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TOWARDS A LARGE SET OF STEINER QUADRUPLE SYSTEMS*

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Abstract. Let $D(v)$ be the number of pairwise disjoint Steiner quadruple systems. A simple counting argument shows that $D(v) \leqq v - 3$. In this paper it is proved that $D(2^k n) \geqq (2^k - 1)n$, $k \geqq 2$, if there exists a set of $3n$ pairwise disjoint Steiner quadruple systems of order $4n$ with a certain structure. This implies that $D(v) \geqq v - o(v)$ for infinitely many values of v. New lower bounds on $D(v)$ for many values of v that are not divisible by 4 are also given, and it is proved that $D(v) \geqq 2$ for all $v \equiv 2$ or $4 (\bmod 6)$, $v \geqq 8$.

Key words. Steiner quadruple system, orthogonal array, one-factorization, disjoint Steiner systems

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1. Introduction. A *Steiner quadruple system* (SQS) is a pair (Q, q) where Q is a finite set of *points* and q is a collection of 4-element subsets of Q called *blocks* such that every 3-element subset of Q is contained in exactly one block of q. The number of points in Q is the order of the SQS, and it is well known that an SQS of order v, denoted SQS(v), has $b_v = \frac{1}{4}\binom{v}{3}$ blocks. Hanani [5] proved that Steiner quadruple systems of order v exist if and only if $v \equiv 2$ or $4 \pmod 6$. Two SQS (Q, q_1) and (Q, q_2) are disjoint if $q_1 \cap q_2 = \varnothing$. A coloring of an SQS is a partition of the set of points into color classes such that no block is properly contained in any color class. An SQS is k-chromatic if it can be k-colored, but no proper coloring having fewer than k color classes exists. A set of p pairwise disjoint SQSs (PDQs) is mutually 2-chromatic if the same partition of Q is a 2-coloring of all the PDQs. If (Q, q) is a 2-chromatic SQS$(2v)$, with 2-coloring A, B, then by Doyen and Vandensavel [3], $|A| = |B| = v$, and the number of blocks n_1, n_2, n_3 that meet A in 1, 2, and 3 points, respectively, is

$$n_1 = n_3 = \binom{v}{3}, \qquad n_2 = \binom{v}{2}.$$

Hence, the maximum size of a set of disjoint mutually 2-chromatic SQS$(2v)$ is v since the number of 4-subsets intersecting A in 3 points is $\binom{v}{3}v$.

Let $D(v)$ denote the maximum number of PDQs. Since $\binom{v}{4} = (v-3)b_v$, we have that $D(v) \leqq v - 3$. A set of $v - 3$ PDQs of order v is called a *large set*. An application of sets of PDQs is in the construction of constant weight codes with distance 4 [1].

It is well known that $D(4) = 1$, $D(8) = 2$, and Kramer and Mesner [11] proved that $D(10) = 5$. Phelps [17] proved that $D(2.5^t) \geqq 5^t$, Phelps and Rosa [19] proved that $D(2.5^a \cdot 13^b \cdot 17^c) \geqq 5^a \cdot 13^b \cdot 17^c$, for all $a, b, c \geqq 0$, $a + b + c > 0$, and Lindner [12] proved that $D(2v) \geqq v$ for $v \equiv 2$ or $4 \pmod 6$, $v \geqq 8$. Recently, Phelps [18] has shown that $D(22) \geqq 11$. All the PDQs of these four constructions are mutually 2-chromatic. Lindner [13] proved that $D(4v) \geqq 3v$ for $v \equiv 2$ or $4 \pmod 6$, $v \geqq 8$ by using his v PDQs of order $2v$ [12].

In § 2 we use a construction with a similar structure to the one of Lindner [13] to obtain $D(4v) \geqq 3p$, where $v \equiv 1$ or $5 \pmod 6$ and a set of p mutually 2-chromatic PDQs of order $2v$ exists. If $p = v$ then our set of $3p$ PDQs is maximal.

In § 3 we use the PDQs of Lindner [13], our PDQs of § 2, and a set of $2^k - 1$ Boolean SQSs of order 2^k to obtain $D(2^k v) \geqq (2^k - 1)v$ for $v \equiv 2$ or $4 \pmod 6$, $v \geqq 8$,

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for $v = 5^a \cdot 13^b \cdot 17^c$, for all $a, b, c \geqq 0$, $a + b + c > 0$, and also for $v = 11$. This result implies, for example, $D(5.2^t) \geqq 5.2^t - 5$, and this is almost best possible, since if there exist $5.2^t - 4$ PDQs of order 5.2^t, the unused quadruples must form an additional disjoint system, and thus a large set exists. No large set of PDQs has been constructed yet.

In § 4 we give various recursive bounds on $D(v)$ in order to improve the state of knowledge when v is not divisible by 4. We also prove the long standing conjecture, due to Lindner and Rosa [14], that $D(v) \geqq 2$ for all $v \equiv 2$ or 4 (mod 6) with $v \geqq 8$.

2. The construction for orders $4v$, v odd. An *orthogonal array* $OA(t, k, n)$ is an $n^t \times k$ matrix M, with entries from the set $\{0, 1, \cdots, n-1\}$, such that the submatrix generated by any t columns contains each ordered t-tuple exactly once as a row. If $M(i, 0) = j_0$, $M(i, 1) = j_1, \cdots, M(i, k-1) = j_{k-1}$, we can also write $(j_0, j_1, \cdots, j_{k-1}) \in M$ or $\{(j_0, 0), (j_1, 1), \cdots, (j_{k-1}, k-1)\} \in M$.

Two orthogonal arrays M_1 and M_2 are disjoint if they have no row in common.

A set $M_0, M_1, \cdots, M_{n-1}$ of n disjoint $OA(t, k, n)$ is said to have property X if the first $t+1$ columns of the arrays cover each ordered $(t+1)$-tuple exactly once. Note, that an orthogonal array $OA(t+1, k+1, n)$ implies the existence of a set with property X, but the contrary does not follow.

LEMMA 1. *For $n \equiv 1$ or 5 (mod 6) there exists an $OA(3, 5, n)$.*

Proof. Raghavarao [20] proved the existence of $OA(3, 5, p)$ for all primes p such that $p \geqq 5$. The proof then follows from the direct product construction for orthogonal arrays, since the smallest prime factor of n is at least 5.

LEMMA 2. *If there exists an $OA(3, 5, n)$ then there exists a set of n disjoint $OA(3, 5, n)$ with property X.*

Proof. Given an $OA(3, 5, n)$, M_0, the rows of M_r, $0 \leqq r \leqq n-1$ are defined by $(a+r, b, c, d, e)$, where $(a, b, c, d, e) \in M_0$ and $a + r$ is taken modulo n. It is obvious that each M_r, $0 \leqq r \leqq n-1$ is an $OA(3, 5, n)$. Property X follows from the construction.

We now construct a set of PDQs of order $4v$ using as input, a set of mutually 2-chromatic PDQs of order $2v$, an orthogonal array, a near one-factorization of K_v, and a fixed partition of a set of size 4. Let $v \equiv 1$ or 5 (mod 6).

- Let D_k, $0 \leqq k \leqq p-1$, be the block sets of p mutually 2-chromatic PDQs of order $2v$, whose point set is $Z_v \times Z_2$ and whose color classes are $Z_v \times \{i\}$, $i = 0, 1$.
- Let M be an $OA(3, 5, v)$.
- Let $F = \{F_0, F_1, \cdots, F_{v-1}\}$ be a near-one-factorization of K_v such that in F_i vertex i is isolated.
- Let $\pi = \{(i, j), (s, t)\}$ be a fixed partition of Z_4 into two ordered pairs.

The point set of our quadruple systems is $Z_v \times Z_4$ and we construct the block sets $S_k(M, F, \pi)$ with $0 \leqq k \leqq p-1$ as the union of the blocks of Types A, B, and C defined below.

Type A.

$$[(a,i),(b,i),(c,j),(d,j)], \qquad [(a,s),(b,s),(c,t),(d,t)],$$

where
$$[(a,0),(b,0),(c,1),(d,1)] \in D_k,$$

$$[(a,i),(b,j),(c,j),(d,j)], \qquad [(a,s),(b,t),(c,t),(d,t)],$$

where
$$[(a,0),(b,1),(c,1),(d,1)] \in D_k,$$

$$[(a,i),(b,i),(c,i),(d,j)], \qquad [(a,s),(b,s),(c,s),(d,t)],$$

where
$$[(a,0),(b,0),(c,0),(d,1)] \in D_k.$$

There are $2b_{2v}$ blocks of this type, which consist of two isomorphic copies of D_k—one on the point set $Z_v \times \{i, j\}$ and the other with point set $Z_v \times \{s, t\}$.

Type B.

$$[(a,i),(b,j),(e,s),(f,s)], \qquad [(a,i),(b,j),(g,t),(h,t)],$$

where $\{e,f\} \in F_c, \quad \{g,h\} \in F_d, \quad \{(a,i),(b,j),(c,s),(d,t),(k,4)\} \in M,$

$$[(a,s),(b,t),(e,i),(f,i)], \qquad [(a,s),(b,t),(g,j),(h,j)],$$

where $\{e,f\} \in F_c, \quad \{g,h\} \in F_d, \quad \{(c,i),(d,j),(a,s),(b,t),(k,4)\} \in M.$

There are $4v^2(v-1)/2$ blocks of Type B, and each of the blocks of Type B intersect three of the sets $Z_v \times \{x\}$ with $x \in Z_4$.

Type C.

$$[(a,i),(b,j),(c,s),(d,t)],$$

where $\{(a,i),(b,j),(c,s),(d,t),(k,4)\} \in M.$

There are v^2 blocks of Type C, and each of the blocks of Type C intersect all four of the sets $Z_v \times \{x\}$ with $x \in Z_4$.

Each $S_k(M, F, \pi)$, $0 \leqq k \leqq p-1$, has $2b_{2v} + 4v^2(v-1)/2 + v^2 = b_{4v}$ blocks.

To show that each $S_k(M, F, \pi)$ is indeed an SQS we note that any 3-subset T of $Z_v \times \{0, 1, 2, 3\}$ is contained in a unique block by the following arguments:

1) If $T \subset Z_v \times \{i, j\}$ or $T \subset Z_v \times \{s, t\}$ then T is contained in a unique block of Type A.
2) If T is of the form $\{(x, i), (y, i), (z, s)\}$, then $\{x, y\}$ is contained in a unique factor F_c and there is a unique row of M containing (c, i), (z, s), $(k, 4)$—and thus T is in a unique block of Type B.
3) If T is of the form $\{(x, i), (y, j), (z, s)\}$ then, either (x, i), (y, j), (z, s), $(k, 4)$ is contained in a row of M—in which case T is in a block of Type C. Otherwise, the row containing (x, i), (y, j), $(k, 4)$ contains (w, s) with $w \neq z$. In this case a unique pair in F_w contains z—and thus T is in a unique block of Type B.

All the other triples T are covered by the symmetries of the construction and analogous arguments.

The fact that all the p SQSs are disjoint follows from:

1) The disjointness of the D_k, for blocks contained in $Z_v \times \{i, j\}$ or $Z_v \times \{s, t\}$.
2) The disjointness of the $OA(2, 4, v)$ induced by the rows of M containing the element $(k, 4)$, for blocks of Types B and C.

Now let $\pi_0 = \{(0, 1), (2, 3)\}$, $\pi_1 = \{(0, 2), (1, 3)\}$, and $\pi_2 = \{(0, 3), (1, 2)\}$. Let M_0, M_1, M_2 be three $OA(3, 5, v)$ from a set of v $OA(3, 5, v)$ with property X. We construct the following block sets of $3p$ SQSs, $S_k(M_0, F, \pi_0)$, $S_k(M_1, F, \pi_1)$, and $S_k(M_2, F, \pi_2)$, with $0 \leqq k < p$. We denote this set of SQSs by $B((D_k), (M_i), F)$.

We claim that this is a set of PDQs, and to prove this it only remains to show that $S_k(M_i, F, \pi_i) \cap S_m(M_j, F, \pi_j) = \varnothing$, for any $i \neq j$. But this holds since the only possible blocks in common are those of Type C, and these are disjoint by property X. Hence, we have the following theorem.

THEOREM 1. *If there exists a set of p mutually* 2-*chromatic* PDQs *of order* $2v$ *then* $D(4v) \geqq 3p$.

Phelps and Rosa [19] proved that there exists a set of n mutually 2-chromatic PDQs of order $2n$, for $n = 5^a.13^b.17^c$, $a, b, c \geqq 0$, and $a + b + c > 0$. Phelps [18] has recently proved the same result for $n = 11$. Hence, we have the following corollary.

COROLLARY 1. *$D(4.5^a \cdot 13^b \cdot 17^c) \geqq 3.5^a \cdot 13^b \cdot 17^c$, for all $a, b, c \geqq 0$ and $a + b + c > 0$. Furthermore, $D(44) \geqq 33$.*

Before moving on to our next construction we list some of the properties of the set $B((D_k), (M_i), F)$ of PDQs.

- No block in any of the systems is entirely contained in any of the sets $Z_v \times \{i\}$, $i \in Z_4$, so these sets are a 4-coloring of each of the systems.
- The blocks in $S_k(M_0, F, \pi_0)$ that are entirely contained in $Z_v \times \{0, 1\}$ form the block sets of p mutually 2-chromatic PDQs of order $2v$ with 2-coloring $Z_v \times \{0\}$, $Z_v \times \{1\}$. Likewise, those blocks induced on the set $Z_v \times \{2, 3\}$ also form a set of mutually 2-chromatic PDQs. Furthermore, there are no other bichromatic blocks in any of the $S_k(M_0, F, \pi_0)$.
- Analogous statements can be made about the sets of block sets $S_k(M_1, F, \pi_1)$ with color classes 0, 2 and 1, 3, and also the sets $S_k(M_2, F, \pi_2)$ with color classes 0, 3 and 1, 2.
- For fixed k and all i there is an isomorphism between the quadruple system D_k and the subsystems of order $2v$ described above. Thus if $(a_0, a_1) \in \pi_i$ then the induced SQS on $Z_v \times \{a_0, a_1\}$ is isomorphic to the subdesign induced on $Z_v \times \{0, 1\}$. Furthermore, the isomorphism is given by $(x, a_0) \to (x, 0)$ and $(x, a_1) \to (x, 1)$.
- The block sets $S_k(M_1, F, \pi_1)$, and $S_k(M_2, F, \pi_2)$, with $0 \leqq k < p$ form a set of $2p$ mutually 2-chromatic PDQs of order $4v$ with two coloring $Z_v \times \{0, 1\}$, $Z_v \times \{2, 3\}$.
- If $p = v$ then our $3v$ PDQs are maximal (nonextendable).

3. The construction for orders $2^k v$, $k \geqq 3$. Our construction for $(2^k - 1)v$ PDQs of order $2^k v$ uses a set of Boolean Steiner quadruple systems, which are defined below. An order of Z_2^k is induced by identifying $x \in Z_2^k$ with the integer $i < 2^k$ whose binary representation is x.

We define a set of $2^k - 1$ SQSs all with point set Z_2^k and block sets B_i with $i \in Z_2^k - \{0\}$. These will be called the Boolean Steiner quadruple systems. The block set B_i, $i \in Z_2^k - \{0\}$ is defined to be the union of the blocks of types (B.1) and (B.2) specified below:

(B.1) $$[x, y, z, w],$$

where

$$x + y + z + w = i \quad \text{and} \quad |\{x, y, z, w\}| = 4,$$

(B.2) $$[x, y, z, w],$$

where

$$x + y = z + w = i \quad \text{and} \quad |\{x, y, z, w\}| = 4.$$

To show that (Z_2^k, B_i) is an SQS, let $T = \{q, r, s\}$ be a 3-subset of Z_2^k, and let $t = q + r + s + i$.

- If $t \notin \{q, r, s\}$ then $[q, r, s, t]$ is the unique block of type (B.1) containing T. Furthermore, no block of type (B.2) contains T, since if any 2-subset of T has sum i this implies that the third member equals t, a contradiction.
- If $t \in T$, say $t = q$, then we have $r + s = i$. We also observe that $q + i \notin T$, since $q + i = q$ is impossible, and $q + i = r$ implies $q = s$ contradicting the cardinality of T. Hence, $[q, r, s, q + i]$ is the unique block of type (B.2) containing T, and it is easy to see that no block of type (B.1) contains T.

The number of blocks of type (B.2) is $\binom{2^{k-1}}{2}$ since the number of pairs of distinct members of Z_2^k which sum to i is 2^{k-1}. Hence, the number of blocks of type (B.1) is $b_{2^k} - \binom{2^{k-1}}{2}$.

Note that each block of (B.2) has zero sum. Also note that $B_i \cap B_j$ contains the quadruple $[q, r, s, t]$ if and only if $q + r + s + t = 0$, and

$$\{i,j\} \subset \{q+r, q+s, q+t\}.$$

Hence, each zero sum 4-subset of Z_2^k is contained in precisely three of the Boolean SQSs, and each quadruple with nonzero sum is contained in a unique Boolean system.

The other main ingredient in our construction for $(2^k - 1)v$ PDQs of order $2^k v$ is a set of $3v$ PDQs of order $4v$ with a series of additional properties. These properties are satisfied by the set $B((D_j), (M_i), F)$ constructed in Corollary 1, and also by Lindner's set of PDQs constructed in [13].

Let D_{ij}, $i \in Z_3$, $j \in Z_p$, be the block sets of $3p$ PDQs with point set $Z_v \times Z_4$. We will refer to the subsets $Z_v \times \{x\}$ of the point set as color classes. We will say that the set D_{ij} has *Property Y*, if the following properties hold.

Property Y.1. None of the D_{ij} contains a monochromatic quadruple.

Property Y.2. Each set D_{0j} contains two 2-chromatic subdesigns of order $2v$, one with colors 0 and 1, and the other with colors 2 and 3. Furthermore, no other bichromatic quadruples are contained in D_{0j}.

Each set D_{1j} contains two 2-chromatic subdesigns of order $2v$, one with colors 0 and 2, and the other with colors 1 and 3. Furthermore, no other bichromatic quadruples are contained in D_{1j}.

Each set D_{2j} contains two 2-chromatic subdesigns of order $2v$, one with colors 0 and 3, and the other with colors 1 and 2. Furthermore, no other bichromatic quadruples are contained in D_{2j}.

Property Y.3. Let $c(0, 1) = c(2, 3) = 0$, $c(0, 2) = c(1, 3) = 1$, and $c(0, 3) = c(1, 2) = 2$. For each $j \in Z_p$ and all $m, n \in Z_4$ with $m < n$, the mapping that sends $(x, 0) \to (x, m)$ and $(x, 1) \to (x, n)$ is an isomorphism between the bichromatic quadruples in D_{0j} with colors 0 and 1, and the bichromatic quadruples in $D_{c(m,n)j}$ with colors m and n. In other words,

$$[(x,0),(y,0),(z,0),(t,1)] \in D_{0j} \text{ iff } [(x,m),(y,m),(z,m),(t,n)] \in D_{c(m,n)j},$$

$$[(x,0),(y,0),(z,1),(t,1)] \in D_{0j} \text{ iff } [(x,m),(y,m),(z,n),(t,n)] \in D_{c(m,n)j},$$

$$[(x,0),(y,1),(z,1),(t,1)] \in D_{0j} \text{ iff } [(x,m),(y,n),(z,n),(t,n)] \in D_{c(m,n)j}.$$

If $p = v$ then the sets of bichromatic quadruples in D_{0j} with colors 0 and 1 must form a set of v mutually 2-chromatic PDQs of order $2v$. Thus $3p = 3v$ is the maximum possible size of a set satisfying Property Y.

We now define a set of $(2^k - 1)p$ SQSs with point set $Z_v \times Z_2^k$. Let D_{ij} be a set of $3p$ PDQs of order $4v$ on the point set $Z_v \times Z_4$ with Property Y, and let B_i be the block set of the ith Boolean SQS of order 2^k. We define the block set S_{ij}, $i \in Z_2^k - \{0\}$, $j \in Z_p$, to be the union of the blocks of Types A and B defined below.

Type A.

$$[(x,q),(y,r),(z,s),(w,t)],$$

where

$$x+y+z+w \equiv j (\bmod v) \quad \text{and} \quad q+r+s+t=i, \qquad |\{q,r,s,t\}| = 4.$$

There are $v^3(b_{2^k} - \binom{2^{k-1}}{2})$ blocks of this type, i.e., v^3 blocks for each block of type (B.1) in B_i.

Type B. Let $[q, r, s, t]$ be a block of type (B.2) in B_i with $q < r < s < t$, and define $h(r) = 0$, $h(s) = 1$, and $h(t) = 2$. Since we have a block of type (B.2), we must have $q + i \in \{r, s, t\}$. Now define $g(0) = q$, $g(1) = r$, $g(2) = s$, and $g(3) = t$. For each block $[q, r, s, t]$ of type (B.2) in B_i, and for each block $[(x, a), (y, b), (z, c), (w, d)]$ in $D_{h(q+i)j}$ construct the block

$$[(x,g(a)),(y,g(b)),(z,g(c)),(w,g(d))].$$

There are $\binom{2^{k-1}}{2}(b_{4v} - 2b_{2v}) + 2^{k-1}b_{2v}$ blocks of Type B in S_{ij}, since the number of quadruples in $D_{h(q+i)j}$ is b_{4v}, but the 2-chromatic subsystems on $Z_v \times \{x, x+i\}$, $x \in Z_2^k$, (having b_{2v} blocks) only occur once, because of Property Y.3.

A little algebraic manipulation shows that each S_{ij} has $b_{2^k v}$ blocks.

Note that the blocks of Type B are the blocks of SQSs of order $4v$ on the points $Z_v \times \{q, r, s, t\}$, where $[q, r, s, t]$ is a block in (B.2) of the $(q + r)$th Boolean Steiner quadruple system. As noted above, $[q, r, s, t]$ is also a block of the $(q + s)$th and $(q + t)$th system, but in these three systems the values taken by $h(r)$, $h(s)$, and $h(t)$ are distinct members of Z_3.

To show that each S_{ij} is indeed an SQS we note that any 3-subset $T = \{(x, q), (y, r), (z, s)\}$ of $Z_v \times Z_2^k$ is contained in a unique block by the following arguments:

1) If $|\{q, r, s, q + r + s + i\}| = 4$ then $\{q, r, s\}$ is contained in a unique (B.1) block of the ith Boolean SQS and the block $[(x, q), (y, r), (z, s), (w, t)]$, where $x + y + z + w \equiv j (\mathrm{mod}\ v)$ and $q + r + s + t = i$ is the unique block of Type A in S_{ij}, which contains T. Furthermore, no block of Type B contains T since no pair of members of $\{q, r, s\}$ sums to i.
2) If $|\{q, r, s\}| = 3$ and $q + r + s + i \in \{q, r, s\}$, say $q + r + s + i = s$ (which implies that $q + r = i$) then $\{q, r, s\}$ is contained in a unique (B.2) block of the ith Boolean SQS, namely $[q, r, s, s + i]$. Let t be the minimum element of this block. Now there is a unique block in $D_{h(t+i)j}$ that contains the 3-subset $\{(x, g^{-1}(q)), (y, g^{-1}(r)), (z, g^{-1}(s))\}$ and this generates a unique block of Type B in S_{ij} that contains T. Furthermore, no block of Type A contains T since $|\{q, r, s, q + r + s + i\}| < 4$.
3) If $|\{q, r, s\}| = 2$, say $r = s$, and $q + r \neq i$, then there is a unique (B.2) block of the ith Boolean SQS, namely $[q, q + i, r, r + i]$, that contains the pair $\{q, r\}$. Let t be the minimum element of this block. Now there is a unique block in $D_{h(t+i)j}$ that contains the 3-subset $\{(x, g^{-1}(q)), (y, g^{-1}(r)), (z, g^{-1}(r))\}$ and this generates a unique block of Type B in S_{ij}, which contains T. Furthermore, no block of Type A contains T since $|\{q, r, s\}| < 3$.
4) If $|\{q, r, s\}| = 2$, say $r = s$, and $q + r = i$, then there are $2^{k-1} - 1$ (B.2) blocks of the ith Boolean SQS that contain the pair $\{q, r\}$. However, each of these blocks generates only a single subdesign of order $2v$ on $Z_v \times \{q, r\}$ by property Y.3. This subdesign contains a unique block whose pre-image in D_{0j} under one of the isomorphisms contains the 3-subset $\{(x, 0), (y, 1), (z, 1)\}$ (in the case where $q < r$) or $\{(x, 1), (y, 0), (z, 0)\}$ (in the case where $q > r$). This generates a unique block of Type B in S_{ij}, which contains T. Furthermore, no block of Type A contains T since $|\{q, r, s\}| < 3$.
5) If $|\{q, r, s\}| = 1$, then the argument is similar to the previous case, considering the (B.2) blocks of the ith Boolean SQS that contain the pair $\{q, q + i\}$.

We argue the disjointness of distinct S_{ij}s as follows:

1) Two distinct S_{ij} can only intersect in blocks of the same type, since if $[(x, q), (y, r), (z, s), (w, t)]$ is a block of Type B with $|\{q, r, s, t\}| = 4$ then $q + r + s + t = 0$ and no block of Type A has this property. Furthermore, all blocks of Type A have four distinct second coordinates.
2) The disjointness of $\{x, y, z, w\}$, defined by $x + y + z + w \equiv j (\bmod v)$, for different j's, and the disjointness of $\{q, r, s, t\}$, defined by $q + r + s + t = i$, for different i's guarantees the disjointness of the blocks of Type A.
3) The sets of blocks of Type B are disjoint since, by construction, no block is monochromatic. If $u = [(x, q), (y, r), (z, s), (w, t)]$ is a 2-chromatic block of Type B and say, $q \neq r$, then u can only be a member of S_{ij}, where $i = q + r$. This follows from the fact that all 2-chromatic blocks in any of the S_{ij} have color classes that sum to i. The uniqueness of the subscript j is then implied by the disjointness of the $D_{0j}, j \in Z_p$. Property Y.2 ensures that the only blocks of this form come from the induced subsystems of order $2v$.

   If $|\{q, r, s, t\}| > 2$ with say, $|\{q, r, s\}| = 3$, then u can only be a member of S_{ij}, where $i = q + r$ or $q + s$ or $r + s$. Again the uniqueness of the system S_{ij} containing u is guaranteed by the disjointness of the D_{nj}, $n \in Z_3, j \in Z_p$.

Hence, we have the following theorem.

THEOREM 2. *If there exists a set of* $3p$ PDQs *of order* $4v$ *with Property* Y *then* $D(2^k v) \geqq (2^k - 1)p$ *for all* $k \geqq 2$.

Since the $3v$ PDQs constructed in the previous section, and also those of Lindner [13], satisfy property Y, we have the following corollary.

COROLLARY 2. $D(2^k v) \geqq (2^k - 1)v$ *for* $v \equiv 2$ *or* $4 \pmod 6$, $v \geqq 8$, *for* $v = 5^a.13^b.17^c$, $a, b, c \geqq 0$, *and* $a + b + c > 0$, *for* $v = 11$, *and for all* $k \geqq 2$.

Finally, we note that if the set of $3v$ PDQs of order $4v$ is maximal then the set of $(2^k - 1)v$ PDQs of order $2^k v$ is also maximal.

4. Other recursive constructions. In this section we show that some of the recursive constructions for SQS can be utilized to give lower bounds on $D(v)$. The bounds we obtain are not very good, but aside from the constructions of Phelps [17], [18], and Phelps and Rosa [19] no other constructions are known that give nontrivial bounds on $D(v)$ when v is not divisible by 4.

The first construction we give is a version of the tripling construction in [7]. A quadruple system of order v with a *hole* of order s, denoted by SQS(v:s), is a triple (X, S, q), where X is a set of size v, S is a subset of X of size s, and q is a set of 4-subsets of X, called blocks, such that every 3-subset $T \subset X$ with $|T \cap S| < 3$ is contained in a unique block, and *no* 3-subset $T \subset S$ is contained in any block. Two systems (X, S, q_1) and (X, S, q_2) are disjoint if $q_1 \cap q_2 = \varnothing$. Let $D(v{:}s)$ denote the number of pairwise disjoint quadruple systems of order v with a hole of order s. (Note that $D(v{:}s) = D(v)$ since a 2-subset S contains no 3-subsets.)

Let $v = 6n + 2s$, let $q_1, q_2, \cdots, q_{D(v)}$ be a set of PDQs of order v, and let $r_1, r_2, \cdots, r_{D(v:s)}$ be a set of pairwise disjoint systems of order v with a hole of size s.

For $x \in Z_n = \{0, 1, \cdots, n-1\}$, we define the notation $|x| = \min(x, n - x)$. Let $F_1, F_2, \cdots, F_{4n+s-1}$ be a one-factorization of the graph with vertex set Z_{6n+s} and edge set containing all pairs $\{x, y\}$ such that $x \equiv y \pmod 2$ or $|x - y| \geqq 2n + 1$. (This graph has a one-factorization by a result of Stern and Lenz [21], as do all other graphs that we factorize in this section.) Finally, let $\alpha(i, j)$ be a Latin square of order $4n + s - 1$ with symbol set $\{1, 2, \cdots, 4n + s - 1\}$, and let $S = \{\infty_0, \infty_1, \cdots, \infty_{s-1}\}$.

We now construct a set of $\min(D(v), D(v{:}s), (2v - s - 3)/3)$ PDQs of order $3v - 2s$ with point set $S \cup (Z_{6n+s} \times Z_3)$. (Note that $(2v - s - 3)/3 = 4n + s - 1$.) The block set of the jth SQS is constructed as follows.

Type 1a. Construct the blocks of q_j on the point set $S \cup (Z_{6n+s} \times \{0\})$.

Type 1b. Construct the blocks of r_j on the point sets $S \cup (Z_{6n+s} \times \{i\})$ for each $i = 1, 2$.

Type 2. For each $i = 0, 1, \cdots, s - 1$, and each $a, b, c \in Z_{6n+s}$ such that $a + b + c \equiv 6n + i + j \pmod{6n + s}$, construct the block

$$[\infty_i, (a,0), (b,1), (c,2)].$$

Type 3. For each $i = 0, 1, 2$ and each $a, b, c \in Z_{6n+s}$ such that $a + b + c \equiv 2ni + j \pmod{6n + s}$ and each $k = 0, 1, \cdots, n - 1$, construct the block

$$[(a+k,i), (a+2n-1-k,i), (b,i+1), (c,i+2)].$$

Type 4. For each $i = 0, 1, 2$ and each $k = 1, 2, \cdots, 4n + s - 1$ and each $\{a, b\} \in F_k$ and each $\{c, d\} \in F_{\alpha(j,k)}$ construct the block

$$[(a,i), (b,i), (c,i+1), (d,i+1)].$$

To prove that we have indeed constructed a set of PDQs of order $3v - 2s$ we note that the systems constructed can only intersect in blocks of the same type. Furthermore, as we verify that each system has a unique block containing each triple, we verify that this block is different for distinct values of j. This will establish the disjointness of the systems. We remark that the number of disjoint sets of blocks of Type 4 is limited by the number of rows in a Latin square of side $4n + s - 1$.

We now verify that the jth system has a unique block containing each 3-subset, T, of its point set.

1) If $T \subset S \cup (Z_{6n+s} \times \{0\})$ then T is contained in a unique block of Type 1a in q_j.
2) If $T \subset S \cup (Z_{6n+s} \times \{i\})$ with $i = 1$ or 2, and $T \not\subset S$, then T is contained in a unique block of Type 1b in r_j.
3) If $T = \{\infty_i, (a, k), (b, k + 1)\}$ then T is contained in a unique block of Type 2 whose fourth point is $(6n + i + j - a - b, k + 2)$.
4) If $T = \{(a, 0), (b, 1), (c, 2)\}$ then
   a) If $a + b + c = 6n + j + x$ with $0 \leqq x \leqq s - 1$, then T is contained in a unique block of Type 2 whose fourth point is ∞_x.
   b) If $a + b + c \neq 6n + j + x$ with $0 \leqq x \leqq s - 1$, then T is contained in a unique block of Type 3 whose fourth point is (d, i) where i is given by the interval $[j + 2ni, j + 2ni + 2n - 1]$ that contains $a + b + c$. Now we compute d as follows. Say $i = 0$ and $a + b + c = j + k$ $(0 \leqq k < n)$ then $d = a + 2n - 1 - 2k$. If $a + b + c = j + 2n - 1 - k$ then $d = a + 2k + 1 - 2n$. A similar argument works for $i = 1, 2$.
5) If $T = \{(a, i), (b, i), (c, i + 1)\}$ then
   a) If $|b - a| = 2k + 1$ for some $0 \leqq k < n$ then T is contained in a unique block of Type 3 whose fourth point is $(j + 2ni - c - a + k, i + 2)$, where $b - a = 2k + 1$.
   b) If $|b - a| \neq 2k + 1$ for some $0 \leqq k < n$ then T is contained in a unique block of Type 4 whose fourth point $(d, i + 1)$ is computed by noting that $\{a, b\}$ is in a unique one-factor, say F_m, and d is the second point of the unique edge containing c in the factor $F_{\alpha(j,m)}$.

6) If $T = \{(a, i), (b, i), (c, i-1)\}$ then the argument is similar to the previous case.

This proves the following theorem.

THEOREM 3. *If* $s \equiv 2$ *or* $4 \pmod 6$ *and* $v \equiv 2s \pmod 6$ *then*

$$D(3v-2s) \geqq \min(D(v), D(v{:}s), (2v-s-3)/3).$$

When $s = 2$ this gives the inequality $D(3v-4) \geqq \min(D(v), (2v-5)/3)$ for all $v \equiv 4 \pmod 6$.

Using systems with holes in place of the blocks of Type 1a yields the inequality $D(3v-2s{:}s) \geqq \min(D(v{:}s), (2v-s-3)/3)$. Omitting the blocks of Type 1a gives the same lower bound for $D(3v-2s{:}v)$.

We now show that there exists a pair of PDQs of order 38, each of which contains three subsystems of order 14 that intersect in two common points. This construction is vital to our proof of the Lindner and Rosa conjecture.

To simplify the discussion we introduce the notation $G(n, L)$ to denote the graph with vertex set Z_n and edges $\{x, y\}$ for all $|x-y| \in L$.

Example 1. We construct two disjoint SQS(38) with point set $\{\infty_0, \infty_1\} \cup (Z_{12} \times Z_3)$. The block set of the jth SQS(38) ($j = 0, 1$) is constructed as follows:

Type 1. Let q_0 and q_1 be two disjoint SQS(14). (It is known [1] that $D(14) \geqq 4$.) Construct the blocks of q_j on the point sets $\{\infty_0, \infty_1\} \cup (Z_{12} \times \{i\})$ for each $i \in Z_3$.

Type 2. Let $X(0) = (0, 3)$ and let $X(1) = (3, 6)$. For each $i = 0, 1$ and each $a, b, c \in Z_{12}$ such that $a+b+c \equiv x \pmod{12}$, and x is the ith coordinate of $X(j)$, construct the block

$$[\infty_i, (a,0), (b,1), (c,2)].$$

Type 3. Let $P(i, 0) = \{\{6+i, 9+i\}\}$ and $P(i, 1) = \{\{8+i, 11+i\}\}$ for $i = 0, 1, 2$. For each $i = 0, 1, 2$ and each $a, b, c \in Z_{12}$ such that $a+b+c \equiv 0 \pmod{12}$, and $\{x, y\} \in P(i, j)$, construct the block

$$[(a+x,i), (a+y,i), (b,i+1), (c,i+2)].$$

Type 4. We first define two one-factorizations F_k^j of the graphs $G(12, \{1, 5, 6\})$ (for $j = 0$) and $G(12, \{2, 4, 6\})$ (for $j = 1$). Let F_0^0, F_1^0 be the one-factorization of $G(12, \{1\})$, let F_2^0, F_3^0 be the one-factorization of $G(12, \{5\})$, and let F_4^0 be the edge set of $G(12, \{6\})$. Let F_0^1, F_1^1, F_2^1 be the one-factorization of $G(12, \{4, 6\})$, defined by $F_i^1 = \{\{3\varepsilon+i, 3\varepsilon+i+4\}, \{3n+i+2, 3n+i+8\} : \varepsilon = 0, 1, 2, 3, n = 0, 1\}$, and let F_3^1, F_4^1 be either of the two one-factorizations of $G(12, \{2\})$.

For each $i = 0, 1, 2$ and each $k = 0, 1, \cdots, 4$ and each $\{a, b\} \in F_k^j$ and each $\{c, d\} \in F_{k+2}^j$ (addition mod 5 in the subscript) construct the block

$$[(a,i), (b,i), (c,i+1), (d,i+1)].$$

Type 5. Let $H_0 = \{\{1, 5\}, \{2, 4\}\}$ and $H_1 = \{\{2, 7\}, \{4, 5\}\}$. For each $i = 0, 1, 2$ and each $a \in Z_{12}$ and each $\varepsilon = 0, 1, 2, 3$ and each $\{x, y\} \in H_j$ construct the block

$$[(a,i+1), (a+3\varepsilon, i+2), (x-2a-3\varepsilon, i), (y-2a-3\varepsilon, i)].$$

Type 6. Let $D_0 = \{2, 4\}$ and $D_1 = \{1, 5\}$. For each $i = 0, 1, 2$ and each $a \in Z_{12}$ and each $\varepsilon = 0, 1, 2, 3$ and each $d \in D_j$ construct the block

$$[(a,i), (a+d,i), (a+3\varepsilon, i+1), (a+d+3\varepsilon, i+1)].$$

Verification that the systems constructed are both SQS(38)s is similar to the previous construction, and full details can be found in [8]. The disjointness of the systems is a

little more complicated since there is the possibility of conflicts between blocks of Types 4 and 6. These conflicts are avoided by our careful construction of the one-factorizations and their ordering in the Type 4 quadruples.

We have obtained a generalization of this example using the tripling construction of [8] and obtained a proof of the following theorem.

THEOREM 4. *If* $s \equiv 2$ *or* $4 \pmod 6$ *and* $v \equiv s \pmod 6$ *then* $D(3v-2s) \geqq \min(D(v), D(v{:}s), 2)$.

This theorem is not necessary for our proof of the Lindner and Rosa conjecture, and since the proof is messy, we omit it. Theorem 4 can probably be strengthened to give the inequality $D(3v-2s) \geqq \min(D(v), D(v{:}s), f(v, s))$ for $v \equiv s \pmod 6$, and some function satisfying $(v-s)/6 \leqq f(v, s) \leqq (v-s)/3$. However, the proof of this inequality would be extremely tedious.

Colbourn and Hartman [2] have used the construction of Theorem 4 (with $v = 10$, $s = 4$, and omitting the blocks of Type 1a) to construct a pair of SQS(22:10) designs that intersect in precisely two blocks. Hartman and Yehudai [10] were able to modify this construction to produce a pair of disjoint SQS(22:10)s. The importance of this construction is that Colbourn and Hartman proved that if $D(22{:}10) \geqq 2$ then $D(v) \geqq 2$ for all $v \equiv 10 \pmod{12}$ with $v \geqq 46$. Thus, we have the following theorem.

THEOREM 5 (Colbourn, Hartman, and Yehudai). $D(v) \geqq 2$ *for all* $v \equiv 10 \pmod{12}$ *with* $v \geqq 46$.

We turn now to the quadrupling construction (construction 3.5 of [5]), and we will show that $D(4v-6) \geqq \min(D(v), (v-2)/2)$. Throughout this section we let $v = 2f+2$ and we will construct $\min(D(v), f)$ PDQs with point set $\{\infty_0, \infty_1\} \cup (Z_{2f} \times Z_4)$. A major ingredient of the construction is a partition of the edge set of K_{2f} into $2f$ parts $G_0, G_1, \cdots, G_{f-1}$ and $H_0, H_1, \cdots, H_{f-1}$ with the property that each H_i is a one-factor, and $G_i \cup \{\{2i, 2i+1\}\}$ is a one-factor for all $0 \leqq i < f$. These partitions were constructed by Hanani for all $f \geqq 1$ in [5]. The other ingredient is a standard one-factorization $F_0, F_1, \cdots, F_{2f-2}$ of K_{2f}. Let $q_1, q_2, \cdots, q_{D(v)}$ be a set of PDQs of order v. The block set of the jth SQS is constructed as follows:

Type 1. Construct the blocks of q_j on the point sets $\{\infty_0, \infty_1\} \cup (Z_{2f} \times \{i\})$ for each $i \in Z_4$.

Type 2. The other blocks containing ∞_0 and ∞_1 are the following:

$$\{[\infty_0,(a,0),(b,1),(c,2)]: a+b+c \equiv 2j \pmod{2f} (a,b,c) \equiv (0,0,0) \pmod 2\},$$

$$\{[\infty_0,(a,0),(b,1),(c,2)]: a+b+c \equiv 2j+1 \pmod{2f} (a,b,c) \equiv (1,1,1) \pmod 2\},$$

$$\{[\infty_0,(a,0),(b,1),(c,3)]: a+b+c \equiv 2j+2 \pmod{2f} (a,b,c) \equiv (1,0,1) \pmod 2\},$$

$$\{[\infty_0,(a,0),(b,1),(c,3)]: a+b+c \equiv 2j+1 \pmod{2f} (a,b,c) \equiv (0,1,0) \pmod 2\},$$

$$\{[\infty_0,(a,0),(b,2),(c,3)]: a+b+c \equiv 2j \pmod{2f} (a,b,c) \equiv (0,1,1) \pmod 2\},$$

$$\{[\infty_0,(a,0),(b,2),(c,3)]: a+b+c \equiv 2j+1 \pmod{2f} (a,b,c) \equiv (1,0,0) \pmod 2\},$$

$$\{[\infty_0,(a,1),(b,2),(c,3)]: a+b+c \equiv 2j+2 \pmod{2f} (a,b,c) \equiv (1,0,1) \pmod 2\},$$

$$\{[\infty_0,(a,1),(b,2),(c,3)]: a+b+c \equiv 2j+1 \pmod{2f} (a,b,c) \equiv (0,1,0) \pmod 2\},$$

$$\{[\infty_1,(a,0),(b,1),(c,2)]: a+b+c \equiv 2j \pmod{2f} (a,b,c) \equiv (1,1,0) \pmod 2\},$$

$$\{[\infty_1,(a,0),(b,1),(c,2)]: a+b+c \equiv 2j+1 \pmod{2f} (a,b,c) \equiv (0,0,1) \pmod 2\},$$

$$\{[\infty_1,(a,0),(b,1),(c,3)]: a+b+c \equiv 2j+2 \pmod{2f} (a,b,c) \equiv (0,1,1) \pmod 2\},$$

$\{[\infty_1,(a,0),(b,1),(c,3)]: a+b+c\equiv 2j+1 \pmod{2f}(a,b,c)\equiv(1,0,0)\pmod 2\}$,

$\{[\infty_1,(a,0),(b,2),(c,3)]: a+b+c\equiv 2j \pmod{2f}(a,b,c)\equiv(0,0,0)\pmod 2\}$,

$\{[\infty_1,(a,0),(b,2),(c,3)]: a+b+c\equiv 2j+1 \pmod{2f}(a,b,c)\equiv(1,1,1)\pmod 2\}$,

$\{[\infty_1,(a,1),(b,2),(c,3)]: a+b+c\equiv 2j+2 \pmod{2f}(a,b,c)\equiv(1,1,0)\pmod 2\}$,

$\{[\infty_1(a,1),(b,2),(c,3)]: a+b+c\equiv 2j+1 \pmod{2f}(a,b,c)\equiv(0,0,1)\pmod 2\}$.

Type 3. Construct the following blocks using the Hanani factorization:

$\{[(a,0),(b,1),(x,2),(y,2)]: a\equiv b \pmod 2 a+b+2c\equiv 2j \pmod{2f}\{x,y\}\in G_c\}$,

$\{[(a,0),(b,1),(x,3),(y,3)]: a\not\equiv b \pmod 2 a+b+2c\equiv 2j+1 \pmod{2f}\{x,y\}\in G_c\}$,

$\{[(a,2),(b,3),(x,0),(y,0)]: a\equiv b \pmod 2 a+b+2c\equiv 2j \pmod{2f}\{x,y\}\in G_c\}$,

$\{[(a,2),(b,3),(x,1),(y,1)]: a\not\equiv b \pmod 2 a+b+2c\equiv 2j+1 \pmod{2f}\{x,y\}\in G_c\}$,

$\{[(a,0),(b,1),(x,2),(y,2)]: a\not\equiv b \pmod 2 a+b+2c\equiv 2j+1 \pmod{2f}\{x,y\}\in H_c\}$,

$\{[(a,0),(b,1),(x,3),(y,3)]: a\equiv b \pmod 2 a+b+2c\equiv 2j \pmod{2f}\{x,y\}\in H_c\}$,

$\{[(a,2),(b,3),(x,0),(y,0)]: a\not\equiv b \pmod 2 a+b+2c\equiv 2j+1 \pmod{2f}\{x,y\}\in H_c\}$,

$\{[(a,2),(b,3),(x,1),(y,1)]: a\equiv b \pmod 2 a+b+2c\equiv 2j \pmod{2f}\{x,y\}\in H_c\}$.

Type 4. Let $\alpha(i,j)$ be a Latin square of order $2f-1$ over the symbol set $\{0, 1, \cdots, 2f-2\}$, and construct the following blocks using the one-factorization:

$$\{[(a,0),(b,0),(x,1),(y,1)]: \{a,b\}\in F_i\{x,y\}\in F_{\alpha(i,j)} 0\leqq i<2f-1\}$$

$$\{[(a,2),(b,2),(x,3),(y,3)]: \{a,b\}\in F_i\{x,y\}\in F_{\alpha(i,j)} 0\leqq i<2f-1\}.$$

The full details of verification that the $j = 0$th system is actually an SQS are contained in Hanani's paper, and the verification for $j > 0$ is almost identical. The disjointness of the systems is guaranteed by the dependence of the constructions on the parameter j and can be easily verified. To assist the reader we give the verification argument for a few representative cases and note that the symmetries of the construction make the argument for the remaining cases a simple exercise.

If T is a 3-subset of $\{\infty_0, \infty_1\} \cup (Z_{2f}\times\{i\})$ for some $i\in Z_4$ then T is contained in a unique block of Type 1.

If $T = \{\infty_0, (a, 0), (b, 1)\}$ then according to the parities of a and b there is a unique block of Type 2 containing T.

If $T = \{(a, 0), (b, 1), (c, 2)\}$, then

1) If $a \equiv b \equiv i \pmod 2$ then
   a) If $a + b + c \equiv 2j \pmod{2f}$ then the fourth point of the block of Type 2 containing T is ∞_i.
   b) If $a + b + c \equiv 2j + 1 \pmod{2f}$ then the fourth point of the block of Type 2 containing T is ∞_{1-i}.
   c) If $a + b + c \not\equiv 2j$ or $2j + 1 \pmod{2f}$ then the fourth point, $(d, 2)$, of the block of Type 3 containing T is computed as follows: Let x be the unique solution to the equation $a + b + x \equiv 2j \pmod{2f}$. Now x is even, and there is a unique edge $\{c, d\}$ in $G_{x/2}$ that contains c since, by assumption $c \neq x$ or $x + 1$.

2) If $a \not\equiv b \pmod 2$ then the fourth point, $(d, 2)$, of the block of Type 3 containing T is given by the unique edge $\{c, d\}$ in $H_{x/2}$ that contains c, where x is the unique solution to the equation $a + b + x \equiv 2j + 1 \pmod{2f}$.

If $T = \{(a, 0), (b, 0), (c, 2)\}$ then the edge $\{a, b\}$ is contained either in G_x or H_x for some x, and the fourth point $(d, 3)$, of the block of Type 3 containing T is given by the solution to $2x + c + d \equiv 2j \pmod{2f}$, (if $\{a, b\} \in G_x$) or $2x + c + d \equiv 2j + 1 \pmod{2f}$, (if $\{a, b\} \in H_x$).

If $T = \{(a, 0), (b, 0), (c, 1)\}$ then the edge $\{a, b\}$ is contained in a unique one-factor F_x and the fourth point $(d, 1)$ of the block of Type 4 containing T is given by the unique edge $\{c, d\}$ in $F_{\alpha(x,j)}$ that contains c.

We have thus indicated the proof of the following theorem.

THEOREM 6. *If* $v \equiv 2$ *or* $4 \pmod 6$ *then* $D(4v - 6) \geqq \min(D(v), (v - 2)/2)$.

Theorem 6 can certainly be generalized to give the inequality $D(4v - 3s) \geqq \min(D(v), D(v{:}s), (v - s)/2)$ but, again, the proof is tedious.

We now apply the singular direct product construction for quadruple systems (Proposition 8 of [6]) to obtain other recursive bounds on $D(v)$ as follows.

THEOREM 7. *If* $n \equiv 1$ *or* $3 \pmod 6$ *and* $v \equiv 4 \pmod 6$ *then* $D(n(v - 2) + 2) \geqq \min(D(v), (2v - 5)/3)$.

Proof. Let $(Z_n \cup \{\infty\}, q)$ be an SQS of order $n + 1$. We construct the jth quadruple system on the point set $\{\infty_0, \infty_1\} \cup (Z_{v-2} \times Z_n)$ as follows: For each point $x \in Z_n$ construct the blocks of the jth quadruple system in a set of PDQs of order v on the point set $\{\infty_0, \infty_1\} \cup (Z_{v-2} \times \{x\})$. For each block $[\infty, x, y, z] \in q$ that contains ∞ use the construction of Theorem 3 on the point set $\{\infty_0, \infty_1\} \cup (Z_{v-2} \times \{x, y, z\})$, omitting the blocks of Type 1. For each block $[x, y, z, t] \in q$ that does not contain ∞ construct the blocks:

$$\{[(a,x),(b,y),(c,z),(d,t)] : a+b+c+d \equiv j \pmod{v-2}\}.$$

Theorem 7 implies, for example, that $D(142) \geqq 11$ and $D(302) \geqq 11$ using $v = 22$ and $n = 7$ and 15, respectively.

A similar proof also gives the following generalization of Construction 3.6 of [5].

THEOREM 8. *If* $n \equiv 1$ *or* $3 \pmod 6$ *then* $D(12n + 2) \geqq 2$.

The proof of Theorem 8 is identical to the proof of the previous theorem, using the systems of order 38 constructed in Example 1 in place if the systems constructed in Theorem 3.

We are now in a position to prove the Lindner and Rosa conjecture.

THEOREM 9. $D(v) \geqq 2$ *for all* $v \equiv 2$ *or* $4 \pmod 6$, $v \geqq 8$.

Proof. If $v \equiv 4$ or $8 \pmod{12}$ and $v \geqq 16$ then Lindner [12] proved that $D(v) \geqq v/2$; furthermore, it is well known that $D(8) = 2$. If $v \equiv 10 \pmod{12}$ and $v \geqq 46$, the result follows from Theorem 5; the remaining values $v \in \{10, 22, 34\}$ are covered by [11], [18], and [19]. If $v \equiv 2 \pmod{24}$ then $8 \leqq (v + 6)/4 \equiv 2 \pmod 6$ and the result follows from Theorem 6 and the induction hypothesis. If $v \equiv 14$ or $38 \pmod{72}$ then $v = 12n + 2$ for some $n \equiv 1$ or $3 \pmod 6$ and the result follows from Theorem 8. Finally, if $v \equiv 62 \pmod{72}$ then $22 \leqq (v + 4)/3 \equiv 4 \pmod 6$ and the result follows from Theorem 3 with $s = 2$, and the induction hypothesis.

One final result for improving the known bounds on $D(v)$ uses the notion of an $H(m, g, k, t)$ design. An $H(m, g, k, t)$ design is a triple $(X, \mathbf{G}, \mathbf{B})$, where X is a set of points whose cardinality is mg, and $\mathbf{G} = \{G_1, G_2, \cdots, G_m\}$ is a partition of X into m sets of cardinality g; the members of $\mathbf{G}$ are called *groups*. A *transverse* of $\mathbf{G}$ is a subset of X that meets each group in at most one point. The set $\mathbf{B}$ contains k-element transverses

of $\mathbf{G}$, called *blocks*, with the property that each t-element transverse of $\mathbf{G}$ is contained in precisely one block.

When $g = 1$ then an $H(m, 1, k, t)$ is just a Steiner system $S(t, k, m)$ and when $k = m$ then an $H(k, g, k, t)$ is equivalent to an $OA(t, k, g)$.

The technique for enlarging the group size of an H-design gives sets of (block) disjoint designs as follows.

LEMMA 3. *If there exists an $H(m, g, 4, 3)$ then there exist n disjoint $H(m, ng, 4, 3)$ designs.*

Proof. If $(X, \mathbf{G}, \mathbf{B})$ is an $H(m, g, 4, 3)$ design then form the new designs on the point set $X \times Z_n$, with groups $G_i \times Z_n$ and, for each block $\{x, y, z, t\} \in \mathbf{B}$, form the n^3 blocks $[(x, a), (y, b), (z, c), (t, d)]$, where $a + b + c + d \equiv j \pmod n$. Letting j range over Z_n gives n disjoint designs.

This lemma is used in the proof of the following theorem.

THEOREM 10. *If there exists an $H(m, g, 4, 3)$ and $ng \equiv 2$ or $4 \pmod 6$ then $D(nmg) \geqq \min(D(ng), n)$.*

Proof. Let $\mathbf{B}_1, \mathbf{B}_2, \cdots, \mathbf{B}_n$ be the block sets of n disjoint $H(m, ng, 4, 3)$ designs, let $q_1, q_2, \cdots, q_{D(ng)}$ be the block sets of PDQs of order ng, and let $F_1, F_2, \cdots, F_{ng-1}$ be a one-factorization of K_{ng}. Finally, let $\alpha(j, k)$ be a Latin square of side $ng - 1$. (In fact, it is sufficient to have an $(ng - 1) \times D(ng)$ Latin rectangle.)

The block set of the j-th SQS consists of $\mathbf{B}_j$, and a copy of q_j on each of the groups of the $H(m, ng, 4, 3)$. A final group of blocks is given by constructing a copy of the one-factorization on each of the groups, and for each pair of distinct groups G_x and G_y $(x < y)$ forming the blocks $\{a, b, c, d\}$, where $\{a, b\}$ is a member of F_k on G_x and $\{c, d\}$ is a member of $F_{\alpha(j,k)}$ on G_y.

We can apply Theorem 10 to give new bounds on $D(v)$ if, for example, we have an $H(m, 2, 4, 3)$ with $m \equiv 1$ or $5 \pmod 6$. Hartman, Mills, and Mullin [9] have shown

TABLE 1
The best known lower bounds for $D(v)$.

v	$D(v) \geq$	Reference	v	$D(v) \geq$	Reference	v	$D(v) \geq$	Reference
4	1	trivial	38	2	Example 1	70	5	Thm. 10
8	2	folklore	40	35	Thm. 2	74	9	Thm. 6
10	5	[11]	44	33	Thm. 1	76	40	[4]
14	4	[1]	46	2	Thm. 5	80	75	Thm. 2
16	8	[12]	50	25	[17]	82	10	Thm. 6
20	15	Thm. 1	52	39	Thm. 1	86	2	Thm. 8
22	11	[18]	56	42	[13]	88	77	Thm. 2
26	13	[19]	58	7	Thm. 6	92	48	[4]
28	18	[4]	62	11	Thm. 3	94	2	Thm. 5
32	24	[13]	64	56	Thm. 2	98	12	Thm. 6
34	17	[19]	68	51	Thm. 1	100	75	Thm. 1

that there exists an $H(m, 2, 4, 3)$ for all $m \equiv 1$ or $7 \pmod{18}$. So we have $D(70) \geqq 5$ using $m = 7$, $g = 2$, and $n = 5$ in Theorem 10.

Mills [16] has also constructed the designs $H(11, 2, 4, 3)$ and $H(13, 2, 4, 3)$. Using the techniques of [9] and [15] together with an $H(13, 2, 4, 3)$ one can show the existence of $H(m, 2, 4, 3)$ for all $m \equiv 1 \pmod 6$. Thus Theorem 10 is applicable to all orders $v = 2nm$ with $m \equiv 1 \pmod 6$ and $n > 1$. The cases where $m \equiv 5 \pmod 6$ are more complicated, since no $H(5, 2, 4, 3)$ exists.

Conclusions. Two major open problems, originally posed by Lindner and Rosa [14] in 1978, have been tackled here. The first problem is the construction of a large set of PDQs of some order v. We have shown that one can get $v - 5$ PDQs of order $v = 5.2^n$ $n \geqq 1$. We have also shown that $D(v) \geqq (1 - \varepsilon)v$ for infinitely many v and any $\varepsilon > 0$. Unfortunately, the existence of a large set still remains an open problem.

The second problem is to show that $D(v) \geqq 2$ for all admissible values of $v \geqq 8$. We have solved this problem, and in many cases we have given even better lower bounds on $D(v)$. The state of the art for $v \leqq 100$ is given in Table 1.

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