

Covering of subspaces by subspaces

Tuvi Etzion

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Abstract Lower and upper bounds on the size of a covering of subspaces in the Grassmann graph $\mathcal{G}_q(n, r)$ by subspaces from the Grassmann graph $\mathcal{G}_q(n, k)$, $k \geq r$, are discussed. The problem is of interest from four points of view: coding theory, combinatorial designs, q -analogs, and projective geometry. In particular we examine coverings based on lifted maximum rank distance codes, combined with spreads and a recursive construction. New constructions are given for $q = 2$ with $r = 2$ or $r = 3$. We discuss the density for some of these coverings. Tables for the best known coverings, for $q = 2$ and $5 \leq n \leq 10$, are presented. We present some questions concerning possible constructions of new coverings of smaller size.

Keywords Covering designs · Lifted MRD codes · Projective geometry · q -analog · Spreads · Subspace transversal design

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1 Introduction

Let \mathbb{F}_q be a finite field with q elements. For given integers $n \geq k \geq 0$, let $\mathcal{G}_q(n, k)$ denote the set of all k -dimensional subspaces of \mathbb{F}_q^n . $\mathcal{G}_q(n, k)$ is often referred to as Grassmannian. It is well known that

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Tuvi Etzion (✉)
Department of Computer Science, Technion, 32000 Haifa, Israel
e-mail: etzion@cs.technion.ac.il

$$|\mathcal{G}_q(n, k)| = \begin{bmatrix} n \\ k \end{bmatrix}_q \stackrel{\text{def}}{=} \frac{(q^n - 1)(q^{n-1} - 1) \cdots (q^{n-k+1} - 1)}{(q^k - 1)(q^{k-1} - 1) \cdots (q - 1)}$$

where $\begin{bmatrix} n \\ k \end{bmatrix}_q$ is the q -ary Gaussian coefficient [32].

A code \mathbb{C} over the Grassmannian is a subset of $\mathcal{G}_q(n, k)$. In recent years there has been an increasing interest in codes over the Grassmannian as a result of their application to error-correction in random network coding as was demonstrated by Koetter and Kschischang [19]. But, the interest in these codes has been also before this application, since these codes are q -analogs of constant weight codes. The well-known concept of q -analogs replaces subsets by subspaces of a vector space over a finite field and their orders by the dimensions of the subspaces. In particular, the q -analog of a constant weight code in the Johnson space is a constant dimension code in the Grassmannian space. q -analogs of various combinatorial objects are well known [32, pp. 325–332]. Design theory is a well studied area in combinatorics related to coding theory. Related to constant dimension codes are q -analogs of block designs. q -analogs of t -designs were studied in various papers and connections [8, 18, 23, 26, 28–31]. Only some of these designs had also some interest in coding theory [2, 26]. These designs are known as Steiner structures which are q -analogs of Steiner systems.

A Steiner structure $\mathbb{S}_q(r, k, n)$ is a collection \mathbb{S} of elements from $\mathcal{G}_q(n, k)$ such that each element from $\mathcal{G}_q(n, r)$ is contained in exactly one element of \mathbb{S} .

The condition that “each element from $\mathcal{G}_q(n, r)$ is contained in exactly one element of \mathbb{S} ” can be relaxed. If “each element from $\mathcal{G}_q(n, r)$ is contained in at most one element of \mathbb{S} ” then the structure is a q -packing design better known as a constant dimension code or a Grassmannian code. These codes were considered in many papers in the last 5 years, e.g. [13–15, 19, 20, 27].

A q -covering design $\mathbb{C}_q(n, k, r)$ is a collection \mathbb{S} of elements from $\mathcal{G}_q(n, k)$ such that each element of $\mathcal{G}_q(n, r)$ is contained in at least one element of \mathbb{S} .

Let $C_q(n, k, r)$ denote the minimum number of subspaces in a q -covering design $\mathbb{C}_q(n, k, r)$. Lower and upper bounds on $C_q(n, k, r)$ were considered in [16]. Lower bounds are obtained by analytical methods, and upper bounds are obtained by constructions of the related q -covering designs.

Grassmannian codes and q -covering designs are also of interest in the context of projective geometry. A k -spread in $\text{PG}(n, q)$ is a q -covering design $\mathbb{C}_q(n + 1, k + 1, 1)$. Similarly, the values of $C_q(n, n - 1, r)$ and $C_q(n, n - 2, r)$, as well as some related values, were studied in the context of projective geometry [3–5, 7, 12, 21, 22]. The related structure in projective geometry is a dual structure to a q -covering design and it is called a blocking set. A set \mathbb{T} of t -subspaces in $\text{PG}(n, q)$ such that every s -subspace is incident with at least one element of \mathbb{T} is called a blocking set. Such a design is a q -analog of the well-known Turán design [9, 10]. The dual subspaces of the blocking set form a q -covering design $\mathbb{C}_q(n + 1, n - t, n - s)$. Blocking sets were considered for example in [22]. We note that there is a difference of one in the dimension of subspaces in the Grassmannian and the dimension of the same subspace the projective geometry. In other words, r -subspaces in $\text{PG}(n, q)$ are $(r + 1)$ -dimensional subspaces in \mathbb{F}_q^{n+1} .

In this paper we consider upper bounds on $C_q(n, k, r)$ based mainly on lifting of maximum rank distance codes combined with other combinatorial structures. The rest of this paper is organized as follows. In Sect. 2 we discuss the the known bounds on $C_q(n, k, r)$ and their implications on the behavior of the value $C_q(n, k, r)$. In Sect. 3 we introduce the ingredients for our constructions, lifting of maximum rank distance codes, subspace transversal designs, and partitions of $\mathcal{G}_2(4, 2)$ into spreads. In Sects. 4 and 5 we present bounds on $C_2(n, k, 2)$

and $C_2(n, k, 3)$ and discuss the density of the obtained q -covering designs compared to the well known covering bound. In Sect. 6 we present tables of the currently known bounds on $C_2(n, k, r)$ for $5 \leq n \leq 10$. In Sect. 7 we present a sequence of problems for further research. The problems suggest various construction methods for q -covering designs. Before we proceed to the results of this paper we want to make a small important remark. Although we will assume throughout that the ambient space is \mathbb{F}_q^n , we point out that our results hold for an arbitrary n -dimensional vector space over \mathbb{F}_q .

2 Known bounds

In this section we present known bounds on $C_q(n, k, r)$. Most of these bounds will be used later to obtain specific bounds, mainly because of the recursive nature of the new bounds, or since the known bounds can be used as initial conditions for the recursive equations. Even so, our paper is devoted to new upper bounds on $C_q(n, k, r)$, we will consider also the lower bounds as we can use the lower bounds to examine how good are the upper bounds. The first bound is the q -analog Schönheim bound [25] given in [16]. This is the lower bound which will be frequently used in our tables.

Theorem 1 $C_q(n, k, r) \geq \left\lceil \frac{q^n - 1}{q^k - 1} C_q(n - 1, k - 1, r - 1) \right\rceil$.

The next theorem can be obtained by iterating Theorem 1 or just by noting that each r -dimensional subspace of \mathbb{F}_q^n must be contained in at least one element of a q -covering design $C_q(n, k, r)$. This bound is known as the *covering bound*.

Theorem 2 $C_q(n, k, r) \geq \frac{\begin{bmatrix} n \\ r \end{bmatrix}_q}{\begin{bmatrix} k \\ r \end{bmatrix}_q}$ with equality holds if and only if a Steiner structure $\mathbb{S}_q(r, k, n)$ exists.

Another lower bound given in [16] is a q -analog of a theorem given by de Caen in [9, 10].

Theorem 3 $C_q(n, k, k - 1) \geq \frac{(q^k - 1)(q - 1)}{(q^{n-k} - 1)^2} \begin{bmatrix} n \\ k + 1 \end{bmatrix}_q$.

The next theorem given in [16] is used infinitely many times.

Theorem 4 $C_q(n + 1, k + 1, r) \leq C_q(n, k, r)$.

Theorem 4 implies a very interesting property on the behavior of optimal q -design coverings.

Corollary 1 For any given $r > 0$ and $\delta > 0$ there exists a constant $c_{q,\delta,r}$ and an integer n_0 such that for each $n > n_0$, $C_q(n, n - \delta, r) = c_{q,\delta,r}$.

Remark 1 The value of $c_{q,\delta,r}$ in Corollary 1 can be derived from the results which follow in this section (see Theorem 8).

The next two theorems [16] can be viewed as complementary results. They provide two families for which we know the exact value of $C(n, k, r)$.

Theorem 5 If $1 \leq k \leq n$, then $C_q(n, k, 1) = \left\lceil \frac{q^n - 1}{q^k - 1} \right\rceil$.

Theorem 6 *If $1 \leq r \leq n - 1$, then $C_q(n, n - 1, r) = \frac{q^{r+1}-1}{q-1}$.*

The next theorem is a consequence of the main recursive construction. It is given in [16] and it is frequently used in our tables. As the q -covering designs obtained by this construction might be the building blocks for other constructions it is presented for completeness and understanding the other constructions. In the sequel we denote by $\langle A \rangle$ the subspace of \mathbb{F}_q^n which is spanned by the the set of elements in $A \subset \mathbb{F}_q^n$.

Construction 1 *Let us represent \mathbb{F}_q^n as $\{(x, \alpha) : x \in \mathbb{F}_q^{n-1}, \alpha \in \mathbb{F}_q\}$. Suppose that \mathbb{S}_1 is a q -covering design $C_q(n - 1, k, r)$ in \mathbb{F}_q^{n-1} and \mathbb{S}_2 is a q -covering design $C_q(n - 1, k - 1, r - 1)$ in \mathbb{F}_q^{n-1} . Given a subspace X of \mathbb{F}_q^{n-1} , we define a corresponding subspace $X \times \{0\}$ of \mathbb{F}_q^n as follows: $X \times \{0\} = \{(v, 0) \in \mathbb{F}_q^n : v \in X\}$. Note that if $\dim X = k - 1$, then there are exactly q^{n-k} distinct subspaces of the form $(X \times \{0\}) \oplus \langle \{(x, 1)\} \rangle$, each of dimension k (since we can choose x from any one of the q^{n-k} cosets of X in \mathbb{F}_q^{n-1}). With this, we now define the sets \mathbb{S}'_1 and \mathbb{S}'_2 as follows:*

$$\mathbb{S}'_1 \stackrel{\text{def}}{=} \{X \times \{0\} \subset \mathbb{F}_q^n : X \in \mathbb{S}_1\},$$

$$\mathbb{S}'_2 \stackrel{\text{def}}{=} \{(X \times \{0\}) \oplus \langle \{(x, 1)\} \rangle \subset \mathbb{F}_q^n : X \in \mathbb{S}_2, x \in \mathbb{F}_q^{n-1}\}.$$

Let $\mathbb{S}' = \mathbb{S}'_1 \cup \mathbb{S}'_2$.

Theorem 7 *\mathbb{S}' is a q -covering design $C_q(n, k, r)$, and hence $C_q(n, k, r) \leq q^{n-k}C_q(n - 1, k - 1, r - 1) + C_q(n - 1, k, r)$.*

Normal spreads [21], also known as geometric spreads [5], are used to prove the following values of $C_q(n, k, r)$ [6].

Theorem 8 *$C_q(vm + \delta, vm - m + \delta, v - 1) = \frac{q^{vm}-1}{q^m-1}$ for all $v \geq 2, m \geq 2$, and $\delta \geq 0$.*

Theorem 8 is one of the general results on the minimal size of a q -covering design which was solved in the context of projective geometry [5]. Some of the results stated in this section were proved before in terms of projective geometry. In the notation of projective geometry we have that $C_q(n, k, r)$ is the smallest size of a set \mathbb{T} which contains $(k - 1)$ -subspaces of $\text{PG}(n - 1, q)$ such that every $(r - 1)$ -subspace is contained in at least one element of \mathbb{T} . It is also equal the smallest size of a set \mathbb{T}' which contains $(n - k - 1)$ -subspaces of $\text{PG}(n - 1, q)$, such that every $(n - r - 1)$ -subspace contains at least one element of \mathbb{T}' .

Remark 2 The corresponding q -covering design $C_q(n, k, r)$ has k -dimensional subspaces of \mathbb{F}_q^n . The dual $(n - k)$ -dimensional subspaces of \mathbb{F}_q^n are the $(n - k - 1)$ -subspaces of \mathbb{T}' from the projective geometry $\text{PG}(n - 1, q)$. We note again that there is a difference of one in the definition of dimension between the Grassmannian and the projective geometry.

Theorem 6 was proved before in the context of projective geometry by Bose and Burton in [7]. Another lower bound was given in [12] by considering sets of lines in $\text{PG}(2s, q)$ contained in s -subspaces.

Theorem 9 *$C_q(2s + 1, 2s - 1, s) \geq \frac{q^{2s+2}-q^2}{q^2-1} + \frac{q^{s+1}-1}{q-1}$ for every integer $s \geq 2$.*

Metsch [22] also gave a construction for a set of lines in $PG(2s + x - 1, q)$, for every $1 \leq x \leq s$, contained in s -subspaces, which yields the following theorem:

Theorem 10 *For any given integers $q \geq 2, 1 \leq x \leq s$ we have $C_q(2s + x, 2s + x - 2, s + x - 1) \leq \frac{q^{2s+2x}-q^{2x}}{q^2-1} + \frac{q^x-1}{q-1} \cdot \frac{q^{s+x}-q^{x-1}}{q-1}$.*

Finally, also Theorem 5 was proved in terms of projective geometry. In projective geometry $C_q(n + 1, k + 1, 1)$ is the minimal number of k -subspaces in $PG(n, q)$ such that each point of $PG(n, q)$ is contained in at least one of these subspaces. The solution obtained in Theorem 5 was obtained before by Beutelspacher [4].

The proofs for all these results and related results in projective geometry were also given by Metsch in [22].

3 Basic known concepts for the construction

In this section we introduce a few concepts which are used in our constructions: lifting of maximum rank distance (MRD in short) codes, subspace transversal designs, spreads, and partition of the Grassmannian $\mathcal{G}_2(4, 2)$ into disjoint spreads.

3.1 MRD codes and subspace transversal designs

For two $k \times \ell$ matrices A and B over \mathbb{F}_q the *rank distance* is defined by

$$d_R(A, B) \stackrel{\text{def}}{=} \text{rank}(A - B) .$$

A $[k \times \ell, \varrho, \delta]$ *rank-metric code* \mathcal{C} is a linear code, whose codewords are $k \times \ell$ matrices over \mathbb{F}_q ; they form a linear subspace with dimension ϱ of $\mathbb{F}_q^{k \times \ell}$, and for each two distinct codewords A and B we have that $d_R(A, B) \geq \delta$ (clearly, $\delta \leq \min\{k, \ell\}$). For a $[k \times \ell, \varrho, \delta]$ rank-metric code \mathcal{C} it was proved in [11, 17, 24] that

$$\varrho \leq \min \{k(\ell - \delta + 1), \ell(k - \delta + 1)\} . \tag{1}$$

This bound is attained for all possible parameters and the codes which attain it are called *maximum rank distance* codes (or MRD codes in short).

A subset \mathbb{C} of $\mathcal{G}_q(n, k)$ is called an $(n, M, d_S, k)_q$ ((n, M, d_S, k) if $q = 2$) *constant dimension code* if it has size M and minimum distance d_S , where the distance function in $\mathcal{G}_q(n, k)$ is defined by

$$d_S(X, Y) \stackrel{\text{def}}{=} 2k - 2 \dim(X \cap Y),$$

for any two subspaces X and Y in $\mathcal{G}_q(n, k)$.

There is a close connection between constant dimension codes and rank-metric codes [13, 27]. Let A be a $k \times \ell$ matrix over \mathbb{F}_q and let I_k be the $k \times k$ identity matrix. The matrix $[I_k \ A]$ can be viewed as a generator matrix of a k -dimensional subspace of $\mathbb{F}_q^{k+\ell}$, and it is called the *lifting* of A [27].

Example 1 Let A and $[I_3 \ A]$ be the following matrices over \mathbb{F}_2

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad [I_3 \ A] = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix},$$

then the subspace obtained by the lifting of A is given by the following 8 vectors:

$$(100110), (010011), (001001), (110101), \\ (101111), (011010), (111100), (000000).$$

A constant dimension code \mathbb{C} such that all its codewords are lifted codewords of an MRD code is called a *lifted MRD code* [27]. This code will be denoted by \mathbb{C}^{MRD} .

Theorem 11 [27] *If \mathcal{C} is a $[k \times (n - k), (n - k)(k - \delta + 1), \delta]$ MRD code then \mathbb{C}^{MRD} is an $(n, q^{(n-k)(k-\delta+1)}, 2\delta, k)_q$ code.*

Remark 3 The parameters of the $[k \times (n - k), (n - k)(k - \delta + 1), \delta]$ MRD code \mathcal{C} in Theorem 11 imply that $k \leq n - k$, by (1).

In the sequel we will assume that $q = 2$, even so some of the results can be generalized for general q , where q is a power of a prime number.

A *subspace transversal design* of groupsize 2^{n-k} , block dimension k , and strength t , denoted by $\text{STD}(t, k, n - k)$, is a triple $(\mathbb{V}^{(n,k)}, \mathbb{G}, \mathbb{B})$, where $\mathbb{V}^{(n,k)}$ is a set of *points*, \mathbb{G} is a set of groups, and \mathbb{B} is a set of blocks. These three sets must satisfy the following five properties:

1. $\mathbb{V}^{(n,k)}$ is the set of all vectors from \mathbb{F}_2^n , which do not start with k zeroes; $|\mathbb{V}^{(n,k)}| = (2^k - 1)2^{n-k}$ (the *points*);
2. \mathbb{G} is a partition of $\mathbb{V}^{(n,k)}$ into $2^k - 1$ classes of size 2^{n-k} (the *groups*);
3. \mathbb{B} is a collection of k -dimensional subspaces of \mathbb{F}_2^n which contain nonzero vectors only from $\mathbb{V}^{(n,k)}$ (the *blocks*);
4. each block meets each group in exactly one point;
5. every t -dimensional subspace (with points from $\mathbb{V}^{(n,k)}$) which meets each group in at most one point is contained in exactly one block.

Let $\mathbb{V}_x^{(n,\ell)}, x \in \mathbb{F}_2^\ell$ denote the set of all vectors from \mathbb{F}_2^n which start with the binary vector x of length ℓ . Clearly, $\mathbb{V}_0^{(n,\ell)}$ is isomorphic to $\mathbb{F}_2^{n-\ell}$, $\mathbb{V}^{(n,\ell)} \cup \mathbb{V}_0^{(n,\ell)}$ is isomorphic to \mathbb{F}_2^n , and $\mathbb{V}_0^{(n,\ell)} \cup \mathbb{V}_x^{(n,\ell)}$ is isomorphic to $\mathbb{F}_2^{n-\ell+1}$.

Theorem 12 [14] *The codewords of an $(n, 2^{(n-k)(k-\delta+1)}, 2\delta, k)$ code \mathbb{C}^{MRD} form the blocks of a $\text{STD}(k - \delta + 1, k, n - k)$, with the set of points $\mathbb{V}^{(n,k)}$ and the set of groups $\mathbb{V}_x^{(n,k)}, x \in \mathbb{F}_2^k$.*

By Theorem 12 we have that if the code \mathbb{C}^{MRD} is part of a q -covering design $\mathbb{C}_2(n, k, r)$, $r = k - \delta + 1$, then each r -dimensional subspace X , for which $\dim(X \cap \mathbb{V}_0^{(n,k)}) = 0$, meets each group of the corresponding subspace transversal design in at most one point and thus, is contained in an element of \mathbb{C}^{MRD} . Hence, if \mathbb{C}^{MRD} is part of the final minimal q -covering design \mathbb{C} then for each element $Z \in \mathbb{C}$ not in \mathbb{C}^{MRD} we must have $\dim(Z \cap \mathbb{V}_0^{(n,k)}) > 0$.

3.2 Partition of $\mathcal{G}_2(4, 2)$ into disjoint spreads

The description in this subsection will be very specific in its parameters, although the known results presented are generalized for other parameters. But, we have restricted ourself only for those parameters which are needed in the sequel.

A k -spread \mathbb{S} in $\mathcal{G}_q(n, k)$ (or \mathbb{F}_q^n) is a set of k -dimensional subspaces of \mathbb{F}_q^n , for which each one-dimensional subspaces of $\mathcal{G}_q(n, 1)$ is contained in exactly one element of \mathbb{S} . Clearly, \mathbb{S} is a Steiner structure $\mathbb{S}_q(1, k, n)$ and it exists if and only if k divides n .

There are $35 = \binom{4}{2}_2$ two-dimensional subspaces in \mathbb{F}_2^4 . These subspaces can be partitioned into seven 2-spreads $\mathbb{P}_0, \mathbb{P}_1, \dots, \mathbb{P}_6$ each one of size 5 [3,33]. This is well known and it relates to the well known fifteen schoolgirls problem [1], to the disjoint translates of the Preparata code [33], and it is a 2-parallelism of $\mathcal{G}_2(n, 2)$, n even, which is also generalized for $q > 2$ [3]. The properties which follow are also well known.

Each subspace of $\mathbb{P}_i, 0 \leq i \leq 6$, partitions \mathbb{F}_2^4 into four additive cosets of itself. Consider the set of all such cosets for \mathbb{P}_i , namely:

$$P_i \stackrel{\text{def}}{=} \{ \{ \mathbf{u}, \mathbf{u} + \mathbf{v}_1, \mathbf{u} + \mathbf{v}_2, \mathbf{u} + \mathbf{v}_3 \} : \{ \mathbf{0}, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \} \in \mathbb{P}_i, \mathbf{u} \in \mathbb{F}_2^4 \}.$$

The set P_i will be called a *spread translate*. Since the size of a 2-spread in \mathbb{F}_2^4 is 5, and each 2-dimensional subspace of \mathbb{F}_2^4 has 4 cosets it follows that P_i consists of 20 distinct 4-subsets for each $i, 0 \leq i \leq 6$. These 20 subsets are partitioned into 5 parallel classes of size 4. Each parallel class contains the cosets of a different 2-dimensional subspace of \mathbb{P}_i .

Lemma 1 For any given $i, 0 \leq i \leq 6$, each pair $\{ \mathbf{u}, \mathbf{v} \} \subset \mathbb{F}_2^4, \mathbf{u} \neq \mathbf{v}$, appears in exactly one element of P_i .

Lemma 2 For any given $i, 0 \leq i \leq 6$ and an element $\{ a_1, a_2, a_3, a_4 \} \in P_i$, the subspace $\{ \{ (z, a_1), (z, a_2), (z, a_3), (z, a_4) \} \}, z \in \mathbb{F}_2^\ell, \ell \in \mathbb{N}$, is a 3-dimensional subspace defined on $\mathbb{V}_0^{(\ell+4, \ell)} \cup \mathbb{V}_z^{(\ell+4, \ell)}$.

Lemma 3 The set

$$\{ \{ (z, a_1), (z, a_2), (z, a_3), (z, a_4) \} \cap \mathbb{V}_0^{(\ell+4, 4)} : \{ a_1, a_2, a_3, a_4 \} \in P_i, 0 \leq i \leq 6 \}, \\ z \in \mathbb{F}_2^\ell, \ell \in \mathbb{N},$$

contains all the 2-dimensional subspaces of $\mathbb{V}_0^{(\ell+4, \ell)}$ (which is isomorphic to \mathbb{F}_2^4).

4 On the value of $\mathcal{C}_2(n, k, 2)$

In this section we will consider lower and upper bounds on $\mathcal{C}_2(n, k, 2)$. By Theorem 8, $\mathcal{C}_q(3m + \delta, 2m + \delta, 2) = \frac{q^{3m} - 1}{q^m - 1}$ for all $m \geq 2$ and $\delta \geq 0$. This implies a sequence of exact values of $\mathcal{C}_2(n, k, 2)$. For values of k not covered by this theorem we will present one general construction and one specific one which will be also used as a basis for another recursive construction. We will also use the recursions implied by Theorems 4 and 7.

In the sequel we will represent nonzero elements of the finite field \mathbb{F}_{2^n} in two different ways. The first one is by n -tuples over \mathbb{F}_2 (in other words, \mathbb{F}_{2^n} is represented by \mathbb{F}_2^n) and the second one is by powers of a primitive element α in \mathbb{F}_{2^n} . We will not distinguish between these two isomorphic representations. When n -tuple z over \mathbb{F}_2 will be multiplied by an element $\beta \in \mathbb{F}_{2^n}$ we will view z as an element in \mathbb{F}_{2^n} and the result will be an element in \mathbb{F}_{2^n} which is also represented by an n -tuple over \mathbb{F}_2 (an element in \mathbb{F}_2^n). Also, when we write $\mathbb{V}_\gamma^{(n, \ell)}$, where $\gamma \in \mathbb{F}_{2^\ell}$, it is the same as writing $\mathbb{V}_x^{(n, \ell)}, x \in \mathbb{F}_2^\ell$, where x is the binary ℓ -tuple which represents γ .

For a set $S \subseteq \mathbb{F}_2^n$ and a nonzero element $\beta \in \mathbb{F}_{2^n}$, we define $\beta S \stackrel{\text{def}}{=} \{ \beta x : x \in S \}$. We note that we can take the set S to be a subspace. The following lemma is a simple observation.

Lemma 4 If X is a k -dimensional subspace of \mathbb{F}_2^n and β is a nonzero element of \mathbb{F}_2 then βX is also a k -dimensional subspace of \mathbb{F}_2^n .

Another important and simple observation is the following result.

Lemma 5 *Let α be a primitive element in \mathbb{F}_{2^k} , X be an r -dimensional subspace of \mathbb{F}_2^k . If $2^k - 1$ and $2^r - 1$ are relatively primes then the set $\{\alpha^j X : 0 \leq j \leq 2^k - 2\}$ contains $2^k - 1$ distinct r -dimensional subspaces.*

For $k \geq 3$ we define the following 6 sets of size 2^{k-1} in \mathbb{F}_2^k . B_1 contains all the elements of \mathbb{F}_2^k which start with a zero; B_2 contains all the elements of \mathbb{F}_2^k which start with an one; B_3 contains all the elements of \mathbb{F}_2^k which start with 00 or 10; B_4 contains all the elements of \mathbb{F}_2^k which start with 01 or 11; B_5 contains all the elements of \mathbb{F}_2^k which start with 00 or 11; B_6 contains all the elements of \mathbb{F}_2^k which start with 01 or 10. The following two lemmas are readily verified.

Lemma 6 *If $\{x, y\}$ is a pair of elements from \mathbb{F}_2^k then there exists at least one $i, 1 \leq i \leq 6$, such that $\{x, y\} \subset B_i$.*

Lemma 7 *For any given $i, 1 \leq i \leq 6$, and $z \in \mathbb{F}_2^k \setminus \{0\}$, the subspace $\langle\{(z, x) : x \in B_i\}\rangle$, is a k -dimensional subspace of \mathbb{F}_2^{2k} .*

Let α be a primitive element in \mathbb{F}_{2^k} and assume that the elements of $B_i, 1 \leq i \leq 6$ are viewed as elements of \mathbb{F}_{2^k} . For each $j, 0 \leq j \leq 2^k - 2$ we define the following set with six subsets of $\mathbb{V}_{\alpha^j}^{(2k,k)}$ (the elements are from \mathbb{F}_2^{2k}),

$$S_j \stackrel{\text{def}}{=} \{(\alpha^j, \alpha^j x) : x \in B_i\} : 1 \leq i \leq 6\}$$

An immediate consequence of Lemmas 4, 6, and 7 is the following lemma.

Lemma 8 *The set $\mathbb{S}_j \stackrel{\text{def}}{=} \{A : A \in S_j\}$ contains six k -dimensional subspaces of $\mathbb{V}_0^{(2k,k)} \cup \mathbb{V}_{\alpha^j}^{(2k,k)}$. Each 2-dimensional subspace X of $\mathbb{V}_0^{(2k,k)} \cup \mathbb{V}_{\alpha^j}^{(2k,k)}$, for which X is not contained in $\mathbb{V}_0^{(2k,k)}$, is a subspace of at least one k -dimensional subspace of \mathbb{S}_j .*

Theorem 13 $C_2(2k, k, 2) \leq 2^{2k} + 6 \cdot (2^k - 1)$.

Proof Let \mathbb{C}^{MRD} be a $(2k, 2^{2k}, 2(k - 1), k)$ lifted MRD code, and let \mathbb{T} be the set of blocks in the corresponding STD(2, k, k). Define

$$\mathbb{C} \stackrel{\text{def}}{=} \mathbb{T} \cup \bigcup_{j=0}^{2^k-2} \mathbb{S}_j .$$

We claim that \mathbb{C} , which contains $2^{2k} + 6 \cdot (2^k - 1)$ subspaces, is a q -covering design $C_2(2k, k, 2)$.

By Theorem 12, each 2-dimensional subspace X of \mathbb{F}_2^{2k} , for which $\dim(X \cap \mathbb{V}_0^{(2k,k)}) = 0$, is contained in an element of \mathbb{T} . By Lemma 8, each 2-dimensional subspace X of \mathbb{F}_2^{2k} , for which $\dim(X \cap \mathbb{V}_0^{(2k,k)}) = 1$, is contained in an element of $\bigcup_{j=0}^{2^k-2} \mathbb{S}_j$. Therefore, we only have to prove that for each 2-dimensional subspace X of $\mathbb{V}_0^{(2k,k)}$, there exists a k -dimensional subspace Y of \mathbb{C} such that $X \subset Y$.

Let Z be a k -dimensional subspace such that $Z \in \mathbb{C}$ and $Z \subset \mathbb{V}_0^{(2k,k)} \cup \mathbb{V}_1^{(2k,k)}$, where $1 = \alpha^0$ (thus $Z \in \mathbb{S}_0$). Z can be written as $Z_0 \cup Z_1$, where $Z_0 \subset \mathbb{V}_0^{(2k,k)}$ and $Z_1 \subset \mathbb{V}_1^{(2k,k)}$. For each $j, 0 \leq j \leq 2^k - 2, \alpha^j Z$ is also a k -dimensional subspace of \mathbb{C} , and $\alpha^j Z \subset \mathbb{V}_0^{(2k,k)} \cup \mathbb{V}_{\alpha^j}^{(2k,k)}$. Thus, $\alpha^j Z$ can be written as $Z_{0,j} \cup Z_{\alpha^j}$ ($Z_{0,0} = Z_0$), where $Z_{0,j} \subset \mathbb{V}_0^{(2k,k)}$ and $Z_{\alpha^j} \subset \mathbb{V}_{\alpha^j}^{(2k,k)}$. Clearly, Z_0 is a $(k - 1)$ -dimensional subspace of $\mathbb{V}_0^{(2k,k)}$. By Lemma 5

the set $\mathbb{Z} \stackrel{\text{def}}{=} \{Z_{0,j} : 0 \leq j \leq 2^k - 2\}$ contains $2^k - 1$ distinct $(k - 1)$ -dimensional subspaces of $\mathbb{V}_0^{(2k,k)}$. Since $\mathbb{V}_0^{(2k,k)}$ is a k -dimensional subspace it contains $2^k - 1$ distinct $(k - 1)$ -dimensional subspaces. Hence, \mathbb{Z} contains all the $(k - 1)$ -dimensional subspaces of $\mathbb{V}_0^{(2k,k)}$. Each 2-dimensional subspace of $\mathbb{V}_0^{(2k,k)}$ is a subspace of some $(k - 1)$ -dimensional subspace of $\mathbb{V}_0^{(2k,k)}$ and since $\alpha^j Z \in \mathbb{C}$ the proof is completed. \square

Theorem 14 $C_2(7, 3, 2) \leq 396$.

Proof Let \mathbb{C}^{MRD} be a $(7, 256, 4, 3)$ lifted MRD code, and let \mathbb{T} be the set of blocks in the corresponding STD(2, 3, 4). Let α be a primitive element in \mathbb{F}_8 . Recall the definition of $P_i, 0 \leq i \leq 6$, given in Sect. 3.2. For each $0 \leq i \leq 6$, we construct the following set of 3-dimensional subspaces

$$\mathbb{B}_{\alpha^i} \stackrel{\text{def}}{=} \left\{ \{(\alpha^i, a_1), (\alpha^i, a_2), (\alpha^i, a_3), (\alpha^i, a_4)\} : \{a_1, a_2, a_3, a_4\} \in P_i \right\}$$

which are contained in $\mathbb{V}_0^{(7,3)} \cup \mathbb{V}_{\alpha^i}^{(7,3)}$, where the elements of $\{(\alpha^i, a_1), (\alpha^i, a_2), (\alpha^i, a_3), (\alpha^i, a_4)\}, \{a_1, a_2, a_3, a_4\} \in P_i$, are embedded on $\mathbb{V}_{\alpha^i}^{(7,3)}$. Let

$$\mathbb{C} \stackrel{\text{def}}{=} \mathbb{T} \cup \bigcup_{0 \leq i \leq 6} \mathbb{B}_{\alpha^i} .$$

We claim that \mathbb{C} is a q -covering design $C_2(7, 3, 2)$.

By Theorem 12, each 2-dimensional subspace X of \mathbb{F}_2^7 , for which $\dim(X \cap \mathbb{V}_0^{(7,3)}) = 0$, is contained in an element of \mathbb{T} . By Lemmas 1 and 2, each 2-dimensional subspace X of \mathbb{F}_2^7 , for which $\dim(X \cap \mathbb{V}_0^{(7,3)}) = 1$ and $X \subset \mathbb{V}_0^{(7,3)} \cup \mathbb{V}_{\alpha^i}^{(7,3)}$, is contained in at least one subspace of \mathbb{B}_{α^i} and hence it is covered by \mathbb{C} . By Lemma 3, each 2-dimensional subspace X of \mathbb{F}_2^7 , for which $\dim(X \cap \mathbb{V}_0^{(7,3)}) = 2$ is contained in at least one subspace of $\bigcup_{0 \leq i \leq 6} \mathbb{B}_{\alpha^i}$ and hence it is covered by \mathbb{C} .

Thus, \mathbb{C} is a q -covering design $C_2(7, 3, 2)$. \mathbb{C} contains $256 + 7 \cdot 20 = 396$ subspaces and hence $C_2(7, 3, 2) \leq 396$. \square

The constructions in the proofs of Theorems 13 and 14, and other q -covering designs $C_2(n, k, 2)$, can be used in a recursive construction implied by the following theorem.

Theorem 15 *Let $n \geq 2k$ and let \mathbb{S} be a q -covering design $C_2(n - k + 1, k, 2)$ in which there exists an $(n - k)$ -dimensional subspace $U \subset \mathbb{F}_2^{n-k+1}$ and a set $\mathbb{S}_1 = \{X : X \in \mathbb{S}, X \subset U\}$, $|\mathbb{S}_1| = c$. Then $C_2(n, k, 2) \leq 2^{2(n-k)} + (2^k - 1)|\mathbb{S}| - (2^k - 2)c$.*

Proof Let \mathbb{C}^{MRD} be an $(n, 2^{2(n-k)}, 2(k - 1), k)$ lifted MRD code, and let \mathbb{T} be the set of blocks in the corresponding STD(2, $k, n - k$).

Let \mathbb{C}_1 consist of the subspaces of $\mathbb{S} \setminus \mathbb{S}_1$ contained in $\mathbb{V}_0^{(n,k)} \cup \mathbb{V}_x^{(n,k)}$, for each $x \in \mathbb{F}_2^n \setminus \{\mathbf{0}\}$, where $\mathbb{V}_0^{(n,k)}$ coincides with U .

Let \mathbb{C}_2 consist of the subspaces of \mathbb{S}_1 on the points of $\mathbb{V}_0^{(n,k)}$.

We define

$$\mathbb{C} \stackrel{\text{def}}{=} \mathbb{T} \cup \mathbb{C}_1 \cup \mathbb{C}_2 .$$

It can be easily verified that \mathbb{C} is a $C_2(n, k, 2)$ covering design of size $2^{2(n-k)} + (2^k - 1)|\mathbb{S}| - (2^k - 2)c$, and the theorem follows. \square

4.1 On the density of q -covering designs

The density of a q -covering design $\mathbb{C}_q(n, k, r)$, \mathbb{C} , is defined as the ratio between $|\mathbb{C}|$ and the

covering bound $\left\lceil \frac{\binom{n}{r}_q}{\binom{k}{r}_q} \right\rceil$ as n tends to infinity. Constructions for which this ratio is equal 1

were considered in [6]. We will consider now this ratio for two cases, q -covering designs $\mathbb{C}_2(2k, k, 2)$ and q -covering designs $\mathbb{C}_2(2n + 1, 3, 2)$.

By Theorem 13 we have $\mathbb{C}_2(2k, k, 2) \leq 2^{2k} + 6 \cdot (2^k - 1)$. The covering bound in this case is equal to

$$\left\lceil \frac{\binom{2k}{2}_2}{\binom{k}{2}_2} \right\rceil = \left\lceil \frac{(2^{2k} - 1)(2^{2k-1} - 1)}{(2^k - 1)(2^{k-1} - 1)} \right\rceil = \left\lceil \frac{2^{3k-1} + 2^{2k-1} - 2^k - 1}{2^{k-1} - 1} \right\rceil = 2^{2k} + 3 \cdot 2^k + 5.$$

The ratio between the size of the q -covering design and the covering bound is $\frac{2^{2k} + 6 \cdot 2^k - 6}{2^{2k} + 3 \cdot 2^k + 5}$ which approaches 1, when k tends to infinity. Thus, the construction in the proof of Theorem 13 yields a q -covering design which is asymptotically optimal.

Now, we consider the value of $\mathbb{C}_2(2n + 1, 3, 2)$. The covering bound in this case is

$$\left\lceil \frac{\binom{2n+1}{3}_2}{\binom{2}{2}_2} \right\rceil = \left\lceil \frac{(2^{2n+1} - 1)(2^{2n} - 1)}{21} \right\rceil. \tag{2}$$

As for the upper bound on $\mathbb{C}_2(2n + 1, 3, 2)$ we use Theorem 15 iteratively with the initial condition $\mathbb{C}_2(7, 3, 2) \leq 396$ (see Theorem 14). Without going into all the specific details of the proofs we can verify the following properties concerning this upper bound.

Lemma 9 *Let Theorem 15 be applied iteratively to obtain a q -covering design $\mathbb{C}_2(2n+1, 3, 2)$, \mathbb{S} , starting with the q -covering design $\mathbb{C}_2(7, 3, 2)$ of size 396, obtained in the proof of Theorem 14. Then each 2-dimensional subspace Y of $\mathbb{V}_0^{(2n+1,3)} \cup \mathbb{V}_x^{(2n+1,3)}$, $x \in \mathbb{F}_2^3 \setminus \{\mathbf{0}\}$, such that $\dim(Y \cap \mathbb{V}_0^{(2n+1,3)}) = 1$, is contained in exactly one 3-dimensional subspaces of \mathbb{S} .*

Proof The proof is by induction on n . The basis, $n = 3$, is the q -covering design $\mathbb{C}_2(7, 3, 2)$ of size 396, obtained in the proof of Theorem 14, and the claim is immediate by Lemma 1. In the induction step we note that $\mathbb{V}_0^{(2n+1,3)}$ is isomorphic to a union of $\mathbb{V}_0^{(2n-1,3)}$, $\mathbb{V}_x^{(2n-1,3)}$, $\mathbb{V}_y^{(2n-1,3)}$, and $\mathbb{V}_z^{(2n-1,3)}$, where $\{\mathbf{0}, x, y, z\}$ is a 2-dimensional subspace of \mathbb{F}_2^3 . For each $u \in \mathbb{F}_2^3 \setminus \{\mathbf{0}\}$ we have that $\mathbb{V}_u^{(2n+1,3)}$ is isomorphic to $\bigcup_{v \in \mathbb{F}_2^3 \setminus \{\mathbf{0}, x, y, z\}} \mathbb{V}_v^{(2n-1,3)}$. Now, the claim follows from the induction hypothesis and the fifth property in the definition of subspace transversal design. □

Corollary 2 *Let Theorem 15 be applied iteratively to obtain a q -covering design $\mathbb{C}_2(2n + 1, 3, 2)$, \mathbb{S} , starting with the q -covering design $\mathbb{C}_2(7, 3, 2)$ of size 396, obtained in the proof of Theorem 14. Then the number of 3-dimensional subspaces of \mathbb{S} which are contained in $\mathbb{V}_0^{(2n+1,3)} \cup \mathbb{V}_x^{(2n+1,3)}$, $x \in \mathbb{F}_2^3 \setminus \{\mathbf{0}\}$, and are not contained in $\mathbb{V}_0^{(2n+1,3)}$, is $\frac{2^{2n-2}(2^{2n-2}-1)}{3 \cdot 4}$.*

Proof Again, the proof of this result is by induction on n . The basis is $n = 3$ and in the construction given in the proof of Theorem 14 the number of such 3-dimensional subspaces is 20. Assume now that the number of 3-dimensional subspaces of \mathbb{S}' (of a q -covering design $\mathbb{C}_2(2n - 1, 3, 2)$) which are contained in $\mathbb{V}_0^{(2n-1,3)} \cup \mathbb{V}_x^{(2n-1,3)}$, $x \in \mathbb{F}_2^3 \setminus \{\mathbf{0}\}$, and are not contained in $\mathbb{V}_0^{(2n-1,3)}$, is $\frac{2^{2n-4}(2^{2n-4}-1)}{3 \cdot 4}$. In the induction step the 3-dimensional subspaces of \mathbb{S} which are contained in $\mathbb{V}_0^{(2n+1,3)} \cup \mathbb{V}_x^{(2n+1,3)}$, $x \in \mathbb{F}_2^3 \setminus \{\mathbf{0}\}$, and are not contained in $\mathbb{V}_0^{(2n+1,3)}$ consist first of all the $2^{2(2n-4)}$ subspaces of the $\text{STD}(2, 3, 2n - 4)$ obtained from the $(2n - 1, 2^{2(2n-4)}, 4, 3)$ lifted MRD code. These subspaces are joined by the 3-dimensional subspaces of \mathbb{S}' which are contained in $\mathbb{V}_0^{(2n-1,3)} \cup \mathbb{V}_u^{(2n-1,3)}$ and are not contained in $\mathbb{V}_0^{(2n-1,3)}$, for each $u \in \mathbb{F}_2^3 \setminus \{\mathbf{0}, x, y, z\}$, where $\{\mathbf{0}, x, y, z\}$ is any subspace of \mathbb{F}_2^3 . By the induction assumption the number of these subspaces is $4 \cdot \frac{2^{2n-4}(2^{2n-4}-1)}{3 \cdot 4}$. Thus, the total number of these subspaces is $2^{2(2n-4)} + 4 \cdot \frac{2^{2n-4}(2^{2n-4}-1)}{3 \cdot 4} = \frac{2^{2n-2}(2^{2n-2}-1)}{3 \cdot 4}$. \square

Lemma 10 *Let Theorem 15 be applied iteratively to obtain a q -covering design $\mathbb{C}_2(2n + 1, 3, 2)$, \mathbb{S} , starting with the q -covering design $\mathbb{C}_2(7, 3, 2)$ of size 396, obtained in the proof of Theorem 14. Then the number of 3-dimensional subspaces of \mathbb{S} which are contained in $\mathbb{V}_0^{(2n+1,3)}$ is $3 \sum_{i=0}^{n-4} \frac{2^{2i+4}(2^{2i+4}-1)}{3 \cdot 4}$.*

Proof It follows from the fact that in each iteration of Theorem 15, the 3-dimensional subspaces contained in $\mathbb{V}_0^{(2n+1,3)}$ of \mathbb{F}_2^{2n+1} , from the q -covering design $\mathbb{C}_2(2n + 1, 3, 2)$, are exactly those 3-dimensional subspaces contained in $\mathbb{V}_0^{(2n-1,3)} \cup \mathbb{V}_x^{(2n-1,3)} \cup \mathbb{V}_y^{(2n-1,3)} \cup \mathbb{V}_z^{(2n-1,3)}$ of \mathbb{F}_2^{2n-1} (where $\{\mathbf{0}, x, y, z\}$ is a 2-dimensional subspace of \mathbb{F}_2^3), from the q -covering design $\mathbb{C}_2(2n - 1, 3, 2)$ used by the recursion. Now, the number of 3-dimensional subspaces of \mathbb{S} contained in $\mathbb{V}_0^{(2n+1,3)}$ follows from Corollary 2 and the fact that no element of $\mathbb{C}_2(7, 3, 2)$ is contained in $\mathbb{V}_0^{(2n+1,3)}$. \square

Corollary 3

$$C_2(2n + 1, 3, 2) \leq 2^{4n-4} + 7 \cdot \frac{2^{2n-2}(2^{2n-2}-1)}{12} + \sum_{i=0}^{n-4} \frac{2^{2i+4}(2^{2i+4}-1)}{4}$$

Now, when n tends to infinity the ratio between the upper bound on $C_2(2n + 1, 3, 2)$ given in Corollary 3 and the covering bound given in (2) is

$$\frac{2^{4n-4} + \frac{7}{3}2^{4n-6} + \frac{2^{4n-6}}{15}}{\frac{2^{4n+1}}{21}} = 1.05 .$$

5 On the value of $C_2(n, k, 3)$

In this section we will present a few new upper bounds on $C_2(n, k, 3)$. By Theorem 8, $C_q(4m + \delta, 3m + \delta, 3) = \frac{q^{4m}-1}{q^m-1}$ for all $m \geq 2$ and $\delta \geq 0$. This implies a sequence of exact values of $C_2(n, k, 3)$. For values of k not covered by this theorem we will present two specific constructions which will be also used as a basis for another recursive construction. We will also use the recursion implied by Theorem 7.

Theorem 16 $C_2(8, 4, 3) \leq 6897$.

Proof Let \mathbb{C}^{MRD} be a $(8, 4096, 4, 4)$ lifted MRD code, and let \mathbb{T} be the set of blocks in the corresponding $\text{STD}(3, 4, 4)$.

Recall the definitions of \mathbb{P}_i and P_i , $0 \leq i \leq 6$, given in Sect. 3.2. Consider one \mathbb{P}_i and its spread translate P_i . We consider a partition for the 20 elements of P_i into its 5 parallel classes denoted by $\tilde{P}_{i,j}$, $1 \leq j \leq 5$. For each $\{\mathbf{0}, x, y, z\} \in \mathbb{P}_i$ we form a set of 80 distinct 4-dimensional subspaces of \mathbb{F}_2^8 :

$$\{ \{ (x, a_1), (x, a_2), (x, a_3), (x, a_4), (y, b_1) \} : \{ a_1, a_2, a_3, a_4 \}, \{ b_1, b_2, b_3, b_4 \} \in \tilde{P}_{i,j}, 1 \leq j \leq 5 \}$$

Let \mathbb{C}_1 be the union of these $5 \cdot 7 = 35$ sets. \mathbb{C}_1 has $35 \cdot 80 = 2800$ subspaces formed in this way. Let

$$\mathbb{C} \stackrel{\text{def}}{=} \mathbb{T} \cup \mathbb{C}_1 \cup \{ \mathbb{V}_0^{(8,4)} \}.$$

We claim that \mathbb{C} , which contains $4096 + 2800 + 1 = 6897$ subspaces, is a q -covering design $\mathbb{C}_2(8, 4, 3)$.

To complete the proof we have to show that each 3-dimensional subspace of \mathbb{F}_2^8 is contained in at least one 4-dimensional subspace of \mathbb{C} . By Theorem 12, each 3-dimensional subspace X of \mathbb{F}_2^8 for which $\dim(X \cap \mathbb{V}_0^{(8,4)}) = 0$ is contained in a subspace of \mathbb{T} .

A 3-dimensional subspace X of \mathbb{F}_2^8 for which $\dim(X \cap \mathbb{V}_0^{(8,4)}) = 1$ has the form

$$\{ \mathbf{0}, (0, u), (x, a_1), (x, a_2), (y, b_1), (y, b_2), (z, d_1), (z, d_2) \},$$

where $(0, u) \in \mathbb{V}_0^{(8,4)}$, $u, x, y, z \in \mathbb{F}_2^4 \setminus \{ \mathbf{0} \}$, $x + y + z = \mathbf{0}$, $a_1 + a_2 = b_1 + b_2 = d_1 + d_2 = u$, and $d_1 = a_1 + b_1$. The 2-dimensional subspace $\{ \mathbf{0}, x, y, z \}$ of \mathbb{F}_2^4 is contained in a unique 2-spread \mathbb{P}_i , for some i , $0 \leq i \leq 6$. By Lemma 1, each one of the pairs $\{ a_1, a_2 \}$, $\{ b_1, b_2 \}$, $\{ d_1, d_2 \}$ appears in exactly one 4-subset A, B, D , respectively, of P_i . Since $a_1 + a_2 = b_1 + b_2 = c_1 + c_2 = u$ it follows that these three 4-subsets are cosets of the unique 2-dimensional subspace $\{ \mathbf{0}, u, v, w \} \in \mathbb{P}_i$. Since $d_1 = a_1 + b_1$ it follows that $\{ \mathbf{0}, (0, u), (0, v), (0, w) \} \cup \{ (x, a) : a \in A \} \cup \{ (y, b) : b \in B \} \cup \{ (z, d) : d \in D \}$ is a 4-dimensional subspace of \mathbb{C}_1 , which contains X .

A 3-dimensional subspace X of \mathbb{F}_2^8 for which $\dim(X \cap \mathbb{V}_0^{(8,4)}) = 2$ has the form

$$\{ \mathbf{0}, (0, u_1), (0, u_2), (0, u_3), (x, a_1), (x, a_2), (x, a_3), (x, a_4) \},$$

where $(0, u_1), (0, u_2), (0, u_3) \in \mathbb{V}_0^{(8,4)}$ and $(x, a_1), (x, a_2), (x, a_3), (x, a_4) \in \mathbb{V}_x^{(8,4)}$. By the construction of \mathbb{C}_1 each 2-spread \mathbb{P}_i , $0 \leq i \leq 6$ is used in the construction and hence by Lemma 3 the 2-dimensional subspace $\{ \mathbf{0}, (0, u_1), (0, u_2), (0, u_3) \}$ is a subspace of some elements from \mathbb{C}_1 . There are 80 subspaces in \mathbb{C}_1 which contain 3-dimensional subspaces contained in $\mathbb{V}_0^{(8,4)} \cup \mathbb{V}_x^{(8,4)}$ from which four contain X .

Finally, each 3-dimensional subspace of \mathbb{F}_2^8 for which $\dim(X \cap \mathbb{V}_0^{(8,4)}) = 3$ is contained in $\mathbb{V}_0^{(8,4)} \in \mathbb{C}$.

Thus, \mathbb{C} is a q -covering design $\mathbb{C}_2(8, 4, 3)$ with 6897 subspaces. □

Theorem 15 can be modified to obtain bounds on $\mathbb{C}_2(n, k, 3)$.

Theorem 17 *Let $n \geq 2k$ and let \mathbb{S} be a q -covering design $\mathbb{C}_2(n - k + 2, k, 3)$ in which there exist an $(n - k)$ -dimensional subspaces $U_0 \subset \mathbb{F}_2^{n-k+2}$, and three $(n - k + 1)$ -dimensional subspace $U_i \subset \mathbb{F}_2^{n-k+2}$, $i = 1, 2, 3$ such that $U_i \cap U_j = U_0$, $1 \leq i < j \leq 3$. Assume further that $\mathbb{S}_i = \{ X : X \in \mathbb{S}, X \subset U_i \}$, $|\mathbb{S}_i| = c_i$, $i = 0, 1, 2, 3$, and $c_1 \leq c_2 \leq c_3$. Then,*

$$\mathbb{C}_2(n, k, 3) \leq 2^{3(n-k)} + \begin{bmatrix} k \\ 2 \end{bmatrix}_2 (|\mathbb{S}| - c_1 - c_2 - c_3 + 2c_0) + (2^k - 1)(c_1 - c_0) + \mathbb{C}_2(n - k, k, 3).$$

Proof Let \mathbb{C}^{MRD} be an $(n, 2^{3(n-k)}, 2(k-2), k)$ lifted MRD code, and let \mathbb{T} be the set of blocks in the corresponding $\text{STD}(3, k, n-k)$.

Let \mathbb{C}_1 consist of the subspaces of $\mathbb{S} \setminus (\mathbb{S}_1 \cup \mathbb{S}_2 \cup \mathbb{S}_3)$ on the points of $\mathbb{V}_0^{(n,k)} \cup \mathbb{V}_x^{(n,k)} \cup \mathbb{V}_y^{(n,k)} \cup \mathbb{V}_z^{(n,k)}$, for each 2-dimensional subspace $\{\mathbf{0}, x, y, z\}$ of \mathbb{F}_2^k , where $\mathbb{V}_0^{(n,k)}$ coincides with U_0 , $\mathbb{V}_x^{(n,k)}$ with U_1 , $\mathbb{V}_y^{(n,k)}$ with U_2 , and $\mathbb{V}_z^{(n,k)}$ with U_3 . The choice, in which $\mathbb{V}_x^{(n,k)}$, $\mathbb{V}_y^{(n,k)}$, and $\mathbb{V}_z^{(n,k)}$ are matched with U_1, U_2 , and U_3 , is made in a way that for each $x \in \mathbb{F}_2^k \setminus \{\mathbf{0}\}$, $\mathbb{V}_x^{(n,k)}$ coincides at least once with U_1 .

Let \mathbb{C}_2 consist of the subspaces of $\mathbb{S}_1 \setminus \mathbb{S}_0$ on the points of $\mathbb{V}_0^{(n,k)} \cup \mathbb{V}_x^{(n,k)}$, for each $x \in \mathbb{F}_2^k \setminus \{\mathbf{0}\}$, where $\mathbb{V}_x^{(n,k)}$ coincides with U_1 .

Let \mathbb{C}_3 consist of a q -covering design $\mathbb{C}_2(n-k, k, 3)$ on the points of $\mathbb{V}_0^{(n,k)}$.

We define

$$\mathbb{C} \stackrel{\text{def}}{=} \mathbb{T} \cup \mathbb{C}_1 \cup \mathbb{C}_2 \cup \mathbb{C}_3.$$

It can be easily verified that \mathbb{C} is a q -covering design $\mathbb{C}_2(n, k, 3)$ with $2^{3(n-k)} + \binom{k}{2}_2 (|\mathbb{S}| - c_1 - c_2 - c_3 + 2c_0) + (2^k - 1)(c_1 - c_0) + \mathbb{C}_2(n-k, k, 3)$ distinct k -dimensional subspaces and the theorem follows. \square

Remark 4 Theorem 17 can be slightly improved for specific cases by omitting for example the subspaces of \mathbb{C}_3 , or taking much less subspaces than required by $\mathbb{C}_2(n-k, k, 3)$. An example will be given in Sect. 6. The messy general proof is omitted as it is of less interest

6 Tables

This section is devoted for tables with the known lower and upper bounds on $\mathbb{C}_2(n, k, r)$ for $1 \leq r \leq k < n, 5 \leq n \leq 10$. At the end of the section we will demonstrate how the upper bound on $\mathbb{C}_2(10, 5, 3)$ was obtained.

Bounds on $\mathbb{C}_2(5, k, r)$					Bounds on $\mathbb{C}_2(6, k, r)$					
r	k				r	k				
	4	3	2	1		5	4	3	2	1
4	$q31^q$				5	$q63^q$				
3	$q15^q$	$a155^a$			4	$q31^q$	$a651^a$			
2	$q7^q$	$c27^m$	$a155^a$		3	$q15^q$	$s114-122^m$	$a1395^a$		
1	$p3^p$	$p5^p$	$p11^p$	$p31^p$	2	$q7^q$	$s21^n$	$s99-106^c$	$a651^a$	
					1	$p3^p$	$p5^p$	$p9^p$	$p21^p$	$p63^p$

Bounds on $\mathbb{C}_2(7, k, r)$						
r	k					
	6	5	4	3	2	1
6	$q127^q$					
5	$q63^q$	$a2667^a$				
4	$q31^q$	$s468-519^r$	$a11811^a$			
3	$q15^q$	$c99^r$	$s839-970^r$	$a11811^a$		
2	$q7^q$	$s21^n$	$s77-93^r$	$s381-396^f$	$a2667^a$	
1	$p3^p$	$p5^p$	$p9^p$	$p19^p$	$p43^p$	$p127^p$

Bounds on $\mathcal{C}_2(8, k, r)$

r	k						
	7	6	5	4	3	2	1
7	${}^q255^q$						
6	${}^q127^q$	${}^a10795^a$					
5	${}^q63^q$	${}^s1895-2139^r$	${}^a97155^a$				
4	${}^q31^q$	${}^s401-426^m$	${}^s6902-8279^r$	${}^a200787^a$			
3	${}^q15^q$	${}^s85^n$	${}^s634-843^r$	${}^s6477-6897^g$	${}^a97155^a$		
2	${}^q7^q$	${}^s21^n$	${}^s75-93^\ell$	${}^s323-346^c$	${}^s1567-1658^i$	${}^a10795^a$	
1	${}^p3^p$	${}^p5^p$	${}^p9^p$	${}^p17^p$	${}^p37^p$	${}^p85^p$	${}^p255^p$

Bounds on $\mathcal{C}_2(9, k, r)$

r	k							
	8	7	6	5	4	3	2	1
8	${}^q511^q$							
7	${}^q255^q$	${}^a43435^a$						
6	${}^q127^q$	${}^s7625-8683^r$	${}^a788035^a$					
5	${}^q63^q$	${}^s1614-1767^r$	${}^s55983-68371^r$	${}^a3309747^a$				
4	${}^q31^q$	${}^c371^r$	${}^s5143-7170^r$	${}^d108574-118631^r$	${}^a3309747^a$			
3	${}^q15^q$	${}^s85^n$	${}^s609-829^r$	${}^s5325-6379^r$	${}^s53383-59953^r$	${}^a788035^a$		
2	${}^q7^q$	${}^s21^n$	${}^s73^n$	${}^s281-346^\ell$	${}^s1261-1325^i$	${}^s6205-6508^i$	${}^a43435^a$	
1	${}^p3^p$	${}^p5^p$	${}^p9^p$	${}^p17^p$	${}^p35^p$	${}^p73^p$	${}^p171^p$	${}^p511^p$

Bounds on $\mathcal{C}_2(10, k, r)$

r	k								
	9	8	7	6	5	4	3	2	1
9	${}^q1023^q$								
8	${}^q511^q$	${}^a174251^a$							
7	${}^q255^q$	${}^s30590-34987^r$	${}^a6347715^a$						
6	${}^q127^q$	${}^s6475-7195^m$	${}^d451631-555651^r$	${}^a53743987^a$					
5	${}^q63^q$	${}^s1489-1546^m$	${}^s41428-59127^r$	${}^d1777360-1966467^r$	${}^a109221651^a$				
4	${}^q31^q$	${}^s341^n$	${}^s4906-7003^r$	${}^s86468-109234^r$	${}^s1761639-1937127^r$	${}^a53743987^a$			
3	${}^q15^q$	${}^s85^n$	${}^s589-669^r$	${}^s4563-6365^r$	${}^s41613-45230^i$	${}^s423181-476465^r$	${}^a6347715^a$		
2	${}^q7^q$	${}^s21^n$	${}^s73^n$	${}^s277-345^r$	${}^s1155-1210^c$	${}^s4979-5197^i$	${}^s24991-26298^a$	${}^a174251^a$	
1	${}^p3^p$	${}^p5^p$	${}^p9^p$	${}^p17^p$	${}^p33^p$	${}^p69^p$	${}^p147^p$	${}^p341^p$	${}^p1023^p$

- a —all $\begin{bmatrix} n \\ k \end{bmatrix}_2$ k -dimensional subspaces of \mathbb{F}_2^n .
- c —simple construction from \mathbb{C}^{MRD} (Theorem 13).
- d —de Caen Theorem (Theorem 3)
- e —a bound on the size of set of lines by Eisfeld and Metsch (Theorem 9).
- f —Theorem 14.
- g —Theorem 16.
- i —improved construction from \mathbb{C}^{MRD} (Theorems 15 and 17).
- ℓ —lengthening theorem (Theorem 4)
- m —a set of lines in projective geometry by Metsch (Theorem 10).
- n — q -covering design based on normal spreads (Theorem 8).
- p —covering of single points (Theorem 5).
- q —Theorem 6.
- r —recursive construction (Theorem 7).
- s —Schönheim bound (Theorem 1).

Theorem 18 $\mathcal{C}_2(10, 5, 3) \leq 45230$

Proof To apply Theorem 17 we should consider the structure of an appropriate q -covering design $\mathbb{C}_2(7, 5, 3)$. We start with a q -covering design $\mathbb{C}_2(6, 4, 2)$ of size 21. This q -covering design is obtained from the orthogonal complement of a normal 2-spread in $\mathcal{G}_2(6, 2)$, where the *orthogonal complement* \mathbb{S}^\perp , of a set \mathbb{S} of subspaces from \mathbb{F}_q^n , is defined by

$$\mathbb{S}^\perp \stackrel{\text{def}}{=} \{X^\perp : X \in \mathbb{S}\},$$

where X^\perp is the dual subspace of X . Hence, we can take a q -covering design $\mathbb{C}_2(6, 4, 2)$ on $\mathbb{F}_2^6 = \mathbb{V}_0^{(6,1)} \cup \mathbb{V}_1^{(6,1)}$ of size 21 with exactly one subspace contained in $\mathbb{V}_0^{(6,1)}$. By applying Construction 1 with this q -covering design $\mathbb{C}_2(6, 4, 2)$ and a q -covering design $\mathbb{C}_2(6, 5, 3)$ of size 15, we obtain a q -covering design $\mathbb{C}_2(7, 5, 3)$ on $\mathbb{F}_2^7 = \mathbb{V}_0^{(7,2)} \cup \mathbb{V}_1^{(7,2)} \cup \mathbb{V}_2^{(7,2)} \cup \mathbb{V}_3^{(7,2)}$. In this design of size 99, there is no element contained in $\mathbb{V}_0^{(7,2)}$, 80 elements meet $\mathbb{V}_0^{(7,2)}$, $\mathbb{V}_1^{(7,2)}$, $\mathbb{V}_2^{(7,2)}$, and $\mathbb{V}_3^{(7,2)}$, each in at least one vector, two elements lie in $\mathbb{V}_0^{(7,2)} \cup \mathbb{V}_2^{(7,2)}$, two elements lie in $\mathbb{V}_0^{(7,2)} \cup \mathbb{V}_3^{(7,2)}$, and 15 elements lie in $\mathbb{V}_0^{(7,2)} \cup \mathbb{V}_1^{(7,2)}$,

Applying the construction in Theorem 17 with this q -covering design $\mathbb{C}_2(7, 5, 3)$, yields a q -covering design $\mathbb{C}_2(10, 5, 3)$ of size $2^{15} + \binom{5}{2}_2 \cdot 80 + 31 \cdot 2 + \mathbb{C}_2(5, 5, 3) = 45231$. But, the unique 5-dimensional subspace of the q -covering design $\mathbb{C}_2(5, 5, 3)$ can be omitted if we will make our choice of the other subspaces in a similar way to the construction in Theorem 13. □

7 Conclusion and problems for future research

The minimal size, $\mathcal{C}_q(n, k, r)$, of a q -covering design $\mathbb{C}_q(n, k, r)$ was considered. A few techniques to obtain bounds on $\mathcal{C}_q(n, k, r)$ were discussed. Some of the results were given for general q , while other were given only for $q = 2$, even so some of them can be definitely generalized. Tables with the lower and upper bounds on $\mathcal{C}_2(n, k, r)$, $1 \leq r \leq k < 10$, were given.

There are many problems for future research and we will mention a few. We will also propose some construction methods for which we are unable to determine, at this point, how much successful they would be. We would like to emphasize that a computer search which will improve some of the specific results seems to be quite difficult at this point of time.

1. We have given only new upper bounds on $\mathcal{C}_q(n, k, r)$ with related constructions. To reduce the gap between the lower and the upper bounds also lower bounds should be considered. We can suggest two methods to obtain new lower bounds. The first one is to find analog theorems to the ones known for covering designs on sets (as done in Theorem 3 [16]). A second method is to examine the number of subspaces for each type, when sets of the form $\mathbb{V}_x^{(n,\ell)}$, are considered. Inequalities related to the way that r -dimensional subspaces are covered by k -dimensional subspaces should be developed and solved to minimize the number of k -dimensional subspaces in the q -covering design $\mathbb{C}_q(n, k, r)$.
2. Theorems 15 and 17 can be further generalized to obtain upper bounds $\mathbb{C}_2(n, k, r)$, $r > 3$. How the constructions related to these theorems can be applied to obtain the best bounds? How can a general bound for all $r \geq 2$ be formulated? Can we obtain better q -covering designs, to be applied for the recursion in the proofs of these theorems?
3. Usually, the upper bound implied by Theorem 4 can be improved by a better construction. In some cases we couldn't produce a better bound or we only obtained a minor

improvement. Can a more significant improvement be made in these cases ($C_2(8, 5, 2)$, $C_2(9, 5, 2)$, $C_2(10, 6, 2)$).

4. The constructions in which a subspace transversal design based on lifting of an MRD code seems to be very powerful in obtaining good bounds on $C_2(n, k, r)$. Unfortunately, a subspace transversal design exists if and only if $k \leq n - k$. Therefore, we ask the following question. Given $k > n - k$ and the related sets $\mathbb{V}_0^{(n,k)}$, $\mathbb{V}_1^{(n,k)}$, \dots , what is the size of the smallest set \mathbb{S} of k -dimensional subspaces such that for each $X \in \mathbb{S}$, X meets each $\mathbb{V}_y^{(n,k)}$, $y \neq 0$, in exactly one point; and each r -dimensional subspace of \mathbb{F}_q^n which meets each $\mathbb{V}_y^{(n,k)}$ in at most one point, is contained in at least one k -dimensional subspace of \mathbb{S} ?
5. Let $\{X_1, X_2, \dots, X_t\}$ be a set of k -dimensional subspaces of \mathbb{F}_q^n such that $Y = X_i \cap X_j$ for each $1 \leq i < j \leq t$ and for each one-dimensional subspace $Z \in \mathcal{G}_q(n, 1)$ there exists at least one i , $1 \leq i \leq t$, such that $Z \subset X_i$. Let $X_i = Y \cup U_i$, such that $Y \cap U_i = \emptyset$. What is the smallest set \mathbb{S} of k -dimensional subspaces such that for each $X \in \mathbb{S}$, X meets each U_i , in at most one point; and each two-dimensional subspace of \mathbb{F}_q^n , disjoint from Y , which meets each U_i in at most one point is contained in at least one k -dimensional subspace of \mathbb{S} ? This is somewhat a generalization of a subspace transversal design. Related question for general r -dimensional subspaces instead of two-dimensional subspaces is also of interest.
6. What is the best upper bound on the density of a q -covering design $C_q(n, k, r)$ which can be obtained for any given k and r such that $1 < r < k < n - 1$, when n tends to infinity? What is the best bound for a given r and all $k > r$? What is the best bound for a given k and all $r < k$?

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