ROW-COMPLETE LATIN SQUARES WHICH ARE NOT COLUMN-COMPLETE

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Abstract. We give a construction of a row-complete latin square, which cannot be made column-complete by a suitable permutation of its rows, for every even order greater than 8.

Introduction.

A latin square of order n is an $n \times n$ array in which each row and each column is a permutation of the same n symbols, say $0, 1, \ldots, n-1$. A *row-complete latin square* (RCLS) of order n is a latin square of order n in which the $n(n-1)$ pairs of horizontally adjacent entries in the array are all distinct. Application of RCLSs in experimental designs and frequency hopping pattern are given in [5], [7]. A *column-complete latin square* of order n is a latin square of order n in which each of the $n(n-1)$ pairs of vertically adjacent entries in the array are all distinct. A *complete latin square* (CLS) is an RCLS which is also column-complete. The reader can find more material on RCLSs in [2]. A *quasi-complete latin square* (QCLS) is an RCLS which cannot be made column-complete by a suitable permutation of its rows. Owens [6] proved that there exist QCLSs of order 10 and 14. Archdeacon *et al* [1] showed a QCLS of order 9. In this paper we prove that there exist QCLSs for all even orders greater than 8. Our QCLSs are based on RCLSs obtained from patterns called polygonal paths [3], [4]. We give three constructions, the first for orders $n \equiv 2 \pmod 4$, the second for orders $n \equiv 4 \pmod 8$, and the third for orders $n \equiv 0 \pmod 8$.

In Section 2 we present the concept of polygonal path. We describe the properties of the RCLSs obtained from symmetric polygonal paths. Those RCLSs have to be modified in order to obtain QCLSs, and hence we present the nature of our modification. In Section 3 we give the specific constructions for the QCLSs.

The structure of the QCLS.

A *polygonal path* X of order n is a permutation $x_0, x_1, \ldots, x_{n-1}$ of Z_n. If all the differences $x_{i+1} - x_i$, $0 \leq i \leq n-2$ are distinct modulo n, it implies [3] that n is even and the latin square $A = ||a_{i,j}||$ defined by $a_{i,j} \equiv x_j + i \pmod n$, $i, j \in Z_n$, is an RCLS. Note, that arranging the rows of A, so that the first row is symmetric with the first column, yields a CLS.

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A polygonal path X, of order $2m$, is *symmetric* if $x_{2m-1-i} \equiv x_i + m \pmod{2m}$, $i \in Z_{2m}$. Symmetric polygonal paths are used in [4]. Throughout this paper *the elements of our polygonal paths and latin squares are taken modulo* $2m$, and also *the subscripts of the polygonal paths, rows and columns of the latin squares are taken modulo* $2m$.

A latin square defined from a symmetric polygonal path has a nice property which we call α-symmetry. A latin square $A = ||a_{i,j}||$ has α-symmetry if $a_{i,2m-1-j} = a_{i+m,j} = a_{i,j} + m$, $i, j \in Z_{2m}$. This means that A can be represented by one of its quarters, say the upper left quarter, as depicted in the following example.

Example 1:

A latin square A with α-symmetry

0	2	1	4	5	3
1	3	2	5	0	4
2	4	3	0	1	5
3	5	4	1	2	0
4	0	5	2	3	1
5	1	0	3	4	2

The upper left quarter of A

0	2	1
1	3	2
2	4	3

Owens [6] used properties of α-symmetric RCLSs to obtain QCLSs of orders 10 and 14. Our method is a modification and generalization of Owens method.

Given RCLS $A = ||a_{i,j}||$ of order $2m$, obtained by a symmetric polygonal path, we change some of the entries of A. The *number of changes* in the first half of row i, is denoted by $nc(i)$, $0 \le nc(i) \le m-1$. Now, we obtain a new square $B = ||b_{i,j}||$ as follows. If $nc(\ell) = s$ then $b_{\ell,r} = a_{\ell,r} + m$ for $r \le s-1$ and for $r \ge 2m - s$, and $b_{\ell,r} = a_{\ell,r}$ for $s \le r \le 2m-s-1$. This is the symmetry of the operation induced by nc. We call *A the Initial square* and *B the Induced square*, since B is induced by A and the function nc. By the α-symmetry of A, to make sure that B is a latin square we must have $nc(m+i) = nc(i)$, $i \in Z_{2m}$. This, the symmetry of the operation induced by nc, and the α-symmetry of A imply that also B is α-symmetric. In the sequel, *the parameters of nc are also taken modulo* $2m$. Since $nc(m+i) = nc(i)$, $(b_{i,j}, b_{i,j+1}) = (a_{i,j} + m, a_{i,j+1} + m)$ iff $(b_{i+m,j}, b_{i+m,j+1}) = (a_{i,j}, a_{i,j+1})$. Therefore, by the α-symmetry, to make sure that B is an RCLS we require that if $(b_{i,j}, b_{i,j+1}) = (a_{i,j} + m, a_{i,j+1})$, that is, $nc(i) = j+1$, then for some i', j', $(b_{i',j'}, b_{i',j'+1}) = (a_{i,j}, a_{i,j+1})$. For this purpose our RCLSs will have the property that for some i', j', $j' \le m-2$, $(a_{i',j'}, a_{i',j'+1}) = (a_{i,j} + m, a_{i,j+1})$ and we will set $nc(i') = j'+1$ to obtain $(b_{i',j'}, b_{i',j'+1}) = (a_{i,j}, a_{i,j+1})$.

Now, we have to determine the values of the function nc in such a way that any permutation on the rows of B will not produce a CLS. Note, that if rows ℓ and $\ell + k$ of B are juxtaposed, any ordered vertical pair which is produced is of the form $\langle i,\ i+k\rangle$, or $\langle j,\ j+k+m\rangle$. Let $V(k)$ denote the set of all ordered pairs $\langle i,\ i+k\rangle$, where $i \in Z_{2m}$, $k \in Z_{2m} \setminus \{0\}$, and $i+k$ is taken modulo $2m$. To produce a CLS, that is, to obtain all the $2m(2m-1)$ possible ordered pairs vertically, each element of $V(k)$ for $k \in Z_{2m} \setminus \{0\}$ must appear once as adjacent vertical pair in the permutation on the rows of B . This implies that in order to produce $V(k) \cup V(k+m)$, $k \neq m$, we have to place two couples of rows, i_2 below i_1 and j_2 below j_1, such that $i_2 - i_1 \equiv j_2 - j_1 \equiv k \pmod{m}$. Note that

$$\text{either}\quad i_2 - i_1 \equiv j_2 - j_1 \pmod{2m}$$

$$\text{or}\quad i_2 - i_1 \equiv j_2 - j_1 + m \pmod{2m}$$

A vertically ordered pair $\langle b_{i,r}, b_{j,r}\rangle$ for $i \neq j$, $i, j, r \in Z_{2m}$ is said to be *consistent* if $\langle b_{i,r}, b_{j,r}\rangle$ is either $\langle a_{i,r}, a_{j,r}\rangle$ or $\langle a_{i,r}+m, a_{j,r}+m\rangle$. The pair is *inconsistent* if $\langle b_{i,r}, b_{j,r}\rangle$ is either $\langle a_{i,r}+m, a_{j,r}\rangle$ or $\langle a_{i,r}, a_{j,r}+m\rangle$. Note that if $\langle a_{i,r}, a_{j,r}\rangle \in V(k)$ then $\langle b_{i,r}, b_{j,r}\rangle$ belongs to $V(k)$ if it is consistent and to $V(k+m)$ if it is inconsistent.

Constructions for QCLSs.

A construction for orders $2m$, m odd

For $2m \equiv 2 \pmod{4}, 2m \geq 10$, we use the symmetric polygonal path of Williams [7]

$$0, 2m-1, 1, 2m-2, 2, \ldots$$

to obtain the Initial square A. In other words, for $i \in Z_{2m}$

$$x_i = \begin{cases} \frac{i}{2} & i \text{ even} \\ 2m - \frac{i+1}{2} & i \text{ odd} \end{cases}$$

For $0 \leq i \leq \frac{m-5}{2}$, let $nc(i) = 1$ and $nc(\frac{m-1}{2}+i) = m-1$. Also let $nc(\frac{m-3}{2}) = 2$, $nc(m-2) = 0$, $nc(m-1) = m-2$, and $nc(m+i) = nc(i)$ for $0 \leq i \leq m-1$. We claim that the latin square B induced by A and the function nc is a QCLS.

Example 2: The construction for $2m = 10$ yields

i	the Initial square A:										$nc(i)$	the Induced square B:									
0	0	9	1	8	2	7	3	6	4	5	1	5	9	1	8	2	7	3	6	4	0
1	1	0	2	9	3	8	4	7	5	6	2	6	5	2	9	3	8	4	7	0	1
2	2	1	3	0	4	9	5	8	6	7	4	7	6	8	5	4	9	0	3	1	2
3	3	2	4	1	5	0	6	9	7	8	0	3	2	4	1	5	0	6	9	7	8
4	4	3	5	2	6	1	7	0	8	9	3	9	8	0	2	6	1	7	5	3	4
5	5	4	6	3	7	2	8	1	9	0	1	0	4	6	3	7	2	8	1	9	5
6	6	5	7	4	8	3	9	2	0	1	2	1	0	7	4	8	3	9	2	5	6
7	7	6	8	5	9	4	0	3	1	2	4	2	1	3	0	9	4	5	8	6	7
8	8	7	9	6	0	5	1	4	2	3	0	8	7	9	6	0	5	1	4	2	3
9	9	8	0	7	1	6	2	5	3	4	3	4	3	5	7	1	6	2	0	8	9

Theorem 1. *The latin square B induced by A and nc, that is, $b_{i,j} = a_{i,j} + m$ for $j \leq nc(i) - 1$ and for $j \geq 2m - nc(i)$, and $b_{i,j} = a_{i,j}$ otherwise, is a QCLS.*

Proof: First, we prove that B is an RCLS, by showing that all the horizontal pairs in A appear as horizontal pairs in B. Given a pair $(a_{i,j}, a_{i,j+1})$, we distinguish between four cases:

(1) $b_{i,j} = a_{i,j}$ and $b_{i,j+1} = a_{i,j+1}$ and hence the pair appears in B.

(2) $b_{i,j} = a_{i,j} + m$ and $b_{i,j+1} = a_{i,j+1} + m$. Since $a_{m+i,j} = a_{i,j} + m$, $a_{m+i,j+1} = a_{i,j+1} + m$, and $nc(m+i) = nc(i)$, it follows that $b_{m+i,j} = a_{i,j}$ and $b_{m+i,j+1} = a_{i,j+1}$ and hence the pair appears in B.

(3) $b_{i,j} = a_{i,j} + m$ and $b_{i,j+1} = a_{i,j+1}$. By the definition of nc, $j = 0, 1, m-3$, or $m-2$. For $j = 0$, $a_{i,j} = i$ and $a_{i,j+1} = i - 1$. Since $j = 0$ means $nc(i) = 1$ it follows that $nc(i + \frac{3m-1}{2}) = m - 1$ and therefore $b_{i+\frac{3m-1}{2}, m-2} = a_{i+\frac{3m-1}{2}, m-2} + m = i + \frac{3m-1}{2} + 2m - \frac{(m-2)+1}{2} + m = i = a_{i,j}$, and $b_{i+\frac{3m-1}{2}, m-1} = a_{i+\frac{3m-1}{2}, m-1} = i + \frac{3m-1}{2} + \frac{m-1}{2} = i - 1 = a_{i,j+1}$. Hence $(b_{i+\frac{3m-1}{2}, m-2}, b_{i+\frac{3m-1}{2}, m-1}) = (a_{i,j}, a_{i,j+1})$. The case $j = m-2$ is similar. For $j = 1$, $a_{i,j} = i - 1$, and $a_{i,j+1} = i + 1$. Since $j = 1$ means $nc(i) = 2$, that is, $i \equiv \frac{m-3}{2} \pmod{m}$, it follows that $nc(i + \frac{m+1}{2}) = m - 2$ and therefore $b_{i+\frac{m+1}{2}, m-3} = a_{i+\frac{m+1}{2}, m-3} + m = i + \frac{m+1}{2} + \frac{m-3}{2} + m = i - 1 = a_{i,j}$, and $b_{i+\frac{m+1}{2}, m-2} = a_{i+\frac{m+1}{2}, m-2} = i + \frac{m+1}{2} + 2m - \frac{(m-2)+1}{2} = i + 1 = a_{i,j+1}$. The case $j = m - 3$ is similar. Hence the pair appears in B.

(4) $b_{i,j} = a_{i,j}$ and $b_{i,j+1} = a_{i,j+1} + m$. This case is similar to case 3 since A and B are α-symmetric.

Thus B is an RCLS since A is an RCLS.

Next, to produce $V(k) \cup V(k+m)$, where $k = \frac{m-1}{2}$, we have to place two couples of rows, i_2 below i_1 and j_2 below j_1, such that $i_2 - i_1 \equiv j_2 - j_1 \equiv k \pmod{m}$. We also must satisfy two requirements.

(R.1) If (C.1) holds, we must have exactly $2m$ consistent (also inconsistent) pairs among the pairs $\langle b_{i_1,r}, b_{i_2,r} \rangle$, $\langle b_{j_1,r}, b_{j_2,r} \rangle$. If (C.2) holds, we must have that the number of inconsistent pairs among the pairs $\langle b_{i_1,r}, b_{i_2,r} \rangle$, that is, $2|nc(i_2) - nc(i_1)|$, is the same as among the pairs $\langle b_{j_1,r}, b_{j_2,r} \rangle$.

(R.2) If (C.1) holds, all the consistent pairs form either $V(k)$ or $V(k+m)$. This implies that the set $\{x\colon\ x = b_{i_1,r}$ or $x = b_{j_1,r}$ for consistent pair $\langle b_{i_1,r} b_{i_2,r} \rangle$ or $\langle b_{j_1,r} b_{j_2,r} \rangle\}$ is equal to Z_{2m}. Hence, by the α-symmetry, the symmetry of the operation induced by nc, and the fact that the elements of each row of A is a permutation of Z_{2m} we must have

$$\{x\colon\ x = a_{i_1,r} \text{ for inconsistent pair } \langle b_{i_1,r}, b_{i_2,r} \rangle\} =$$
$$\{x\colon\ x = a_{j_1,r} \text{ for consistent pair } \langle b_{j_1,r}, b_{j_2,r} \rangle\}$$

If (C.2) holds, all the consistent pairs of the form $\langle b_{i_1,r}, b_{i_2,r} \rangle$ together with all the inconsistent pairs of the form $\langle b_{j_1,r}, b_{j_2,r} \rangle$ produce either $V(k)$

or $V(k+m)$. This implies that the set

$$\{x\colon\ x = b_{i_1,r} \text{ for consistent pair } \langle b_{i_1,r}, b_{i_2,r}\rangle \text{ or } x = b_{j_1,r} \text{ for inconsistent pair } \langle b_{j_1,r}, b_{j_2,r}\rangle\ \}$$

is equal to Z_{2m}. Hence, by the α-symmetry, the symmetry of the operation induced by nc, and the fact that each row of A is a permutation of Z_{2m} we must have

$$\{x\colon\ x = a_{i_1,r} \text{ for consistent pair } \langle b_{i_1,r}, b_{i_2,r}\ \} = \{x\colon\ x = a_{j_1,r} \text{ for consistent pair } \langle b_{j_1,r}, b_{j_2,r}\rangle\ \}.$$

Note, that (R.2) has meaning only if (R.1) holds, and also that the discussion on (R.1) and (R.2) is general. We will refer to (R.1) and (R.2) in our other proofs.

If (C.1) holds, that is, $i_2 - i_1 \equiv j_2 - j_1 \pmod{2m}$, then by (R.1) we must have

$$2|nc(i_2) - nc(i_1)| + 2|nc(j_2) - nc(j_1)| = 2m. \tag{1}$$

The ordered pairs $(nc(i_1), nc(i_2))$ and $(nc(j_1), nc(j_2))$ can take the values $(0,1)$, $(1, m-1)$, $(2,0)$, $(m-2,2)$, $(m-1,1)$, and $(m-1, m-2)$. To satisfy (1), w.l.o.g., $\{nc(i_1), nc(i_2)\}$ and $\{nc(j_1), nc(j_2)\}$ must take the values $\{0,2\}$ and $\{1, m-1\}$, respectively. By (R.2) we must have

$$\{a_{i_1,0}, a_{i_1,1}, a_{i_1,2m-2}, a_{i_1,2m-1}\} = \{a_{j_1,0}, a_{j_1,m-1}, a_{j_1,m}, a_{j_1,2m-1}\}$$

which is impossible, by the definition of the polygonal path X.

If (C.2) holds, that is, $i_2 - i_1 \equiv j_2 - j_1 + m \pmod{2m}$, it is clear that

$$i_1 \neq j_1 \text{ and } i_1 \neq j_1 + m \tag{2}$$

and by (R.1) we must have

$$|nc(i_2) - nc(i_1)| = |nc(j_2) - nc(j_1)|. \tag{3}$$

The ordered pairs $(nc(i_1), nc(i_2))$ and $(nc(j_1), nc(j_2))$ can take the values mentioned above. To satisfy (2) and (3) either $\{nc(i_1),\ nc(i_2)\} = \{nc(j_1),\ nc(j_2)\} = \{1, m-1\}$ or $\{(nc(i_1), nc(i_2)),\ (nc(j_1), nc(j_2))\} = \{(0,1),\ (m-1, m-2)\}$. By (R.2), $\{nc(i_1), nc(i_2)\} = \{nc(j_1), nc(j_2)\} = \{1, m-1\}$ implies $\{a_{i_1,0}, a_{i_1,m-1}, a_{i_1,m}, a_{i_1,2m-1}\} = \{a_{j_1,0}, a_{j_1,m-1}, a_{j_1,m}, a_{j_1,2m-1}\}$ which is impossible. For $\{(nc(i_1), nc(i_2)), (nc(j_1), nc(j_2))\} = \{(0,1), (m-1, m-2)\}$, w.l.o.g. assume $nc(i_1) = 0$. Hence, $i_1 \equiv m-2 \pmod{m}$, $j_2 \equiv m-1 \pmod{m}$, which implies $j_1 \equiv j_2 - k \equiv \frac{m-1}{2} \pmod{m}$, but it can be easily verified that for any $p, q \in Z_{2m}$, $p \neq q$,

$$\{a_{p,0}, a_{p,2m-1}\} = \{a_{q,m-2}, a_{q,m+1}\} \text{ iff } q - p \equiv \frac{m-1}{2} \pmod{m},$$

and hence (R.2) cannot be satisfied.

Therefore, we conclude that B is a QCLS. ∎

A construction for orders $2m$, $m \equiv 2 \pmod 4$

For $2m = 8k+4$, $k \geq 1$, we use the polygonal path

$$y_i = \begin{cases} 0 & i = 0 \\ \frac{m-i}{2} & i = 2j, \quad 1 \leq j \leq k \\ \frac{3m+i}{2} & i = 2j, \quad k+1 \leq j \leq 2k \\ \frac{3m+i-1}{2} & i = 2j+1,\ 0 \leq j \leq k \\ \frac{m-i+1}{2} & i = 2j+1,\ k+1 \leq j \leq 2k \end{cases}$$

and $y_{2m-1-i} = y_i + m, 0 \leq i \leq m-1$, to form the Initial square A.

Example 3: For $2m = 20$ we have

$$Y = 0, 15, 4, 16, 3, 17, 18, 2, 19, 1, 11, 9, 12, 8, 7, 13, 6, 14, 5, 10$$

Lemma 1. *Y is a polygonal path for RCLS.*

Proof: First, we prove that Y is a permutation of Z_{2m}. Since

$$\left\{\frac{m-i+1}{2} : i = 2j+1,\ k+1 \leq j \leq 2k\right\} = \{r : 1 \leq r \leq k\},$$

$$\left\{\frac{m-i}{2} : i = 2j,\ 1 \leq j \leq k\right\} = \{r : k+1 \leq r \leq 2k\},$$

$$\left\{\frac{3m+i-1}{2} : i = 2j+1,\ 0 \leq j \leq k\right\} = \{r : 6k+3 \leq r \leq 7k+3\},$$

$$\left\{\frac{3m+i}{2} : i = 2j,\ k+1 \leq j \leq 2k\right\} = \{r : 7k+4 \leq r \leq 8k+3\},$$

and since $y_{2m-1-i} = y_i + m,\ 0 \leq i \leq m-1$, it follows that Y is a permutation of Z_{2m}.

Next, we have to prove that $\{y_{i+1} - y_i :\ 0 \leq i \leq 2m-2\} = Z_{2m} \setminus \{0\}$.

(1) For $i = 0,\ y_{i+1} - y_i = \frac{3m}{2} = 6k+3$.

(2) For $i = 2j+1, 0 \leq j \leq k-1,\ y_{i+1} - y_i = \frac{m-(2j+2)}{2} - \frac{3m+(2j+1)-1}{2} = 4k - 2j + 1$.

(3) For $i = 2j, 1 \leq j \leq k,\ y_{i+1} - y_i = \frac{3m+(2j+1)-1}{2} - \frac{m-2j}{2} = 4k + 2j + 2$.

(4) For $i = 2k+1,\ y_{i+1} - y_i = \frac{3m+2k+2}{2} - \frac{3m+(2k+1)-1}{2} = 1$.

(5) For $i = 2j,\ k+1 \leq j \leq 2k,\ y_{i+1} - y_i = \frac{m-(2j+1)+1}{2} - \frac{3m+2j}{2} = 4k - 2j + 2$.

(6) For $i = 2j+1,\ k+1 \leq j \leq 2k-1,\ y_{i+1} - y_i = \frac{3m+2j+2}{2} - \frac{m-(2j+1)+1}{2} = 4k + 2j + 3$.

(7) For $i = 4k+1,\ y_{i+1} - y_i = 4k+2$.

It is easily verified that the differences defined in (1) through (6) together with their complements ($2m - y$ is the *complement* of y) and $4k + 2$ form the set $Z_{2m} \setminus \{0\}$.

Thus Y is a polygonal path for RCLS. ∎

For $0 \leq i \leq \frac{m-2}{2}$, let $nc(i) = 3$ and $nc(\frac{m}{2} + i) = m - 1$. Also let $nc(m+i) = nc(i)$ for $0 \leq i \leq m-1$.

Example 4: The construction for $2m = 12$ yields the following upper left quarters of the squares A and B

the Initial square A:						the the Induced square B:					
0	9	2	10	11	1	6	3	8	10	11	1
1	10	3	11	0	2	7	4	9	11	0	2
2	11	4	0	1	3	8	5	10	0	1	3
3	0	5	1	2	4	9	6	11	7	8	4
4	1	6	2	3	5	10	7	0	8	9	5
5	2	7	3	4	6	11	8	1	9	10	6

Theorem 2. *The latin square B induced by A and nc is a QCLS.*

Proof: First, the proof that B is an RCLS, is similar to the one of Theorem 1. We only show the non-trivial case of pair $(a_{i,j}, a_{i,j+1})$:

$b_{i,j} = a_{i,j} + m$ and $b_{i,j+1} = a_{i,j+1}$. By the definition of nc, $j = 2$ or $m-2$. For $j = 2$, $a_{i,j} = i + \frac{m-2}{2}$ and $a_{i,j+1} = i + \frac{3m+2}{2}$. Now, since $nc(i + \frac{3m}{2}) = m - 1$ we have that $b_{i+\frac{3m}{2},m-2} = a_{i+\frac{3m}{2},m-2} + m = i + \frac{3m}{2} + \frac{3m+m-2}{2} + m = i + \frac{m-2}{2} = a_{i,j}$ and $b_{i+\frac{3m}{2},m-1} = a_{i+\frac{3m}{2},m-1} = i + \frac{3m}{2} + \frac{m-(m-1)+1}{2} = i + \frac{3m+2}{2} = a_{i,j+1}$. The case $j = m-2$ is handled similarly.

Thus, B is an RCLS.

Next, to produce $V(\frac{m}{2}) \cup V(\frac{3m}{2})$ we have to place two couples of rows, i_2 below i_1 and j_2 below j_1, such that $i_2 - i_1 \equiv j_2 - j_1 \equiv \frac{m}{2} \pmod{m}$.

If (C.1) holds, then by (R.1), $2|nc(i_2) - nc(i_1)| + 2|nc(j_2) - nc(j_1)| = 2m$. But, since $|nc(i_2) - nc(i_1)| = |nc(j_2) - nc(j_1)| = m - 4$ we have that $4(m-4) = 2m$, implying $2m = 16 \not\equiv 4 \pmod{8}$.

If (C.2) holds, it is clear that $i_1 \neq j_1$ and $i_1 \neq j_1 + m$, and by (R.1), $|nc(i_2) - nc(i_1)| = |nc(j_2) - nc(j_1)|$. This implies that $\{nc(i_1), nc(i_2)\} = \{nc(j_1), nc(j_2)\} = \{3, m-1\}$, and (R.2) implies that

$$\{a_{i_1,0}, a_{i_1,1}, a_{i_1,2}, a_{i_1,m-1}, a_{i_1,m}, a_{i_1,2m-3}, a_{i_1,2m-2}, a_{i_1,2m-1}\} =$$
$$\{a_{j_1,0}, a_{j_1,1}, a_{j_1,2}, a_{j_1,m-1}, a_{j_1,m}, a_{j_1,2m-3}, a_{j_1,2m-2}, a_{j_1,2m-1}\}.$$

It is easy to verify that this is impossible by the definition of the polygonal path Y.

Therefore, B is a QCLS. ∎

A construction for orders $2m,\ m \equiv 0 \pmod 4$

For $2m = 8k,\ k \geq 2$, we use the polygonal path

$$z_i = \begin{cases} i & i = 0,\ 1 \\ 2m-3 & i = 2 \\ 2m-1 & i = 3 \\ \frac{i}{2} & i = 2j, \quad 2 \leq j \leq 2k-1 \\ 2m-2-\frac{i+1}{2} & i = 4j+1, 1 \leq j \leq k-2 \\ 2m+2-\frac{i+1}{2} & i = 4j+3, 1 \leq j \leq k-1 \\ \frac{3m}{2} & i = m-3 \end{cases}$$

and $z_{2m-1-i} = z_i + m, 0 \leq i \leq m-1$, to form the Initial square A.

Example 5: For $2m = 24$ we have

$$Z = 0,1,21,23,2,19,3,22,4,18,5,20,8,$$
$$17,6,16,10,15,7,14,11,9,13,12$$

The proof that Z is a polygonal path for RCLS is similar to the proof that Y is a polygonal path for RCLS and hence it is omitted.

For $0 \leq i \leq \frac{m-2}{2}$, let $nc(i) = 4$ and $nc(\frac{m}{2}+i) = m-1$. Also let $nc(m+i) = nc(i)$ for $0 \leq i \leq m-1$.

Theorem 3. *The latin square B induced by A and nc is a QCLS.*

Proof: First, the proof that B is an RCLS, is similar to the one of Theorem 1. We only show the non-trivial case of pair $(a_{i,j}, a_{i,j+1})$:

$b_{i,j} = a_{i,j} + m$ and $b_{i,j+1} = a_{i,j+1}$. By the definition of nc, $j = 3$ or $m-2$. For $j = 3, a_{i,j} = i-1$ and $a_{i,j+1} = i+2$. Now, since $nc(i+\frac{m}{2}) = m-1$ we have that $b_{i+\frac{m}{2},m-2} = a_{i+\frac{m}{2},m-2} + m = i + \frac{m}{2} + \frac{m}{2} - 1 + m = i - 1 = a_{i,j}$ and $b_{i+\frac{m}{2},m-1} = a_{i+\frac{m}{2},m-1} = i + \frac{m}{2} + 2m + 2 - \frac{m}{2} = i+2 = a_{i,j+1}$. The case $j = m-2$ is handled similarly.

Thus, B is an RCLS.

Next, to produce $V(\frac{m}{2}) \cup V(\frac{3m}{2})$ we have to place two couples of rows, i_2 below i_1 and j_2 below j_1, such that $i_2 - i_1 \equiv j_2 - j_1 \equiv \frac{m}{2} \pmod m$.

If (C.1) holds then by (R.1), $2|nc(i_2) - nc(i_1)| + 2|nc(j_2) - nc(j_1)| = 2m$. But, since $|nc(i_2) - nc(i_1)| = |nc(j_2) - nc(j_1)| = m-5$ we have that $4(m-5) = 2m$, implying $2m = 20 \not\equiv 0 \pmod 8$.

If (C.2) holds, it is clear that $i_1 \neq j_1$ *and* $i_1 \neq j_1 + m$, and by (R.1), $|nc(i_2) - nc(i_1)| = |nc(j_2) - nc(j_1)|$. This implies that $\{nc(i_1), nc(i_2)\} = \{nc(j_1), nc(j_2)\} = \{4, m-1\}$, and by (R.2)

$$\{a_{i_1,0}, a_{i_1,1}, a_{i_1,2}, a_{i_1,3}, a_{i_1,m-1}, a_{i_1,m}, a_{i_1,2m-4}, a_{i_1,2m-3}, a_{i_1,2m-2}, a_{i_1,2m-1}\} =$$
$$\{a_{j_1,0}, a_{j_1,1}, a_{j_1,2}, a_{j_1,3}, a_{j_1,m-1}, a_{j_1,m}, a_{j_1,2m-4}, a_{j_1,2m-3}, a_{j_1,2m-2}, a_{j_1,2m-1}\}$$

It is easy to verify that this is impossible by the definition of the polygonal path Z.

Therefore, B is a QCLS. ∎

Conclusion.

We gave constructions for row-complete latin squares of order n, which cannot be made column complete by a suitable permutation of their rows, for all even orders greater than 8.

The only known RCLS of odd order which cannot be made column complete is of order 9 [1]. It is still an open problem for which odd orders greater than 9 there exist RCLSs. For those odd orders for which RCLSs exist, it is interesting to find RCLSs which cannot be made column-complete.

Latin squares that cannot be made neither row-complete nor column-complete are relatively easily constructed. This is left as an exercise to interested readers.

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