Optimal Doubly Constant Weight Codes

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Abstract: A doubly constant weight code is a binary code of length $n_1 + n_2$, with constant weight $w_1 + w_2$, such that the weight of a codeword in the first n_1 coordinates is w_1. Such codes have applications in obtaining bounds on the sizes of constant weight codes with given minimum distance. Lower and upper bounds on the sizes of such codes are derived. In particular, we show tight connections between optimal codes and some known designs such as Howell designs, Kirkman squares, orthogonal arrays, Steiner systems, and large sets of Steiner systems. These optimal codes are natural generalization of Steiner systems and they are also called doubly Steiner systems. © 2007 Wiley Periodicals, Inc. J Combin Designs 16: 137–151, 2008

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1. INTRODUCTION

Constant weight codes play an important role in various areas of coding theory [1,10,11,15,37]. One of their applications is in obtaining lower and upper bounds on the sizes of unrestricted codes for given length and minimum Hamming distance [23,27]. In the same way as constant weight codes play a role in obtaining bounds on the sizes of unrestricted codes, doubly constant weight codes play an important role in obtaining bounds on the sizes of constant weight codes [5]. There are tight connections between constant weight codes and block designs. As an obvious example are Steiner systems which are optimal constant weight codes. The main objective of this article is to examine optimal doubly constant weight codes and to show their tight connections with block designs. New lower and upper bounds on the sizes of such codes are derived. In particular, we show tight connections between optimal codes and some known designs such as Howell designs, Kirkman squares, orthogonal arrays, Steiner systems, and large sets of Steiner systems.

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A binary unrestricted *code* of length n is a set of binary words of length n. The *weight* of a word is the number of *ones* in the word. A *constant weight code* of length n and weight w is a binary code whose codewords have constant weight w. A *doubly constant weight code* of length n and weight w is a constant weight code of length n and weight w, with w_1 ones in the first n_1 positions and w_2 ones in the last n_2 positions, where $n = n_1 + n_2$ and $w = w_1 + w_2$. The *Hamming distance* (or distance in short) between two words of the same length n is the number of coordinates in which they differ. The *minimum Hamming distance* (or minimum distance in short) of a code $\mathcal{C}$ is the minimum distance between any two distinct codewords of $\mathcal{C}$. Let $A(n, d)$ denote the maximum number of codewords in a binary code of length n and minimum distance d. Let $A(n, d, w)$ denote the maximum number of codewords in a constant weight code of length n, weight w, and minimum distance d. A (w_1, n_1, w_2, n_2, d) code is a doubly constant weight code, with w_1 ones in the first n_1 positions, w_2 ones in the last n_2 positions, and minimum distance d. Let $T(w_1, n_1, w_2, n_2, d)$ denote the maximum number of codewords in a (w_1, n_1, w_2, n_2, d) code. Upper bounds on $T(w_1, n_1, w_2, n_2, d)$ were found and used in [5,8] to find upper bounds on $A(n, d, w)$.

This article is organized as follows. In Section 2 we present the elements of block designs and design theory which are used in our discussion. In Section 3 we present the basic known upper bounds which combine the quantities $A(n, d)$, $A(n, d, w)$, and $T(w_1, n_1, w_2, n_2, d)$. We discuss the nature of the codes which attain these bounds and their connections to block designs. We define new types of designs which are appropriate to describe some of the optimal doubly constant weight codes. In Section 4 we prove some bounds on the lengths and weights, n_1, n_2, w_1, and w_2 of optimal doubly constant weight codes. These bounds are derived by using the block designs which are either embedded or can be embedded in the code. Section 5 is devoted to all trivial optimal codes. Codes with minimum distance four are discussed in Section 6. In Sections 7 and 8 we present various constructions for optimal doubly constant weight codes. All constructions are derived from related designs. Conclusion and problems for further research are discussed in Section 9.

2. DESIGNS

Most of our constructions which are described in the following sections will use various designs. Hence, we will devote this section for describing the various designs that we use.

An *orthogonal array* $OA(n, k)$ is a $n^2 \times k$ array of n symbols such that any two columns contain each of the n^2 ordered pairs of symbols exactly once. It is well known [3,30] that $OA(n, k)$ is equivalent to a set with $k - 2$ of mutually orthogonal Latin squares of order n. This leads to bounds on k [3].

Let $\mathcal{S}$ be a set of $2n$ elements called symbols. A *Howell Design* $H(s, 2n)$ on the symbol set $\mathcal{S}$ is an $s \times s$ array $\mathcal{D}$ that satisfies the following properties:

- Every cell of $\mathcal{D}$ either is empty or contains an unordered pair of symbols from $\mathcal{S}$.
- Each symbol of $\mathcal{S}$ occurs once in each row and column of $\mathcal{D}$.
- Every unordered pair of symbols occurs in at most one cell of $\mathcal{D}$.

There is a trivial necessary condition for the existence of a Howell design [14].

Theorem 1. *If there exists a Howell design $H(s, 2n)$ then $n \leq s \leq 2n - 1$.*

If $s = 2n - 1$ then the Howell design is called a *Room square* of side $2n - 1$. The existence of Howell designs has been completely determined [2,34] as follows.

Theorem 2. *Let s and n be positive integers such that* $n \leq s \leq 2n - 1$. *Then there exists an* $H(s, 2n)$ *if and only if* $(s, 2n) \neq (2, 4), (3, 4), (5, 6)$, *or* $(5, 8)$.

A *k-dimensional Howell design* $H_k(s, 2n)$ is a k-dimensional array $\mathcal{D}$ such that

- Every cell of $\mathcal{D}$ either is empty or contains an unordered pair of symbols from $\mathcal{S}$.
- Each two-dimensional projection of $\mathcal{D}$ is an $H(s, 2n)$.

The question, what is the largest k such that a k-dimensional Howell design exists? is important in the design of $(2, n_1, k, n_2, 2k + 2)$ codes. Information on the answer to the question as well as an excellent survey on Room squares and Howell designs appear in [14].

A *Steiner system* $S(t, k, v)$ is a collection of k-subsets (called *blocks*) taken from a v-set $\mathcal{S}$ such that each t-subset of $\mathcal{S}$ is contained in exactly one block.

A *large set* of Steiner systems $S(t, k, v)$ is a partition of all $\binom{v}{k}$ different k-subsets of $\mathcal{S}$ into pairwise disjoint Steiner systems $S(t, k, v)$.

Information on the parameters of known Steiner systems can be found in [9]. Necessary conditions for the existence of Steiner systems is given in the following well known theorem [9].

Theorem 3. *A necessary condition for a Steiner system* $S(t, k, n)$ *to exist, is that the numbers* $\frac{\binom{n-i}{t-i}}{\binom{k-i}{t-i}}$, *must be integers, for all* $0 \leq i \leq t$.

A *Kirkman square* $KS_k(v)$ is a $t \times t$ array $\mathcal{D}$, $t = \frac{v-1}{k-1}$ defined on a v-set $\mathcal{S}$ such that

- Every cell of $\mathcal{D}$ either is empty or contains a k-subset of $\mathcal{S}$.
- Each symbol of $\mathcal{S}$ occurs once in each row and column of $\mathcal{D}$.
- The k-subsets of $\mathcal{D}$ form a Steiner system $S(2, k, v)$.

It was shown in [12] that there exists $KS_3(v)$ for $v \equiv 3 \ (mod\ 6)$, $v = 3$ and $v \geq 27$ with at most 23 possible exceptions for v.

In the same way as Howell design generalizes Room square we will generalize the concept of Kirkman square. We are not aware of such generalization in the literature. A *generalized Kirkman square* $GKS_k(t, v)$ is a $t \times t$ array $\mathcal{D}$ with the following properties:

- Every cell of $\mathcal{D}$ either is empty or contains a k-subset of a v-set $\mathcal{S}$.
- Each symbol of $\mathcal{S}$ occurs once in each row and column of $\mathcal{D}$.
- Each 2-subset of $\mathcal{S}$ is contained in at most one k-subset of $\mathcal{D}$.

Lemma 1. *A necessary condition for the existence of a generalized Kirkman square* $GKS_k(t, v)$ *is* $\frac{v}{k} \leq t \leq \frac{v-1}{k-1}$.

Proof. Since each element of $\mathcal{S}$ appears in a row exactly once it follows that the number of k-subsets in a row is $\frac{v}{k}$ and hence $\frac{v}{k} \leq t$. The total number of k-subsets in a square is $t\frac{v}{k}$ and the total number of k-subsets in a Steiner system $S(2, k, v)$ is $\frac{v(v-1)}{k(k-1)}$ and hence $t\frac{v}{k} \leq \frac{v(v-1)}{k(k-1)}$, that is, $t \leq \frac{v-1}{k-1}$. □

3. CLASSIC UPPER BOUNDS AND OPTIMAL CODE DEFINITION

In this section we present some of the classic upper bounds on $A(n, d)$ and $A(n, d, w)$. These bounds are attained by codes which are either designs or in which designs are embedded. It will lead to a definition of optimal codes which attain $T(w_1, n_1, w_2, n_2, d)$ and could be viewed as a new type of designs. The most basic upper bound is the sphere packing bound (known also as the Hamming bound) [27]. From this bound we have:

$$A(n, 2e + 1) \leq \frac{2^n}{\sum_{i=0}^{e} \binom{n}{i}}. \tag{1}$$

Johnson [23] has improved this bound as follows.

$$A(n, 2e + 1) \leq \frac{2^n}{\sum_{i=0}^{e} \binom{n}{i} + \frac{\binom{n}{e+1} - \binom{2e+1}{e} A(n,2e+2,2e+1)}{A(n,2e+2,e+1)}}.$$

From this bound one can obtain the following bound:

$$A(n, 2e + 1) \leq \frac{2^n}{\sum_{i=0}^{e} \binom{n}{i} + \frac{\binom{n}{e+1}\left(\frac{n-e}{e+1} - \lfloor \frac{n-e}{e+1} \rfloor\right)}{\lfloor \frac{n}{e+1} \rfloor}}. \tag{2}$$

Codes which attain the bound in (1) are called perfect. If the all-zeros word is a codeword, in a perfect code, then the codewords of minimum weight form a Steiner system. Codes which attain the bound in (2) are called nearly perfect. These bounds are generalized for constant weight codes as follows [28].

$$A(n, 4e + 2, w) \leq \frac{\binom{n}{w}}{\sum_{i=0}^{e} \binom{w}{i}\binom{n-w}{i}}. \tag{3}$$

$$A(n, 4e + 2, w) \leq \frac{\binom{n}{w}}{\sum_{i=0}^{e} \sum_{i=0}^{e} \binom{w}{i}\binom{n-w}{i} + \frac{\binom{w}{e+1}\binom{n-w}{e+1} - \binom{2e+1}{e}^2 T(2e+1,w,2e+1,n-w,4e+2)}{T(e+1,w,e+1,n-w,4e+2)}}. \tag{4}$$

No code which attain (3) is known for $2 < 4e + 2 < w$. It was conjectured by Delsarte in 1973 [13] that no such code (known as e-perfect code in the Johnson scheme $J(n, w)$) exist. This conjecture is not proved yet [18]. One of the significant results concerning this conjecture was obtained by Roos [32].

Theorem 4. *If there exist an e-perfect code in $J(n, w)$ then $n \leq (w - 1)\frac{2e+1}{e}$.*

This bound was proved in a different method [17] and improved in [20] to $n < (w - 1)\frac{2e+1}{e}$. In the next section we will prove this bound again by obtaining a general bound on the length of some optimal doubly constant weight code. Generalizing the bound in (2) can be done in several ways. One natural way to generalize the definition of nearly perfect codes was given by Hammond [21] who generalized the properties of codes which attain (2). Again, no non-trivial nearly perfect constant weight code is known. On the other hand there are codes which attain the bound of (4).

Theorem 5. *A Steiner system $S(w-2, w, n)$ attains the bound of (4) with equality.*

Proof. By (4) we have

$$A(n,6,w) \leq \frac{\binom{n}{w}}{1 + w(n-w) + \frac{\binom{w}{2}\binom{n-w}{2} - 9T(3,w,3,n-w,6)}{T(2,w,2,n-w,6)}}$$

Since $T(3, w, 3, n-w, 6) \leq \binom{w}{3}\frac{n-w}{3}$ and $T(2, w, 2, n-w, 6) \leq \binom{w}{2}$ it follows that (4) in this case is just $A(n, 6, w) \leq \frac{\binom{n}{w}}{\binom{n-w+2}{2}}$. Since $|S(w-2, w, n)| = \frac{\binom{n}{w}}{\binom{n-w+2}{2}}$ it follows that a Steiner system $S(w-2, w, n)$ attains the bound of (4) with equality. □

It should be noted that the bound $A(n, 6, w) \leq \frac{\binom{n}{w}}{\binom{n-w+2}{2}}$ is one of the well known Johnson bounds for constant weight codes [27], which also implies that $S(w-2, w, n)$ is an optimal constant weight code.

A code $\mathcal{C}$ of length n and weight w can be described in terms w-subsets taken from an n-set. Each binary word of length n with w ones in positions $i_1, i_2, \ldots, i_w$ is translated into a w-subset $\{i_1, i_2, \ldots, i_w\}$. In the sequel we will use a mixed language of set notation and vector notation. It should be understood from the context which one we are using, and how to translate between the two different notations.

Let $\mathcal{C}$ be a perfect code which attain (3). We can partition the set of coordinates into two subsets $\mathcal{A}$ and $\mathcal{B}$ such that $|\mathcal{A}| = w$ and $|\mathcal{B}| = n - w$. We say that a word is from *configuration* (i, j) if it has weight i in the coordinates of $\mathcal{A}$ and weight j in the coordinates of $\mathcal{B}$. Assume that the word from configuration $(w, 0)$ is a codeword and let $\mathcal{C}_1$ be the set of codewords from configuration $(w-2e-1, 2e+1)$. Let $\mathcal{C}_2 = \{(\mathcal{A} \setminus X) \cup Y : X \cup Y \in \mathcal{C}_1\}$; it was proved in [16] that $\mathcal{C}_2$ is a $(2e+1, w, 2e+1, n-w, 4e+2)$ code in which each word from configuration $(e+1, e+1)$ is contained in exactly one codeword. This is the motivation for the next definition which is the main structure of this article.

Definition. *A* $(w_1, n_1, w_2, n_2, 2(w_1 + w_2 - i - j + 1))$ *code is perfect* (i, j) *cover if every word from configuration* (i, j) *is contained in exactly one codeword.*

Note, that in any (w_1, n_1, w_2, n_2, d) code all codewords are from configuration (w_1, w_2), and if every word from configuration (i, j) is contained in exactly one codeword, as a consequence the maximum possible distance of the code is $2(w_1 - i + w_2 - j + 1)$ if $n_1 - w_1 \geq w_1 - i + 1$ and $n_2 - w_2 \geq w_2 - j + 1$. The definition of doubly constant weight code which is a perfect cover is akin to a constant weight code which is a Steiner system. Hence, one can call such a code *doubly Steiner system* $S(i, j, w_1, w_2, n_1, n_2)$.

Theorem 6. *A doubly Steiner system* $S(i, j, w_1, w_2, n_1, n_2)$ *is an optimal* $(w_1, n_1, w_2, n_2, 2(w_1 + w_2 - i - j + 1))$ *code.*

Proof. If there exists a $(w_1, n_1, w_2, n_2, d = 2(w_1 + w_2 - i - j + 1))$ code $\mathcal{C}'$ which is not a perfect (i, j) cover and has more codewords than in $\mathcal{C}$ then there is a word from configuration (i, j) which is contained in two codewords of $\mathcal{C}'$. The distance between these two codewords is at most $2(w_1 - i + w_2 - j)$ which is less than d. Thus, $\mathcal{C}$ is an optimal $(w_1, n_1, w_2, n_2, 2(w_1 + w_2 - i - j + 1))$ code. □

This partition of the coordinates has appeared in [16] and also handled in [33] is the one required for description of doubly constant weight codes. For the rest of this article, we always assume that for a (w_1, n_1, w_2, n_2, d) code, the first set of n_1 coordinates is $\mathcal{A}$ and the second set of n_2 coordinates is $\mathcal{B}$.

4. BOUNDS ON THE LENGTH OF THE CODE

Theorem 7. *If $\mathcal{C}$ is a $(w_1, n_1, w_2, n_2, 2(w_1 + w_2 - i - j + 1))$ code which is a perfect (i, j) cover, $0 \leq i \leq w_1 - 1$, $1 \leq j \leq w_2 - 1$, then* $\frac{(w_2-j)n_1-(i-1)w_2+jw_1}{w_1-i+1} \leq n_2 \leq \frac{(w_2-j+1)n_1+(j-1)w_1-iw_2}{w_1-i}$.

Proof. Let $\mathcal{C}$ be a $(w_1, n_1, w_2, n_2, 2w)$ code which is a perfect (i, j) cover. The total number of codewords in $\mathcal{C}$ is

$$\frac{\binom{n_1}{i}\binom{n_2}{j}}{\binom{w_1}{i}\binom{w_2}{j}}.$$

If $\mathcal{C}$ were a perfect $(i + 1, j - 1)$ cover than it would have contained

$$\frac{\binom{n_1}{i+1}\binom{n_2}{j-1}}{\binom{w_1}{i+1}\binom{w_2}{j-1}}$$

codewords. No word from configuration $(i + 1, j - 1)$ can be contained in more than one codeword and hence

$$\frac{\binom{n_1}{i}\binom{n_2}{j}}{\binom{w_1}{i}\binom{w_2}{j}} \leq \frac{\binom{n_1}{i+1}\binom{n_2}{j-1}}{\binom{w_1}{i+1}\binom{w_2}{j-1}},$$

which is equivalent to $\frac{w_1-i}{n_1-i} \leq \frac{w_2-j+1}{n_2-j+1}$ and thus, we have the following upper bound on n_2:

$$n_2 \leq \frac{(w_2 - j + 1)n_1 + (j - 1)w_1 - iw_2}{w_1 - i},$$

and the following lower bound on n_1:

$$\frac{(w_1 - i)n_2 - (j - 1)w_1 + iw_2}{w_2 - j + 1} \leq n_1. \tag{5}$$

As we can exchange the roles of n_1 with n_2, w_1 with w_2, and i with j we obtain from (5) a lower bound on n_2:

$$\frac{(w_2 - j)n_1 - (i - 1)w_2 + jw_1}{w_1 - i + 1} \leq n_2.$$

□

By substituting $n_1 = w$, $n_2 = n - w$, $w_1 = w_2 = 2e + 1$, $i = j = e + 1$, in $n_2 \leq \frac{(w_2 - j + 1)n_1 + (j-1)w_1 - iw_2}{w_1 - i}$ we obtain $n - w \leq \frac{(e+1)w - 2e - 1}{e}$ which is equivalent to $n \leq (w - 1)\frac{2e+1}{e}$, that is, we have obtained the Roos bound of Theorem 4.

For a code $\mathcal{C}$ with coordinate set $\mathcal{A}$, and $\mathcal{B} \subset \mathcal{A}$, let $\mathcal{C}(\mathcal{B})$ denote the projection of $\mathcal{C}$ onto the column set $\mathcal{B}$.

Theorem 8. *If $\mathcal{C}$ is a $(w_1, n_1, w_2, n_2, 2(w_1 + w_2 - i - j + 1))$ code which is a perfect (i, j) cover then there exist Steiner systems $S(i, w_1, n_1)$ and $S(j, w_2, n_2)$.*

Proof. Let $\mathcal{C}_1$ be a sub-code of $\mathcal{C}$ which consists of all codewords of $\mathcal{C}$ with *i ones* in the first i positions of $\mathcal{A}$. Since $\mathcal{C}$ is a perfect (i, j) cover it follows that for each j positions in $\mathcal{B}$ there is a codeword of $\mathcal{C}_1$ with *ones* in these j positions. Since the minimum distance of the code is $2(w_1 + w_2 - i - j + 1)$ it follows that there are not two codewords of $\mathcal{C}_1$ which have *j ones* in the same j positions of $\mathcal{B}$. Hence, $\mathcal{C}_1(\mathcal{B})$ is a Steiner system $S(j, w_2, n_2)$. Similarly there exists a Steiner system $S(i, w_1, n_1)$. □

By combining the result of Theorem 8 with the necessary conditions of Theorem 3 we obtain that a $(w_1, n_1, w_2, n_2, 2(w_1 + w_2 - i - j + 1))$ code which is a perfect (i, j) cover can exist only for a restricted number of values n_1 and n_2.

5. TRIVIAL CODES

There are a few families of optimal trivial doubly constant weight codes. First, an optimal $(w_1, n_1, w_2, n_2, 2)$ code consists of all possible different words. Such a code is a perfect (w_1, w_2) cover.

A second family of optimal trivial codes consists of $(w, w, k, v, 2(k - t + 1))$ codes which are formed from Steiner systems $S(t, k, v)$. This is summarized in the following theorem whose trivial proof is omitted.

Theorem 9. *If there exists a Steiner system $S(t, k, v)$ then there exists a $(w, w, k, v, 2(k - t + 1))$ code which is a perfect (w, t) cover.*

It is well known and easy to prove that if a Steiner system $S(t, k, v)$ exists then a Steiner system $S(t - 1, k - 1, n - 1)$ exists. Similarly, we have a result for doubly constant weight codes which are perfect covers.

Theorem 10. *If there exists a $(w_1, n_1, w_2, n_2, 2(w_1 + w_2 - i - j + 1))$ code which is a perfect (i, j) cover then there exists a $(w_1 - 1, n_1 - 1, w_2, n_2, 2(w_1 + w_2 - i - j + 1))$ code which is perfect $(i - 1, j)$ cover.*

6. CODES WITH MINIMUM DISTANCE FOUR

In this section we handle the less trivial case of codes with minimum distance 4.

A *packing* $P(t, k, n)$ is a collection of k-subsets (called blocks) taken from a n-set $\mathcal{S}$ such that each t-subset of $\mathcal{S}$ is contained in at most one block.

A *partition* $\Pi(n, w) = \{X_1, \ldots, X_m\}$ is a partition of all $\binom{n}{w}$ different w-subsets of an n-set into pairwise disjoint packings $P(w - 1, w, n)$. The vector $\pi(n, w) = (|X_1|, \ldots, |X_m|)$ is the *index vector* of the partition $\Pi(n, w)$. We always assume $|X_1| \geq \cdots \geq |X_m|$. $\Pi(n, w)$

is said to be *optimal* if for any other given partition $\Pi'(n, w) = \{Y_1, \ldots, Y_k\}$ with index vector $\pi'(n, w) = (|Y_1|, \ldots, |Y_k|)$ we have for all $\ell \leq m$, $\sum_{i=1}^{\ell} |X_i| \geq \sum_{i=1}^{\ell} |Y_i|$

Theorem 11. *If there exists a partition* $\Pi(n_1, w_1) = \{X_1, \ldots, X_m\}$ *and a partition* $\Pi(n_2, w_2) = \{Y_1, \ldots, Y_k\}$ *then* $T(n_1, w_1, n_2, w_2, 4) \geq \sum_{i=1}^{min(m,k)} |X_i| \cdot |Y_i|$.

Proof. The result is obtained by forming the code $\{(x, y) : x \in X_i, y \in Y_i, 1 \leq i \leq \min(m, k)\}$. □

Theorem 12. *If there exists a partition* $\Pi(n_1, w_1) = \{X_1, \ldots, X_m\}$ *and there exists a set of m pairwise disjoint Steiner systems* $S(w_2 - 1, w_2, n_2)$ *then there exists a* $(w_1, n_1, w_2, n_2, 4)$ *code which is a perfect* $(w_1, w_2 - 1)$ *cover.*

Proof. The code formed in Theorem 11 in this case is a perfect $(w_1, w_2 - 1)$ cover. □

Theorem 13. *Assume that in any union of* $n_1 - w_1 + 1$ *disjoint packings* $P(w_2 - 1, w_2, n_2)$ *there are at most* γ *words. Then* $T(w_1, n_1, w_2, n_2, 4) \leq \frac{\binom{n_1}{w_1-1}\gamma}{w_1}$.

Proof. Let $\mathcal{C}$ be an optimal $(w_1, n_1, w_2, n_2, 4)$ code on the coordinates set $\mathcal{A} \cup \mathcal{B}$. For $\mathcal{I} = \{i_1, \ldots, i_{w_1-1}\} \subset \mathcal{A}$, there are exactly $n_1 - w_1 + 1$ distinct w_1-subsets of $\mathcal{A}$, $\mathcal{S}_1, \ldots, \mathcal{S}_{n_1-w_1+1}$, which contain $\mathcal{I}$. Let $\mathcal{C}_j = \{\mathcal{Q} : \mathcal{Q} \in \mathcal{C} \text{ and } \mathcal{Q} \cap \mathcal{A} = S_j\}$ for each $1 \leq j \leq n_1 - w_1 + 1$. Clearly $\mathcal{C}_j(\mathcal{B})$ is a packing $P(w_2 - 1, w_2, n_2)$, for each $1 \leq j \leq n_1 - w_1 + 1$, and $\mathcal{C}_j(\mathcal{B}) \cap \mathcal{C}_\ell(\mathcal{B}) = \emptyset$ for each $1 \leq j < \ell \leq n_1 - w_1 + 1$. Hence, $|\cup_{j=1}^{n_1-w_1+1} \mathcal{C}_j| \leq \gamma$; summing over all $\binom{n_1}{w_1-1}$ subsets of size $w_1 - 1$ and taking into account that each w_1-subset contains w_1 different $(w_1 - 1)$-subsets we obtain $T(w_1, n_1, w_2, n_2, 4) = |\mathcal{C}| \leq \frac{\binom{n_1}{w_1-1}\gamma}{w_1}$. □

More information about partitions, construction methods involving partitions, and pairwise disjoint Steiner systems, can be found in [11,19,29]. By using Theorem 13 and the index vectors of the following partitions known to be optimal [11] ($\pi(6, 3) = (4, 4, 4, 4, 2, 2)$, $\pi(7, 3) = (7, 7, 6, 6, 5, 4)$, $\pi(8, 4) = (14, 14, 12, 12, 10, 8)$) we improve some of the specific upper bounds on $T(w_1, n_1, w_2, n_2, 4)$. The numbers in parenthesis are the previous known bounds. $T(2, 6, 3, 6, 4) \leq 54$ (60), $T(2, 4, 3, 7, 4) \leq 40$ (42), $T(2, 5, 3, 7, 4) \leq 65$ (70), $T(2, 6, 3, 7, 4) \leq 93$ (105), $T(2, 4, 4, 8, 4) \leq 80$ (84), $T(3, 6, 3, 7, 4) \leq 126$ (140), $T(3, 7, 3, 7, 4) \leq 217$ (245), $T(3, 6, 4, 8, 4) \leq 252$ (280), $T(3, 7, 4, 8, 4) \leq 434$ (490), $T(4, 8, 4, 8, 4) \leq 868$ (980). The current best known upper bounds on $T(w_1, n_1, w_2, n_2, d)$ can be found in [6]. The list we give is partial and by using other partitions more bounds can be improved.

Finally, only three infinite families of large sets with Steiner systems $S(w - 1, w, n)$ are known:

- Trivial large set with one Steiner system $S(n - 1, n, n)$ for $n \geq 2$.
- Large set with $n - 1$ Steiner systems $S(1, 2, n)$, for even n, known also as one-factorization of the complete graph K_n [38].
- Large set with $n - 2$ Steiner systems $S(2, 3, n)$ for $n \equiv 1$ or 3 $(mod\ 6)$, $n > 7$ [22,24,25,35].

7. PERFECT (1, 1) COVERS

A. Optimal $(2, n_1, k, n_2, 2k + 2)$ Codes

By the results we have obtained so far there are some necessary conditions for the existence of a $(2, n_1, k, n_2, 2k + 2)$ code which is a perfect (1, 1) cover. By Theorems 3, 7, 8 we have that $\frac{(k-1)n_1}{2} + 1 \leq n_2 \leq k(n_1 - 1)$, $n_1 \equiv 0$ $(mod\ 2)$, and $n_2 \equiv 0$ $(mod\ k)$, that is, $n_1 = 2n$ and $n_2 = ks$. In this subsection we will prove that for $k = 2$ these conditions are sufficient except for two cases. For $k > 2$ we will prove that if certain designs exist then the necessary condition is sufficient.

Theorem 14. *If there exists a k-dimensional Howell design* $H_k(s, 2n)$ *then there exists a* $(2, 2n, k, ks, 2k + 2)$ *code which is a perfect (1, 1) cover.*

Proof. Let $\mathcal{D}$ be a Howell design $H_k(s, 2n)$. We construct a code $\mathcal{C}$ of length $2n + ks$ on the coordinates set $\mathcal{A} \cup \mathcal{B}$, where $\mathcal{A} = Z_{2n}$ and $\mathcal{B} = Z_k \times Z_s$. For each nonempty entry of $\mathcal{D}$ we have one codeword in $\mathcal{C}$. If $\mathcal{D}(i_1, i_2, \ldots, i_k) = \{\alpha, \beta\}$ then the corresponding codeword of $\mathcal{C}$ has *ones* in coordinates α and β of Z_{2n} and coordinate (j, i_j) for each $1 \leq j \leq k$. It is easy to verify that $\mathcal{C}$ is a $(2, 2n, k, ks, 2k + 2)$ code which is a perfect (1, 1) cover. □

By Theorem 2 there are no Howell designs $H(s, 2n)$ with parameters $(s, 2n) = (2, 4)$, (3, 4), (5, 6), and (5, 8). Even so, some corresponding $(2, 2n, 2, 2s, 6)$ codes which are perfect (1, 1) covers do exist. A (2, 6, 2, 10, 6) code which is a perfect (1, 1) cover is given below:

$$\begin{bmatrix}
1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}.$$

A (2, 8, 2, 10, 6) code which is a perfect (1.1) cover is given below:

$$\begin{bmatrix}
1&1&0&0&0&0&0&0&&1&1&0&0&0&0&0&0&0&0\\
1&0&1&0&0&0&0&0&&0&0&1&1&0&0&0&0&0&0\\
1&0&0&1&0&0&0&0&&0&0&0&0&1&1&0&0&0&0\\
1&0&0&0&1&0&0&0&&0&0&0&0&0&0&1&1&0&0\\
1&0&0&0&0&1&0&0&&0&0&0&0&0&0&0&0&1&1\\
0&1&1&0&0&0&0&0&&0&0&0&0&1&0&1&0&0&0\\
0&1&0&1&0&0&0&0&&0&0&1&0&0&0&0&0&1&0\\
0&1&0&0&0&0&1&0&&0&0&0&1&0&1&0&0&0&0\\
0&1&0&0&0&0&0&1&&0&0&0&0&0&0&0&1&0&1\\
0&0&1&0&1&0&0&0&&1&0&0&0&0&0&0&0&1&0\\
0&0&1&0&0&1&0&0&&0&0&0&0&0&1&0&1&0&0\\
0&0&1&0&0&0&1&0&&0&1&0&0&0&0&0&0&0&1\\
0&0&0&1&1&0&0&0&&0&0&0&1&0&0&0&0&0&1\\
0&0&0&1&0&0&1&0&&1&0&0&0&0&0&0&1&0&0\\
0&0&0&1&0&0&0&1&&0&1&0&0&0&0&1&0&0&0\\
0&0&0&0&1&1&0&0&&0&1&0&0&1&0&0&0&0&0\\
0&0&0&0&1&0&0&1&&0&0&1&0&0&1&0&0&0&0\\
0&0&0&0&0&1&1&0&&0&0&1&0&0&0&1&0&0&0\\
0&0&0&0&0&1&0&1&&1&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&1&1&&0&0&0&0&1&0&0&0&1&0
\end{bmatrix}.$$

But, one can easily verify that a (2, 4, 2, 4, 6) and a (2, 4, 2, 6, 6) codes which are perfect (1, 1) covers do not exist.

By Theorem 1 the necessary conditions for the existence of $H_k(s, 2n)$ is that $n \leq s \leq 2n - 1$, that is, $kn \leq ks \leq k(2n - 1)$. As said before $n_1 = 2n$ and $n_2 = ks$; hence, the existence of Howell design $H_k(s, 2n)$ implies the existence of an $(2, n_1, k, n_2, 2k + 2)$ code which is perfect (1, 1) cover for $k\frac{n_1}{2} \leq n_2 \leq k(n_1 - 1)$. Hence, we now need only to find a codes for the range $\frac{(k-1)n_1}{2} + 1 \leq n_2 < k\frac{n_1}{2}$

Theorem 15. *If there exists a generalized Kirkman square $GKS_k(t, n)$ then there exists a $(2, 2t, k, n, 2k + 2)$ code which is a perfect* (1, 1) *cover.*

Proof. Let $\mathcal{D}$ be a generalized Kirkman square $GKS_k(t, n)$. We construct a code $\mathcal{C}$ of length $2t + n$ on the coordinates set $\mathcal{A} \cup \mathcal{B}$, where $\mathcal{A} = Z_t \times \{0, 1\}$ and $\mathcal{B} = Z_n$. For each nonempty entry of $\mathcal{D}$ we have one codeword in $\mathcal{C}$. If $\mathcal{D}(i, j) = \{\alpha_1, \alpha_2, \ldots, \alpha_k\}$ then the corresponding codeword of $\mathcal{C}$ has *ones* in coordinates $(i, 0)$ and $(j, 1)$ of $\mathcal{A}$ and coordinate α_ℓ of $\mathcal{B}$ for each $1 \leq \ell \leq k$. It is easy to verify that $\mathcal{C}$ is a $(2, 2t, k, n, 2k + 2)$ code which is a perfect (1, 1) cover. □

By Lemma 1 the necessary conditions for the existence of a generalized Kirkman square $GKS_k(t, n_2)$ is that $\frac{n_2}{k} \leq t \leq \frac{n_2-1}{k-1}$, that is, $t(k - 1) + 1 \leq n_2 \leq kt$. Hence, the existence of a generalized Kirkman square $GKS_k(t, n_2)$ implies the existence of an $(2, n_1, k, n_2, 2k + 2)$ code which is perfect (1, 1) cover for $\frac{(k-1)n_1}{2} + 1 \leq n_2 \leq k\frac{n_1}{2}$.

B. Optimal $(k_1, n_1, k_2, n_2, 2(k_1 + k_2 - 1))$ Codes

Theorem 16. *If there exists an orthogonal array* $OA(n, k)$ *then for any* $0 < k_1, k_2 < k$ *such that* $k_1 + k_2 = k$ *there exists a* $(k_1, nk_1, k_2, nk_2, 2k - 2)$ *code which is a perfect* (1, 1) *cover.*

Proof. Let $\mathcal{D}$ be an orthogonal array $OA(n, k)$. We generate a code $\mathcal{C}$ as follows: for each row $(i_1, i_2, \ldots, i_k)$ of $\mathcal{D}$ we construct a codeword of length nk on the coordinates set $Z_k \times Z_n$ with an *one* in position (j, i_j) for each j, $1 \leq j \leq k$. For $0 < k_1 < k$ let $\mathcal{A} = Z_{k_1} \times Z_n$ and $\mathcal{B} = (Z_k \setminus Z_{k_1}) \times Z_n$. It is easy to verify that $\mathcal{C}$ is a $(k_1, nk_1, k_2, nk_2, 2k - 2)$ code which is a perfect (1, 1) cover for $k_1 + k_2 = k$. □

A further generalization of Kirkman squares enable us to have a design from which a $(k_1, n_1k_1, k_2, n_2k_2, 2k - 2)$ code which is a perfect (1, 1) cover for $k_1 + k_2 = k$, can be constructed. A *d-dimensional generalized Kirkman design* $GKS_k(d, t, v)$ is a d-dimensional array $\mathcal{D}$ with the following properties:

- Every cell of $\mathcal{D}$ either is empty or contains a k-subset of a v-set $\mathcal{S}$.
- Each two-dimensional projection of $\mathcal{D}$ is a $GKS_k(t, v)$.

Theorem 17. *If there exists a d-dimensional generalized Kirkman design* $GKS_k(d, t, v)$ *then there exists a* $(d, dt, k, v, 2(k + d) - 2)$ *code which is a perfect* (1, 1) *cover.*

Proof. Let $\mathcal{D}$ be a d-dimensional generalized Kirkman design $GKS_k(d, t, v)$. We construct a code $\mathcal{C}$ of length $dt + v$ on the coordinates set $\mathcal{A} \cup \mathcal{B}$, where $\mathcal{A} = Z_t \times Z_d$ and $\mathcal{B} = Z_v$. For each nonempty entry of $\mathcal{D}$ we have one codeword in $\mathcal{C}$. If $\mathcal{D}(i_1, i_2, \ldots, i_d) = \{\alpha_1, \alpha_2, \ldots, \alpha_k\}$ then the corresponding codeword of $\mathcal{C}$ has *ones* in coordinates (i_j, j) of $\mathcal{A}$ for each $1 \leq j \leq d$ and coordinate α_ℓ of $\mathcal{B}$ for each $1 \leq \ell \leq k$. It is easy to verify that $\mathcal{C}$ is a $(d, dt, k, v, 2(k + d) - 2)$ code which is a perfect (1, 1) cover. □

Our last construction in this section is recursive and can use our previous constructions.

Theorem 18. *If there exists a* $(k_1, n_1, k_2, n_2, 2(k_1 + k_2 - 1))$ *code which is a perfect (1, 1) cover and there exists an orthogonal array* $OA(n, k_1 + k_2)$ *then there exists a* $(k_1, n \cdot n_1, k_2, n \cdot n_2, 2(k_1 + k_2 - 1))$ *code which is a perfect* (1, 1) *cover.*

Proof. Let $\mathcal{C}$ be a $(k_1, n_1, k_2, n_2, 2(k_1 + k_2 - 1))$ code which is a perfect (1, 1) cover on the coordinate set $\mathcal{A} \cup \mathcal{B}$, where $\mathcal{A} = Z_{n_1} \times \{0\}$ and $\mathcal{B} = Z_{n_2} \times \{1\}$. Let $\mathcal{D}$ be an orthogonal array $OA(n, k_1 + k_2)$ with symbols set Z_n. We construct a code $\mathcal{C}_1$ on the coordinates set $\mathcal{A}' \cup \mathcal{B}'$, where $\mathcal{A}' = Z_n \times Z_{n_1} \times \{0\}$ and $\mathcal{B}' = Z_n \times Z_{n_2} \times \{1\}$. For each codeword $\{(i_1, 0), \ldots, (i_{k_1}, 0), (j_1, 1), \ldots, (j_{k_2}, 1)\}$, $i_\ell < i_{\ell+1}$, $j_m < j_{m+1}$, of $\mathcal{C}$ and a row $(\ell_1, \ldots, \ell_{k_1+k_2})$ of $\mathcal{D}$ we construct the codeword $\{(\ell_1, i_1, 0), \ldots, (\ell_{k_1}, i_{k_1}, 0), (\ell_{k_1+1}, j_1, 0), \ldots, (\ell_{k_1+k+2}, j_{k_2}, 0)\}$ in $\mathcal{C}_1$. It is easy to verify that $\mathcal{C}_1$ is a $(k_1, n \cdot n_1, k_2, n \cdot n_2, 2(k_1 + k_2 - 1))$ code which is a perfect (1, 1) cover. □

8. CONSTRUCTION FROM STEINER SYSTEMS

A. Perfect (0, *j*) Covers

Let $\mathcal{C}$ be a $(1, n_1, k, n_2, 2(k-j+2))$ code, on the coordinates set $\mathcal{A} \cup \mathcal{B}$, $j \geq 2$. If $\mathcal{C}$ is a perfect $(0, j)$ cover then clearly $\mathcal{C}(\mathcal{B})$ is a Steiner system $S(j, k, n_2)$. Moreover, if $\{i, m_1, \ldots, m_k\} \in \mathcal{C}$ and $\{i, \ell_1, \ldots, \ell_k\} \in \mathcal{C}$, where $i \in \mathcal{A}$ and $\{m_1, \ldots, m_k\} \subset \mathcal{B}$, $\{\ell_1, \ldots, \ell_k\} \subset \mathcal{B}$ then $|\{m_1, \ldots, m_k\} \cap \{\ell_1, \ldots, \ell_k\}| \leq j-2$. The size of a Steiner system $S(j, k, n_2)$ is $\frac{n_2!(k-j)!}{k!(n_2-j)!}$ and hence, if $n_1 \geq \frac{n_2!(k-j)!}{k!(n_2-j)!}$ such code exists. If $n_1 < \frac{n_2!(k-j)!}{k!(n_2-j)!}$ then such code exists if there exists a Steiner system $S(j, k, n_2)$ that can be partition into n_1 pairwise disjoint packings $P(j-1, k, n_2)$. If $n_1 = \frac{n_2-j+1}{k-j+1}$ then such code exists if there exists a Steiner system $S(j, k, n_2)$ that can be partition into n_1 pairwise disjoint Steiner systems $S(j-1, k, n_2)$.

B. Perfect (*i*, *j*) Covers

Theorem 19. *If there exists a Steiner system $S(t, k, n)$ then there exist a $(t-1, k, k-t+1, n-k, 2(k-t+1))$ code which is a perfect $(t-1, 1)$ cover and a $(k-t+1, k, k-t+1, n-k, 2(k-t+1))$ code which is a perfect $(k-t+1, 1)$ cover.*

Proof. Given a Steiner system $S(t, k, n)$ we partition to coordinates set into two subsets $\mathcal{A}$ and $\mathcal{B}$ such that $|\mathcal{A}| = k$, $|\mathcal{B}| = n-k$, and the word from configuration $(k, 0)$ is a codeword. Clearly, the codewords from configuration $(t-1, k-t+1)$ form a $(t-1, k, k-t+1, n-k, 2(k-t+1))$ code $\mathcal{C}$ which is a perfect $(t-1, 1)$ cover. Similarly, the code $\{(\mathcal{A} \setminus X) \cup Y : X \subset \mathcal{A}, Y \subset \mathcal{B}, X \cup Y \in \mathcal{C}\}$ is a $(k-t+1, k, k-t+1, n-k, 2(k-t+1))$ code which is a perfect $(k-t+1, 1)$ cover. □

From an $(i, k, w, n, 2(w-j+1))$ code which is a perfect (i, j) cover we can derive more codes except from the ones derived by Theorem 10.

Theorem 20. *If there exists an $(i, k, w, n, 2(w-j+1))$ code which is a perfect (i, j) cover then there exists a $(i, k-1, w, n, 2(w-j+1))$ code which is a perfect (i, j) cover.*

Proof. The new code is obtained by taking from the $(i, k, w, n, 2(w-j+1))$ code all codewords which don't have a *one* in the first coordinate and removing the first coordinate. □

Finally, Theorem 12 is generalized in the following way by using a similar proof.

Theorem 21. *Assume that a Steiner system $S(t, k, v)$ can be partitioned into m pairwise disjoint Steiner systems $S(t-1, k, v)$ and some additional k-subsets. If there exists a partition of all w-subsets on an n-set into at most m pairwise disjoint packings $P(w-k+t-1, w, n)$ then there exists a $(w, n, k, v, 2(k-t+2))$ code which is a perfect $(w, t-1)$ cover.*

Except for large sets of Steiner systems $S(k-1, k, v)$ which are partitions of a Steiner system $S(k, k, v)$ into pairwise disjoint Steiner systems $S(k-1, k, v)$, a few more partitions can be used in Theorem 21. Examples are the partitions of Steiner systems $S(3, 4, n)$ into Steiner systems $S(2, 4, n)$ [7,36,39] or partition of $S(2, 3, n)$, $n \equiv 3$ $(mod\ 6)$, into disjoint Steiner systems $S(1, 3, n)$ [26,31]. Survey on partitions of $S(2, k, n)$ and other partitions is given in [4].

9. CONCLUSION AND OPEN PROBLEMS

We have looked on optimal doubly constant weight codes. Bounds on the double lengths of the codes are derived as well as upper bounds on the sizes of codes with minimum distance 4. Various constructions which result in optimal codes are given. The constructions uses various kind of known designs and some generalizations of known designs. Except for the well known problems concerning the designs which we discussed, our discussion leads to many more research problems. We summarize here the five which we find to be the most interesting ones.

1. Do there exist generalized Kirkman squares $GKS_k(t, v)$? We conjecture that the answer is YES. Hence, we would like to see a comprehensive study for constructions of such squares.
2. What is the largest d for which a d-dimensional generalized Kirkman design $GKS_k(d, t, v)$ exists? Again, we would like to see a comprehensive study for this problem.
3. Do there exist $(2e + 1, n_1, 2e + 1, n_2, 4e + 2)$ codes which are perfect $(e + 1, e + 1)$ covers? We conjecture that unless $n_1 = 2e + 1$ and $n_2 = 2e + 1$ such codes do not exist. Negative solution to this problem will imply also the nonexistence of perfect codes in the Johnson schemes.
4. Is the code obtained in Theorem 11 optimal if the partitions are optimal? We conjecture that the answer is YES. If indeed the answer is positive then the bound given in Theorem 13 can be improved.
5. We have no construction for a $(w_1, n_1, w_2, n_2, 2(w_1 + w_2 - i - j) + 2)$ code which is a perfect (i, j) cover for $1 < i < w_1, 1 < j < w_2, w_1 \neq n_1, w_2 \neq n_2$. We conjecture that such codes exist and we would like to see constructions for such codes.

Finally, in order to solve our last question, we would like to propose a construction for a $(2, 2t, k, n, 2k)$ code which is a perfect (1, 2) cover. This construction will use a new type of design which we call a *packing square*. A packing square $PS_k(t, n)$ is a $t \times t$ square $\mathcal{D}$ on a symbol set of an n-set $\mathcal{S}$ such that

- Every cell of $\mathcal{D}$ is either empty or contains disjoint k-subsets of $\mathcal{S}$.
- Each 2-subset of $\mathcal{S}$ is contained in exactly one k-subset in each row and column of $\mathcal{D}$.
- Each 3-subset of $\mathcal{S}$ is contained in at most one k-subset of one cell of $\mathcal{D}$.

Similarly to the proof of Theorem 15 one can prove that if there exists a packing square $PS_k(t, n)$ then there exists a $(2, 2t, k, n, 2k)$ code which is a perfect (1, 2) cover. Hence, we would like to see any construction, even with specific parameters, of a packing square $PS_k(t, n)$.

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