ for $v+1 \equiv 0$ (mod 6). This implies that by taking from the quadruples which contain point i, $0 \leqslant i \leqslant v$, in a PQ of order $v+1$, $v+1 \equiv 0$ (mod 6), the other three points, we obtain an optimal PT of order v. Instead of the $v-2$ pairwise disjoint STSs of order v, we take $v-2$ pairwise disjoint optimal PTs of order v obtained in Section 4. By applying the construction of Rosa [12] for $v = 3k+2$, $k \equiv 1$ (mod 6) we obtain $6k+3$ PTs of order $6k+5$ and optimal size and two more PTs. These two PTs are constructed as follows: The PQs of order $3(k+1) = 6r$ were constructed as in Brouwer [1]. The quadruples are from the set $X = I_6 \times Y$, where $I_6 = \{0, 1, 2, 3, 4, 5\}$ and $|Y| = r$. The triples from X which do not appear in the $6k+3$ PTs can form two disjoint PTs, p_1 and p_2:

p_1 has the blocks

$$\{0, 2, 4\} \times \{y\},\ \{0, 3, 5\} \times \{y\},\ \{1, 2, 5\} \times \{y\},\ \{1, 3, 4\} \times \{y\} \quad (y \in Y)$$

p_2 has the blocks

$$\{0, 2, 5\} \times \{y\},\ \{0, 3, 4\} \times \{y\},\ \{1, 2, 4\} \times \{y\},\ \{1, 3, 5\} \times \{y\} \quad (y \in Y).$$

Let S_{3k} and S_{3k+1} be the last two PTs of the partition of order $3k+2$. To p_1 we join the blocks of S_{3k} and to p_2 we join the blocks of S_{3k+1}. Hence the size of p_1 is $4r+3k=5k+2$ and the size of p_2 is $4r+k=3k+2$. Therefore, we have the following theorem.

THEOREM 3. *For $n=3k+2$, $k=2m+1\equiv 3 \pmod{12}$ we obtain $3k+2$ pairwise disjoint PTs of order $3k+2$, for which the first $3k$ are optimal, and the last two are of sizes $5m+2$ and $3m+2$.*

6. A Recursive Construction from Two Known Partitions

In this section we give a construction which obtains a partition of order $qk+i$, $i=1, 2$, from partitions of orders $q+2$ and $k+i$. The construction is similar to the first construction of Lu [6]. Before describing the construction we give one property associated with each STS and a related definition. Let $ST(q+2)$ be any STS of order $q+2$. It is clear that we can order the elements Z_q in cyclic sequences $a_0, a_1, ..., a_{r-1}$ such that $a_i \neq a_j$ for $i \neq j$, and for each i, $0 \leqslant i \leqslant r-1$, $\{a_i, a_{i+1}, q\} \in ST(q+2)$ or $\{a_i, a_{i+1}, q+1\} \in ST(q+2)$, where subscripts taken modulo r. Note, that each element of Z_q appears in exactly one sequence. Let $\text{index}\{j, m, ST(q+2)) = 1$ if $j=a_i$ and $m=a_{i+1}$ in some cyclic sequence, and $\text{index}(j, m, ST(q+2)) = 0$ for all other ordered pairs.

Property A. A set of q pairwise disjoint STSs $ST_i(q+2)$, $0 \leqslant i \leqslant q-1$, has this property if for $j \neq m$, $0 \leqslant j$, $m \leqslant q-1$, $\text{index}(j, m, ST_d(q+2)) = 1$, for some d, $0 \leqslant d \leqslant q-1$, then for each $s \neq d$, $\text{index}(j, m, ST_s(q+2)) = 0$.

In Lu [6] this property is called C'_{ab}. Let $D(A)$ denote the set of orders for which a set of q STSs, of order $q+2$, with Property A exists. Now, we give the construction for a partition of order $qk+1$. The points of the partition are taken from $Z_k \times Z_q \cup \{qk\}$. In this section we assume that for a point (i, j) addition in the first coordinate is taken modulo k and in the second modulo q.

Let $F=\{F_0, F_1, ..., F_{k-1}\}$ be a near-one-factorization of K_k, such that in F_i vertex i is isolated. Let $ST_i(q+1)$, $0 \leqslant i \leqslant q-1$, $q+2 \in D(A)$, be STS number i from a set of q pairwise STSs of order $q+2$. Let $PT_i(k+1)$, $0 \leqslant i \leqslant k-1$, $k \equiv 3 \pmod 6$, $k=3p>8$, be PT number i from a set of k pairwise disjoint optimal PTs of order $k+1$, and let $PT_k(k+1)$ be the last PT in this partition.

Each set S_{i+dk}, $0 \leqslant i \leqslant k-1$, $0 \leqslant d \leqslant q-1$, has the blocks

$$\{(j, r), (m, s), (m+j+i, t)\}, \qquad (r, s, t\} \in ST_d(q+1)$$

$$\{j, r), (m, r), (c+i, s)\}, \qquad \{j, m\} \in F_c, \qquad \{r, s, q\} \in ST_d(q+2),$$

$$\text{index}(r, s, ST_d(q+2)) = 1$$

$$\{(j, r), (m, r), (c+i, s)\}, \quad \{j, m\} \in F_c, \quad \{r, s, q+1\} \in ST_d(q+2),$$
$$\text{index}(r, s, ST_d(q+2)) = 1$$
$$\{(j, r), (j+i, s), qk\}, \quad \{r, s, q\} \in ST_d(q+2), \quad \text{index}(r, s, ST_d(q+2)) = 1$$
$$\{(j, r), (m, r), (l, r)\}, \quad \{r, q, q+1\} \in ST_d(q+2), \quad \{j, m, l\} \in PT_i(k+1)$$
$$\{(j, r), (m, r), qk\}, \quad \{r, q, q+1\} \in ST_d(q+2), \quad \{j, m, k\} \in PT_i(k+1),$$

where $0 \leqslant c, j, m, l \leqslant k-1$, and $0 \leqslant r, s, t \leqslant q-1$.

Each set S_i, $0 \leqslant i \leqslant qk-1$, has

$$\frac{1}{q}\binom{q}{3}k^2 + 2 \cdot \frac{r-1}{2}\binom{k}{2} + \frac{r-1}{2} \cdot k + \frac{3 \cdot (\frac{1}{3}k)^2 - 1}{2}$$
$$= \frac{q^2k^2-3}{6} = \frac{3(qp)^2-1}{2} \text{ blocks.}$$

The last set, S_{qk}, has the $qk/3$ blocks

$$\{(j, r), (m, r), (l, r)\}, \quad 0 \leqslant j, m, l \leqslant k-1, \quad \{j, m, l\} \in PT_k(k+1),$$
$$0 \leqslant r \leqslant q-1,$$
$$\{(j, r), (m, r), (m, r), qk\}, \quad 0 \leqslant j, m \leqslant k-1, \quad \{j, m, k\} \in PT_k(k+1),$$
$$0 \leqslant r \leqslant q-1.$$

All the $qk+1$ sets defined are PTs and together include all the triples of the set $Z_k \times Z_k \cup \{qk\}$. Also it is easy to verify that each two PTs are disjoint. Hence, we have the following theorem.

THEOREM 4. *The S_i's, $0 \leqslant i \leqslant qk$, define a set of $qk+1$ pairwise disjoint PTs of order $qk+1$, $q+2 \in D(A)$, $k \equiv 3 \pmod 6$ for which the first qk are optimal.*

In order to obtain partitions for orders which were not covered in Section 3 we need optimal partitions of order $k+1$, $k \equiv 3 \pmod 6$. An optimal partition of order 10 is known (see [10] and Section 7). Therefore we have

COROLLARY 1. *For $n = 9q+1$, $q+2 \in D(A)$, there exists a partition of order n with $9q+1$ PTs for which the first $9q$ are optimal.*

For orders $n = qk+2$ the definitions of F and $ST_i(q+2)$, $0 \leqslant i \leqslant q-1$, are the same. Let $PT_i(k+2)$, $0 \leqslant i \leqslant k-1$, $k \equiv 3 \pmod 6$, $k = 3p > 8$, be PT number i from a set of k pairwise disjoint optimal PTs of order $k+2$. The points of the partition are taken from $Z_k \times Z_k \cup \{qk, qk+1\}$.

Each set S_{i+dk}, $0 \leqslant i \leqslant k-1$, $0 \leqslant d \leqslant q-1$, has the blocks

$$\{(j, r), (m, s), (m+j+i, t)\}, \qquad \{r, s, t\} \in ST_d(q+2)$$

$$\{(j, r), (m, r), (c+i, s)\}, \qquad \{j, m\} \in F_c, \qquad \{r, s, q\} \in ST_d(q+2),$$
$$\text{index}(r, s, ST_d(q+2)) = 1$$

$$\{(j, r), (m, r), (c+i, s)\}, \qquad \{j, m\} \in F_c, \qquad \{r, s, q+1\} \in ST_d(q+2),$$
$$\text{index}(r, s, ST_d(q+2)) = 1$$

$$\{(j, r), (j+i, s), qk\}, \qquad \{r, s, q\} \in ST_d(q+2), \qquad \text{index}(r, s, ST_d(q+2)) = 1$$

$$\{(j, r), (j+i, s), qk+1\}, \qquad \{r, s, q+1\} \in ST_d(q+2),$$
$$\text{index}(r, s, ST_d(q+2)) = 1$$

$$\{(j, r), (m, r), (l, r)\}, \quad \{r, q, q+1\} \in ST_d(q+2), \quad \{j, m, l\} \in PT_i(k+2)$$

$$\{(j, r), (m, r), qk\}, \qquad \{r, q, q+1\} \in ST_d(q+2), \qquad \{j, m, k\} \in PT_i(k+2)$$

$$\{(j, r), (m, r), qk+1\}, \ \{r, q, q+1\} \in ST_d(q+2), \ \{j, m, k+1\} \in PT_i(k+2)$$

$$\{(j, r), qk, qk+1\}, \quad \{r, q, q+1\} \in ST_d(q+2), \quad \{j, k, k+1\} \in PT_i(k+2),$$

where $0 \leqslant c, j, m, l \leqslant k-1$ and $0 \leqslant r, s, t \leqslant q-1$.

Each set S_i, $0 \leqslant i \leqslant qk-1$, has

$$\frac{q^2k^2-3}{6} + \frac{q-1}{2} \cdot k + \frac{k-1}{2} = \frac{q^2k^2+3qk-6}{6} = \frac{3(qp)^2+3qp-2}{2} \text{ blocks.}$$

Let $PT_{k+i}(k+2)$, $i=0, 1$, be the last two PTs of the partition of order $k+2$. Note that, since not all the pairs can be covered by these two PTs, we can assume that $\{j, k, k+1\} \notin PT_{k+i}(k+2)$, $i=0, 1$, $0 \leqslant j \leqslant k-1$.

S_{qk+i}, $i=0, 1$ has the blocks

$$\{(j, r), (m, r), (l, r)\}, \qquad 0 \leqslant j, m, l \leqslant k-1, \qquad \{j, m, l\} \in PT_{k+i}(k+2),$$
$$0 \leqslant r \leqslant q-1,$$

$$\{(j, r), (m, r), qk\}, \quad 0 \leqslant j, m \leqslant k-1, \quad \{j, m, k\} \in PT_{k+i}(k+2),$$
$$0 \leqslant r \leqslant q-1,$$

$$\{(j, r), (m, r), qk+1\} \quad 0 \leqslant j, m \leqslant k-1, \quad \{j, m, k+1\} \in PT_{k+i}(k+2),$$
$$0 \leqslant r \leqslant q-1,$$

If $PT_k(k+2)$ has n_0 blocks and $PT_{k+1}(k+2)$ has n_1 blocks then S_{qk} and S_{qk+1} have $q \cdot n_0$ and $q \cdot n_1$ blocks, respectively.

All the $qk+2$ sets defined are PTs and together include all the triples of

the set $Z_k \times Z_q \cup \{qk, qk+1\}$. Also it is easy to verify that each two PTs are disjoint. Hence, we have the following theorem.

THEOREM 5. *The S_i's, $0 \leqslant i \leqslant qk+1$, define a set of $qk+2$ pairwise disjoint PTs of order $qk+2$, $q+2 \in D(A)$, $k \equiv 3 \pmod 6$ for which the first qk are optimal.*

THEOREM 6. *If there exists a partition of order $k+2$, k odd, with $k+1$ PTs then there exists a partition of order $qk+2$, $q+2 \in D(A)$, with $qk+1$ PTs.*

Proof. Follows immediately from the construction and the fact that in each PT there is at least one uncovered pair. Q.E.D.

COROLLARY 2. *If there exists a partition of order $k+2$, $k \equiv 3 \pmod 6$, $k > 3$, with $k+1$ PTs, k of them with optimal size, then there exists a partition of order $qk+2$, $q+2 \in D(A)$, with $qk+1$ PTs, qk of them with optimal size.*

Brouwer *et al.* [2] found a partition of order 11 with 10 PTs. Hence we have

COROLLARY 3. *For $n = 9q+2$, $q+2 \in D(A)$, there exists a partition of order n with $9q+1$ PTs.*

Finally, we show that there are infinitely many values in $D(A)$. First, note that with the same construction one can obtain $(v-2)(w-2)$ pairwise disjoint STSs of order $(v-2)(w-2)+2$, where $v \in D(A)$, $w \equiv 1$ or $3 \pmod 6$ and there exist a set of $w-2$ pairwise disjoint STSs of order w. Moreover, we have the following result.

THEOREM 7 [15]. *If $q+2 \in D(A)$ and $k+2 \in D(A)$ then $qk+2 \in D(A)$, where q and k are not necessarily distinct.*

The set of $n-2$ pairwise disjoint STSs of order n constructed by Schreiber [13] and Wilson [19] has Property A. Hence, we have the following theorem.

THEOREM 8 [13, 19]. *If p is prime such that the order of -2 modulo p is congruent to 2 modulo 4 then $p+2 \in D(A)$.*

The condition of Theorem 8 is satisfied for all primes of the form $8t-1$ and "roughly" $\frac{1}{6}$ of those of the form $8t+1$ [14]. Lu [6] found that also 19. $21 \in D(A)$, and Schreiber [15] found the $15 \in D(A)$ by using the 13 pairwise disjoint STSs of Denniston [4].

7. Partitions of Orders $n=10$, $n=11$, $n=16$, and $n=17$

Since there is no $\mathcal{S}T(n)$ for $n \equiv 5 \pmod 6$ and there is no set of five pairwise STSs of order 7, the constructions in Sections 3 and 4, do not apply for $n=16$, 17 and for $n=3k+1$, $3k+2$, where $k \equiv 3 \pmod 6$. For $n=10$ an optimal partition is given in [10]. In this section we give a different optimal partition for $n=10$ and $n=16$, and a partition with $n-2$ disjoint optimal PTs for $n=11$ and $n=17$. The method used here might be good to obtain partitions for $n=3k+1$, $3k+2$, $k \equiv 3 \pmod 6$. Also it might be good to improve the partitions for $n=3k+2$, $k \equiv 1$ or $5 \pmod 6$. This method is a generalization of the construction used by Denniston [3] to obtain pairwise disjoint STSs. First, we find a PT with $(\binom{3k}{3}-k)/3k$ triples, one from each orbit of length $3k$. For $3k=9$ we take the following triples:

$$\{0, 1, 2\},\ \{0, 5, 7\},\ \{0, 6, 8\},\ \{1, 3, 7\},\ \{1, 5, 6\},$$
$$\{2, 3, 8\},\ \{2, 4, 7\},\ \{3, 4, 6\},\ \{4, 5, 8\}.$$

The pairs which are not covered by these triples are any two consecutive elements from the cyclic sequence 3, 5, 2, 6, 7, 8, 1, 4, 0. Denote the elements of this sequence by $a_0, a_1, ..., a_8$ and let $b_i = \min\{a_i - a_{i-1} \pmod 9, a_{i-1} - a_i \pmod 9\}$, $i=1, ..., 8$. Note, that $\{b_{2i}\}_{i=1}^4 = \{b_{2i-1}\}_{i=1}^4 = \{1, 2, 3, 4\}$. Hence, by taking the triples $\{3, 5, A\}$, $\{2, 6, A\}$, $\{7, 8, A\}$, $\{1, 4, A\}$, $\{5, 2, B\}$, $\{6, 7, B\}$, $\{8, 1, B\}$, $\{4, 0, B\}$ we obtain a PT of order 11 and optimal size 17. With eight cyclic shift on the points $\{0, 1, ..., 8\}$ we obtain nine disjoint PTs of size 17. By taking the blocks which do not contain point B and adding a PT with the blocks $\{0, 3, 6\}$, $\{1, 4, 7\}$, $\{2, 5, 8\}$, we obtain 10 disjoint PTs of order 10, 9 with optimal size 13, and one with size 3. On the nine PTs of size 17 and the remaining 12 triples $\{i, A, B\}$, $0 \leqslant i \leqslant 8$, $\{i, i+3, i+6\}$, $i=0, 1, 2$, we apply a technique similar to Phase 2 of Section 4. The reader can find out that we can obtain 11 disjoint PTs, nine of size 17, one of size 10, and one of size 2.

A similar construction works for $n=16$ and $n=17$. From each orbit of length 15 we have one representative in each PT. For the first PT we have the 30 triples:

$$\{0, 1, 2\},\ \{0, 3, 6\},\ \{0, 5, E\},\ \{0, 8, 9\},\ \{0, A, B\},\ \{0, C, D\},$$
$$\{1, 3, E\},\ \{1, 4, B\},\ \{1, 5, 9\},\ \{1, 6, C\},\ \{1, 7, D\},\ \{2, 3, A\},$$
$$\{2, 4, C\},\ \{2, 6, 8\},\ \{2, 9, B\},\ \{2, D, E\},\ \{3, 5, C\},\ \{3, 7, 8\},$$
$$\{3, B, D\},\ \{4, 5, D\},\ \{4, 6, 7\},\ \{4, 8, E\},\ \{4, 9, A\},\ \{5, 7, B\},$$
$$\{5, 8, A\},\ \{6, 9, E\},\ \{6, A, D\},\ \{7, 9, C\},\ \{7, A, E\},\ \{8, B, C\}.$$

By adding the triples

$$\{0, 4, F\}, \{0, 7, G\}, \{1, 8, F\}, \{1, A, G\}, \{2, 5, G\}, \{2, 7, F\}, \{3, 4, G\},$$
$$\{3, 9, F\}, \{5, 6, F\}, \{6, B, G\}, \{9, D, G\}, \{A, C, F\}, \{B, E, F\}, \{C, E, G\},$$

we obtain an optimal PT of size 44. Fourteen cyclic shifts will obtain 15 pairwise disjoint optimal PTs of size 44 for $n = 17$ and 15 pairwise disjoint optimal PTs of size 37 for $n = 16$. By adding the PT with the five block of the form $\{i, 5+i, 10+i\}$, $i = 0, 1, 2, 3, 4$, we obtain an optimal partition of order 16. A method similar to the one for $n = 11$ yields 15 PTs of size 44, one of size 17, and one of size 3, for $n = 17$. This partition was found using computer search which was done by Doron Cohen.

8. Conclusion

We gave a construction of optimal partitions for triples of orders $n = 3k+1$, $k \equiv 1$ or 5 (mod 6), $k > 5$. For $n = 3k+2$, $k \equiv 1$ or 5 (mod 6), $k > 5$, we obtain $3k$ PTs of optimal size, one of size $3k$, and one of size k. Another construction for $n = 3k+2$, $k \equiv 3$ (mod 12) obtains $3k$ PTs of optimal size, one PT of size $5m+2$, and one PT of size $3m+2$, where $k = 2m+1$. We gave a construction which gets a partition of order $n = qk+i$, $i = 1, 2$, $q+2 \in D(A)$, $k \equiv 3$ (mod 6), with qk disjoint optimal codes, if a partition of order $k+i$ with k optimal codes exists. We also proved that for $n = 9q+2$, $q+2 \in D(A)$, there exists a partition with $9q+1$ PTs.

We end this paper with some questions and conjectures:

(1) Does there exist a partition with $3k+1$ PTs for every $n = 3k+2$, k odd? We conjecture Yes.

(2) Does there exist a partition with $3k$ optimal PTs and one PT of size $4k$ for some $n = 3k+2$, k odd?

(3) Does there exist a partition with $3k$ optimal PTs and one PT of size k for every $n = 3k+1$, k odd? We conjecture Yes.

(4) Can you make exchanges of triples in the partitions of orders $n = 3k+2$, $k \equiv 3$ (mod 12) with $3k$ optimal PTs, for which we did not get an optimal partition of order $3k+1$, in such a way that by taking the triples which do not contain a certain point, say $3k+1$, an optimal partition of order $3k+1$ is obtained?

(5) Find more values which belong to $D(A)$.

(6) Find a method to construct a PT of order $3k$, such that from each orbit of length $3k$ there is one element in the PT. Can you find a PT like this for which the construction of Section 7 works?

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