On the other hand, the effect of reservation overhead on maximum throughput is more profound. Fig. 9 shows S_{max} versus M with h as a parameter. We see that initially S_{max} increases with M (which is expected). However, as M increases beyond

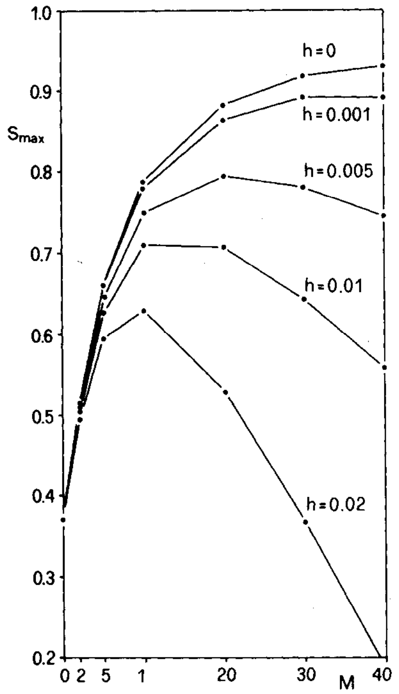

Fig. 10. Maximum throughput of DF as function of M.

some critical value (depending on h), the reservation overhead gets so large that S_{max} starts to drop. Thus for a given value of h, there exists an M that maximizes the throughput. In general, for h not too large, the throughput is maximized for a broad range of M. Fig. 10 shows the same for the DF protocol. We see that for h not to large, say equal to 0.1 percent of the packet length, a maximum throughput of 0.89 can be obtained.

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New Lower Bounds for Constant Weight Codes

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***Abstract*—Some new lower bounds are given for $A(n,4,w)$, the maximum number of codewords in a binary code of length n, minimum distance 4, and constant weight w. In a number of cases the results significantly improve on the best bounds previously known.**

I. Introduction

In this correspondence we present a method of finding lower bounds for $A(n,4,w)$, the maximum number of codewords in a binary code of length n, maximum distance 4, and constant weight w. Graham and Sloane [1] gave the first lower bound for $A(n,4,w)$:

$$A(n,4,w) \geq \frac{1}{n}\binom{n}{w}. \tag{1}$$

Their proof is based on a mapping T: $\boldsymbol{F}_w^n \to \boldsymbol{Z}_n$, where $\boldsymbol{F}_w^n$ denotes the set of $\binom{n}{w}$ binary vectors of length n and weight w and $\boldsymbol{Z}_n = \boldsymbol{Z}/n\boldsymbol{Z}$ denotes the residue classes modulo n. The mapping is $T(b_1, b_2, \cdots, b_n) \equiv \sum_{i=1}^{n} ib_i \pmod{n}$, and for $1 \leq i \leq n$ the code $C_i = T^{-1}(i-1)$ is a constant weight code with distance 4. Clearly,

$$A(n,4,w) \geq \max_{1 \leq i \leq n} |C_i| \tag{2}$$

and (1) is an immediate consequence of (2).

Kløve [2] observed that we can take other additive groups instead of $\boldsymbol{Z}_n$ and found larger codes in a few cases. Other bounds for $A(n,4,w)$ were provided by Kibler [3], Brouwer [4], and Delsarte and Piret [5].

The codes given by (2) and by Kløve [2] are not significantly better than the ones obtained from (1). In the present correspondence we show that in many cases we can partition $\boldsymbol{F}_w^n$ into n classes so that each class i is a constant weight code C_i with distance 4, $\bigcup_{i=1}^{n} C_i = \boldsymbol{F}_w^n$, and $\max_{1 \leq i \leq n}|C_i|$ is fairly large, and that in many cases we can partition $\boldsymbol{F}_w^n$ into $n-1$ classes so that each class i is a constant weight code C_i with distance 4, $\bigcup_{i=1}^{n-1} C_i = \boldsymbol{F}_w^n$, and $\max_{1 \leq i \leq n-1}|C_i|$ is large. Using these partitions we were in many cases able to improve on the lower bounds obtained from (1) and (2).

In Section II we present the new method of obtaining lower bounds for $A(n,4,w)$, a table of new lower bounds, and constructions for partitioning $\boldsymbol{F}_w^n$. In the Appendix we present partitions of $\boldsymbol{F}_w^n$ which were used to obtain the new lower bounds.

II. The New Lower Bounds

For the representation of our results we need some definitions. An (n,d,w) code is a code of length n, constant weight w, and distance d. A set of codes will be called *disjoint* if the intersection of any two different members of the set is empty. Let $\boldsymbol{F}^n$ denote the set of all binary n-tuples. A *partition* $\Pi(n)$ of $\boldsymbol{F}^n$ is a set of k subsets (called classes), $A_1, A_2, \cdots, A_k$, such that each

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set A_i is an $(n,4,w)$ code for some w, $A_i \cap A_j = \varnothing$ for $i \neq j$, and $\bigcup_{i=1}^{k} A_i = F^n$. $\Pi(n,w)$ is the set of classes in $\Pi(n)$ which are $(n,4,w)$ codes. Let $r = |\Pi(n,w)|$ and $\Pi(n,w) = \{A_1, A_2, \cdots, A_r\}$, where the A_i are disjoint $(n,4,w)$ codes; then $\text{index}_{\Pi}(n,w)$ is the vector $(i_1, i_2, \cdots, i_r)$ of length r, with $i_j = |A_j|$. A word $(\alpha,\beta) \in A \times B$, where $\alpha \in A$ and $\beta \in B$, is a (w_1, w_2) configuration if α has weight w_1 and β has weight w_2. First, we give some results on partitions, especially for small weights. The first lemma is an immediate result.

Lemma 1: If in a partition $\Pi(n,w)$ we complement all of the words, we obtain a partition $\Pi(n, n-w)$.

The second lemma is an immediate consequence of the existence of one-factorization of the complete graph [6].

Lemma 2: F_2^{2k} and F_2^{2k-1} can be partitioned into $2k-1$ classes of size k and size $k-1$, respectively.

A partition of F_3^n that contains as many disjoint $(n,4,3)$ codes attaining $A(n,4,3)$ as possible is more difficult to find. For $n = 6m+1$ or $n = 6m+3$ a partition can be obtained by finding a packing with Steiner triple systems. A packing of order 9 was discovered by Kirkman [7]. Schreiber [8] discovered a construction for packing of order $p+2$, whenever the prime factors of p are all congruent to 7 modulo 8. Denniston [9] gave a solution for $n = 13,15,17,21,25,31,33,43,49,61,69$. Teirlinck [10] has shown how to construct a packing of order $3v$, when a packing of order v is given. The following lemmas (which can be easily verified by simple combinatorial arguments) lead to our conclusions.

Lemma 3: If A and B are $(n_1,4,w_1)$ and $(n_2,4,w_2)$ codes, respectively, then $A \times B$ is an $(n_1+n_2, 4, w_1+w_2)$ code.

Lemma 4: Let A_1 and A_2 be disjoint $(n_1,4,w_1)$ codes, and B_1 and B_2 disjoint $(n_2,4,w_2)$ codes. Then $(A_1 \times B_1) \cup (A_2 \times B_2)$ is an $(n_1+n_2,4,w_1+w_2)$ code.

Lemma 5: If A_i and B_i, $1 \le i \le k$, are disjoint $(n_1,4,w_1)$ codes and disjoint $(n_2,4,w_2)$ codes, respectively, then $\bigcup_{i=1}^{k}(A_i \times B_i)$ is an $(n_1+n_2,4,w_1+w_2)$ code.

Lemma 6: If A_i and B_j, $1 \le i \le k_1$, $1 \le j \le k_2$, are disjoint $(n_1,4,w_1)$ codes and disjoint $(n_2,4,w_2)$ codes, respectively, then $\bigcup_{i,j}(A_i \times B_j)$ can be partitioned into $k_3 = \max(k_1,k_2)$ disjoint $(n_1+n_2,4,w_1+w_2)$ codes.

Lemma 7: Let A_1 be an $(n_1,4,w_1)$ code and A_2 an $(n_1,4,w_1+2)$ code. Let B_1 be an $(n_2,4,w_2)$ code and B_2 an $(n_2,4,w_2-2)$ code. Then $(A_1 \times B_1) \cup (A_2 \times B_2)$ is an $(n_1+n_2,4,w_1+w_2)$ code.

The following method makes use of Lemmas 3–7 to obtain lower bounds for $A(n,4,w)$.

Construction A: Take partitions $\Pi(n_1)$ of F^{n_1} and $\Pi(n_2)$ of F^{n_2}. Maximize the size of the code obtained by using a combination of Lemmas 5 and 7. If a $(0, w_1+w_2)$ or a $(w_1+w_2, 0)$ configuration was used, we can take the largest known $(n_2,4,w_1+w_2)$ code and the largest known $(n_1,4,w_1+w_2)$ codes, respectively, for those configurations.

Example 1: From the partitions of F^9 and F^{10}, given in the Appendix, we use the following configurations of $F^9 \times F^{10}$:

- in configuration (1,4) we have 208 codewords;
- in configuration (3,2) we have $84 \cdot 5 = 420$ codewords;
- in configuration (5,0) we have 18 codewords.

Thus $A(19,4,5) \ge 646$ which is better than the previous bound of 612 codewords [1].

Some other interesting results are $A(18,4,8) \ge 2958$ from $F^{10} \times F^8$ (which is compared to 2438 [1]), $A(20,4,8) \ge 7710$ from $F^{10} \times F^{10}$ (which is compared to 6310 [1]), and $A(24,4,10) \ge 93380$ from $F^{12} \times F^{12}$ (which is compared to 81752 [1]). All the other improvements of the lower bounds for $n \le 24$ are given in Table I. These values can be compared to the bounds given in [1] and [2].

TABLE I
LOWER BOUNDS FOR $A(n,4,w)$

n/w	5	6	7	8	9	10	11	12
16			813[a]		813[a]			
17		854[b]	1340[b]	1678[b]	1678[b]	1340[b]	854[b]	
18			2039[c]	2958[c]	3198[d]	2958[c]	2039[c]	
19	646[e]	1510[e]	3170[e]	4640[e]	5936[e]	5936[e]	4640[e]	3170[e]
20	831[f]	2140[f]	4188[f]	7710[f]	9895[f]	11692[f]	9895[f]	7710[f]
21			6064[g]	10613[g]	16644[g]	19605[g]	19605[g]	16644[g]
22			7998[h]	16160[h]	24852[i]	35418[i]	38314[i]	35418[i]
23			11266[j]	22530[j,k]	39761[j]	55649[j]	71792[j]	71792[j]
24	1892[p]		15269[m]	33795[m]	56267[m]	93380[m]	106351[m]	143068[m]

[a]From Construction A, by considering $F^8 \times F^8$.
[b]From Construction A, by considering $F^9 \times F^8$.
[c]From Construction A, by considering $F^{10} \times F^8$.
[d]From Construction A, by considering $F^9 \times F^9$.
[e]From Construction A, by considering $F^{10} \times F^9$.
[f]From Construction A, by considering $F^{10} \times F^{10}$.
[g]From Construction A, by considering $F^{11} \times F^{10}$.
[h]From Construction A, by considering $F^{11} \times F^{11}$.
[i]From Construction A, by considering $F^{12} \times F^{10}$.
[j]From Construction A, by considering $F^{12} \times F^{11}$.
[k]By Lemma 10.
[l]Is like i.
[m]From Construction A, by considering $F^{12} \times F^{12}$.
[n]Is for a length of a code.
[o]Is like O.
[p]From Construction A, by considering $F^{12} \times F^{12}$ [12].

We will now use the lemmas to obtain "good" partitions.

Construction B: Let $\Pi(n_1)$ and $\Pi(n_2)$ be two partitions. Let $\Pi(n_1+n_2)$ be the partition obtained from $\bigcup_{i,j}(A_i \times B_j)$, where $A_i \in \Pi(n_1)$ and $B_j \in \Pi(n_2)$ by using combinations of Lemmas 6 and 7. For each w generate $\Pi(n_1+n_2,w)$ with r classes and $\text{index}_{\Pi} = (i_1, i_2, \cdots, i_r)$ such that $\prod_{j=1}^{r}(i_j)^2$ is maximized.

Since we apply Lemma 5 to obtain part of the $(n_1+n_2,4,w_1+w_2)$ code of Construction A, we use the Cartesian product.

Construction B is a heuristic way of obtaining a large code. Since we use a Cartesian product, our purpose in a partition is to maximize the product of the "indices." Of course sometimes this does not give the best lower bound and hence sometimes we have to use several partitions. Examples of this are presented in the Appendix. Usually, Construction B will be applied for $n_1 = n_2$ or $n_1 = n_2 - 1$. The following lemma can be verified from the combinations taken from the Cartesian product.

Lemma 8: There is a partition Π obtained by Construction B in which

$$|\Pi(n_1+n_2,2w)| = \max\Big(\max_{w_1 \text{ even}} |\Pi(n_1,w_1)|, \max_{w_2 \text{ even}} |\Pi(n_2,w_2)|\Big) + \max\Big(\max_{w_1 \text{ odd}} |\Pi(n_1,w_1)|, \max_{w_2 \text{ odd}} |\Pi(n_2,w_2)|\Big)$$

and

$$|\Pi(n_1+n_2,2w-1)| = \max\Big(\max_{w_1 \text{ even}} |\Pi(n_1,w_1)|, \max_{w_2 \text{ odd}} |\Pi(n_2,w_2)|\Big) + \max\Big(\max_{w_1 \text{ odd}} |\Pi(n_1,w_1)|, \max_{w_2 \text{ even}} |\Pi(n_2,w_2)|\Big).$$

From Lemma 8 and the fact that we can always find a partition Π for which $|\Pi(n,w)| \le n$ for each $0 \le w \le n$, we infer the following result.

Lemma 9: If for a given n, the number of classes in each partition of even weight $2w$ is less than $n-k$, where $k \ge 0$, then for each even weight $2x$ there is a partition, $\Pi(2n)$, of $\boldsymbol{F}^{2n}$ for which the number of classes is less than $2n-k$.

Theorem 1: For $n = 4\cdot 2^i$, for $n = 6 \cdot 2^i$, $i \ge 1$, and for $0 \le w \le n/2$ we have

$$A(n,4,2w) \ge \frac{1}{n-1}\binom{n}{2w} \qquad (3)$$

Proof: $|\Pi(n,0)| = |\Pi(n,n)| = 1$ for every n. By Lemma 2 there is a partition $\Pi(4)$ such that $|\Pi(4,2)| = 3$, and by Lemmas 1 and 2 there is a partition Π such that $|\Pi(6,2)| = |\Pi(6,4)| = 5$. Hence by Lemma 9 we have that, for $n = 4\cdot 2^i$ and for $n = 6\cdot 2^i$, $i \ge 0$, there is a partition $\Pi(n)$, for which $|\Pi(n,2w)| \le n-1$. Thus

$$A(n,4,2w) \ge \frac{1}{n-1}\binom{n}{2w}. \qquad \text{Q.E.D.}$$

We wish to remark that by applying Construction A, we usually obtain lower bounds that are somewhat better than the bounds obtained by Theorem 1. The shortening of codes may lead in some cases to improvement of the lower bounds. By shortening (n,d,w) codes, we can obtain the following results [6].

Lemma 10:

$$A(n-1,d,w-1) \ge \left\lceil \frac{w}{n} A(n,d,w) \right\rceil$$

$$A(n-1,d,w) \ge \left\lceil \frac{n-w}{n} A(n,d,w) \right\rceil.$$

Similarly, we can shorten partitions. This is done in the next construction.

Construction C:

1) Given a partition $\Pi(n,w)$, for a given column k, $1 \le k \le n$, take all the codewords that have a zero in column k. By deleting column k, we obtain a partition $\Pi(n-1,w)$.

2) Given a partition $\Pi(n,w)$, for a given column k, $1 \le k \le n$, take all the codewords which have a one in column k. By deleting column k, we obtain a partition $\Pi(n-1,w-1)$.

Finally, we will give two heuristic methods of obtaining "good" partitions.

Construction D: Given k disjoint $(n,4,w)$ codes $C_1, C_2, \cdots, C_k$, let $a_1, a_2, \cdots, a_r$ be all the distinct words of length n and weight w that do not appear in any code C_i, $1 \le i \le k$. On those words that are randomly ordered use a greedy algorithm to choose the words of the next $(n,4,w)$ code C_{k+1}.

Construction E: Let C_1 be an $(n,4,w)$ code which attains the lower bound. In the general step k, $k > 1$, apply a permutation on the columns of C_1 and from the constructed words remove those that appear in one of the codes C_i, $1 \le i \le k-1$. The remaining words will define C_k.

Table I gives all the improvements of $A(n,4,w)$ made with Construction A, or by applying Lemma 10 to the codes obtained by Construction A. By comparing to the previous bounds, we can see that the improvements are very significant.

The Appendix presents the partitions used to obtain the lower bounds of Table I. Some were generated by the methods described in this section, and some of them by specific combinatorial arguments. Finally, we wish to remark that good partitions also have applications in the design of dc-free codes [13].

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Appendix

Table II summarizes the values of $\text{index}_{11}(n,w)$ which were used to obtain Table I. The partitions $\Pi_1(6,3)$, $\Pi_1(7,3)$, $\Pi_1(8,3)$, $\Pi_1(8,4)$, $\Pi_1(9,3)$, $\Pi_2(9,4)$, $\Pi_1(10,3)$, and $\Pi_1(12,3)$ are explained in [14]. $\Pi_2(11,4)$, $\Pi_1(12,4)$, and $\Pi_1(12,5)$ were obtained by Construction B. Now let $c_1, c_2, \cdots, c_n$ be the n columns of an $(n,4,w)$ code. By applying the permutation $(p_1, p_2, \cdots, p_n)$ on this code we obtain the $(n,4,w)$ code with the columns $c_{p_1}, c_{p_2}, \cdots, c_{p_n}$.

The following (10,4,4) code is the first class of $\Pi_1(10,4)$:

1011010000	1100110000	1010101000	0111001000	0110100100
1001100100	1110000010	0101100010	1101000001	0011100001
0000110101	0000101110	0001001101	0001010011	0010011100
0010001011	0100011010	0100000111	1000010110	1000011001
0001111000	1100001100	0011000110	1000100011	0110010001
0101010100	1010000101	0100101001	0010110010	1001001010.

TABLE II
PARTITIONS USED IN COMPUTING $A(n,4,w)$

n	w	i	$\text{index}_{\Pi_i}(n,w)$
6	3	1	(4,4,4,4,2,2)
7	3	1	(7,7,6,6,5,4)
8	3	1	(8,8,8,8,8,8,8)
8	4	1	(14,14,12,12,10,8)
9	3	1	(12,12,12,12,12,12,12)
9	4	1	(18,18,18,18,18,14,13,7,1,1)
		2	(16,16,16,16,16,16,16,14)
10	3	1	(13,13,13,13,13,13,13,13,13,3)
10	4	1	(30,30,30,30,30,22,22,12,2,2)
10	5	1	(36,36,34,34,30,30,25,19,7,1)
11	3	1	(17,17,17,17,17,17,17,17,15,11,3)
11	4	1	(35,35,35,34,33,32,32,30,26,21,16,1)
		2	(34,34,34,34,34,28,28,26,26,26,26)
11	5	1	(66,66,60,60,55,43,42,36,25,9)
12	3	1	(20,20,20,20,20,20,20,20,20,20,20)
12	4	1	(51,51,51,51,51,40,40,40,40,40,40)
12	5	1	(70,70,70,70,64,64,64,64,64,64,64,64)
12	6	1	(132,132,120,120,110,84,82,66,50,26,2)

By taking the permutations (2,3,4,5,1,7,8,9,10,6), (3,4,5,1,2,8,9,10,6,7), (4,5,1,2,3,9,10,6,7,8), and (5,1,2,3,4,10,6,7,8,9) of this code, we obtain five disjoint (10,4,4) codes of size 30. $\Pi_1(10,4)$ consists of these codes and the following five codes:

0001100101	0010100011	1100000101	0000011011	1110100000
0000011110	0101000011	0111100000	1111000000	0000010111
1000110010	0011000101	0000001111		
0010100110	1011100000	0010101100		
1100001010	1010001100	0011001010		
1010011000	0110001010	0101010001		
1100010100	0110010100	1001000011		
0001101010	0100110001	0110001001		
1101100000	0000011101	0001110010		
0100100011	0001110100	0100111000		
0010110001	0011010010	1010000110		
0110010010	1000100101	1000110100		
0100101100	1001000110			
1001010001	0101001100			
1001001100	0001101001			
0110000101	1100010010			
0011010100	0100100110			
1010000011	1100001001			
1000101001	1010010001			
0011001001	1000101010			
0101011000	0010111000			
0101000110	1001011000.			

By applying Construction C to the last column of $\Pi_1(10,4)$, we obtain $\Pi_1(9,4)$. The following (10,4,5) code of size 36 was used as the first class in $\Pi_1(10,5)$:

1100000111	1001010110	1001100101	1100101100	1000111010	0010111001
0010010111	0101001110	1010100011	1010110100	0110101010	1011010001
0000101111	1000011101	1001001011	0110011100	1101100010	0101101001
1010001110	0110100101	0100011011	1111000100	0011011010	1101011000
0100110110	0101010101	0111000011	0001111100	1110001001	1011101000
0011100110	0011001101	0001110011	1110010010	1100110001	0111110000.

By taking the permutations (5,3,8,6,10,4,9,7,2,1), (3,2,10,9,6,8,4,5,7,1), (9,10,5,1,3,8,6,2,4,7), and (5,9,4,6,2,10,7,8,3,1) of this code and applying Construction E, we obtain five (10,4,5) disjoint codes of sizes 36, 36, 34, 34, and 30, respectively. By applying Construction D with these codes, we obtain $\Pi_1(10,5)$.

$\Pi_1(11,3)$ was obtained by Construction D and the following table presents its eleven classes:

01010000100	10001001000	01000000101	00000010011	01010100000
00001001010	00100001010	11001000000	10100000010	00000011001
00100100100	00000111000	10010010000	00010100100	10100000001
10010001000	01000100100	10100000100	10001100000	00101001000
00011100000	00010000011	00001001001	00001000110	00001100010

00010010001	11010000000	00000011100	00100000101	10000000110
00000001101	00011010000	00001100100	00100110000	00001010100
00000110010	10000000101	00001010010	10000010100	01100000100
00101000001	01000010001	01100010000	00001011000	00010000101
10001000100	01001000010	01010001000	10000001001	10000110000
00110000010	00110100000	00111000000	00000101010	11000001000
10000100001	10100010000	00100000011	01000100001	00000101100
01000000011	10000100010	01000100010	01000001100	00110010000
01000101000	00101000100	00000110001	01010000010	10011000000
11100000000	00001100001	10000001010	00011000001	01001000001
00100011000	00000010110	00010000110	00110001000	00010001010

10100001000	10110000000	10010000010	00100001001	10010000001
00010001001	00001000011	01010010000	00010100001	00100100010
00000110100	00010100010	00000001011	01110000000	00000101001
10000010001	00000100101	00001110000	00001010001	01100000001
01100000010	01100001000	00000010101	00000100110	00000000111
00000100011	00100000110	00010101000	11000010000	11000100000
00010010010	00000011010	11000000001	10010000100	00010110000
00000001110	01010000001	00100010010	00101000010	01000010100
10001000010	00100010001	00100001100	10100100000	00011000010
00101010000	10000101000	01100100000	00001001100	10000010010
00110000100	01001000100	10000100100	01001100000	10000001100
01011000000	11000000010	01000000110	00100010100	
10010100000	00010010100	00011000100	10000000011	
00001101000	00101100000	01001001000	00010011000	00100100001
00001000101	10001010000	10101000000	01000001010	01000001001
01000011000	00011001000	10000011000		10001000001
11000000100	01000110000	00110000001.		

The following 35 codewords make up the (11,4,4) code which is the first class of $\Pi_1(11,4)$:

01000100011	10010001001	10100000110	00100101100	00110011000
10001000011	00010110001	10000101010	10001001100	11000011000
00110000011	01100010001	00011001010	01000110100	01010101000
01000001101	10100100001	01100001010	00011010100	10101010000
10000010101	01011000001	00100110010	10010100100	00111100000
00001100101	00000011110	01001010010	01101000100	11001100000
00101001001	01010000110	10010010010	00001111000	11110000000.

By taking the permutations (2,9,10,3,1,7,8,4,5,6,11), (5,7,11,1,10,4,9,8,6,2,3), and (1,2,10,11,8,5,3,6,4,9,7) of this code and applying Construction E, we obtain four disjoint (11,4,4) codes of sizes 35, 35, 35, and 34 respectively. By applying Construction D with these classes, we obtain $\Pi_1(11,4)$.

The following 66 codewords make up the (11,4,5) code which is the first class of $\Pi_1(11,5)$:

00010001111	10000100111	01001000111	00100010111	00001101011	01000011011
10100001011	10001001101	01100001101	00000111101	11000001110	00100101110
00001011110	10100011100	01001101100	00111001100	10010101100	01010011100
10000111010	01101001010	01010101010	00110011010	10011001010	10101000110
10010010110	01000110110	01110000110	00011100110	11010000011	01100100011
00111000011	00010110011	10001010011	11000010101	01010100101	00011010101
10110000101	00101100101	10010011001	00110101001	00101011001	11000101001
01011001001	11001011000	10101101000	01100111000	11110001000	00011111000
11100100100	10001110100	01101010100	11011000100	00110110100	11100010010
11001100010	00101110010	10110100010	01011010010	01110010001	11101000001
10011100001	01001110001	10100110001	11010110000	10111010000	01111100000.

By taking the permutations (1, 2, 8, 5, 3, 6, 4, 11, 9, 10, 7), (2, 7, 4, 8, 6, 1, 9, 5, 11, 10, 3), (2, 3, 7, 6, 10, 8, 5, 4, 11, 1, 9), and (8,5,10,9,7,11,2,4,3,6,1) of this code and applying Construction E, we obtain five disjoint (11,4,5) codes of sizes 66, 66, 60, 60, and 55, respectively. By applying construction D with these codes, we obtain $\Pi_1(11,5)$. $\Pi_1(12,6)$ was obtained as follows. For each of the first five classes of $\Pi_1(11,5)$ a column with ones was added, and each set was expanded with its complements. All the five new sets are disjoint (12,4,6) codes. Then Construction D was applied in such a way that each class contains words and their complements, and $\Pi_1(12,6)$ was obtained.

Finally, for each of the two cases,

1) $n = 9$, $w = 4$ $(w = 5)$,
2) $n = 11$, $w = 4$ $(w = 7)$,

two different partitions were used. For example, $\Pi_1(9,4)$ should be used for our lower bounds of $A(18,4,9)$ and $A(19,4,8)$, whereas $\Pi_2(9,4)$ should be used for our lower bounds of

$A(17,4,6)$ and $A(19,4,7)$. Similarly, $\Pi_1(11,4)$ should be used for our lower bounds of $A(22,4,7)$ and $A(23,4,10)$, and $\Pi_2(11,4)$ should be used for our lower bounds of $A(22,4,8)$ and $A(23,4,11)$. The interested reader can check to see when Π_1 is used and when Π_2 is used in the other entries of Table I.

Note Added in Proof

In a recent work Brouwer *et al.* [15] found many new partitions and improved most of the lower bounds given here. Their work also includes all recent updates on disjoint Steiner systems.

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Extremal Codes are Homogeneous

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Abstract—**Extremal codes are shown to be homogeneous. This implies that each punctured code of an extremal code has the same weight distribution which can be calculated directly from the weight distribution of the parent extremal code.**

It has been observed that all punctured codes of a certain extremal code have the same weight distribution even though the extremal code in question does not have a transitive group. Our main theorem shows that this must be so for all extremal codes.

Let $\bar{C}$ be a code of length $n+1$, and let M_i be the matrix whose rows are the vectors in $\bar{C}$ of weight i. Then $\bar{C}$ is called *homogeneous* if, for any i, each column of M_i has the same weight. Any code with a transitive group is homogeneous. We show that there are other homogeneous codes.

The code C of length n obtained from $\bar{C}$ by deleting a fixed coordinate is called a *punctured code* of $\bar{C}$. Let A_i denote the number of vectors of weight i in $\bar{C}$ and a_i the number of vectors of weight i in C. The following proposition is due to Prange [6, theorem 80].

Proposition: If $\bar{C}$ is homogeneous, then

$$a_i = \frac{(n+1-i)}{(n+1)} A_i + \frac{(i+1)}{(n+1)} A_{i+1}.$$

It follows from this proposition that we can compute the weight distribution of any code that is punctured from a homogeneous code $\bar{C}$ if we know the weight distribution of $\bar{C}$.

By an *extremal* code is meant one of the following codes:

1) an
$$(n+1,(n+1)/2,4\lfloor(n+1)/24\rfloor+4)$$
doubly even code (these can exist only when $n+1\equiv 0 \pmod 8$);

2) an
$$(n+1,(n+1)/2,3\lfloor(n+1)/12\rfloor+3)$$
ternary self-dual code (here $n+1$ must be $\equiv 0 \pmod 4$);

3) an
$$(n+1,(n+1)/2,2\lfloor(n+1)/6\rfloor+2)$$
quaternary self-dual code (here $n+1$ must be even);

4) an
$$(n+1,(n+1)/2,2\lfloor(n+1)/8\rfloor+2)$$
self-dual binary code (here $n+1$ must be even).

Orthogonality in cases 1), 2) and 4) is with respect to the usual inner product. In the third case the inner product of two vectors $x=(x_1,\cdots,x_n)$ and $y=(y_1,\cdots,y_n)$ is defined to be $\Sigma_{i=1}^n x_i y_i^2$ and orthogonality is with respect to this inner product. Extremal codes are self-dual codes with the largest possible minimum weights [3], [6, theorem 84, cor.]. It is well-known that in the first case all weights in a doubly even code are divisible by four, in the second case all weights are divisible by three, and in the third and fourth cases all weights are even. We demonstrate our main theorem for these four cases.

Theorem: Extremal self-dual codes are homogeneous.

Proof: Case 1) is the easiest. The Assmus–Mattson theorem [6, theorem 92] shows that the vectors of any weight in an extremal doubly even code hold a one-design. For a binary code this is equivalent to the code being homogeneous.

We have to work harder for the other two cases. We show first that every punctured code has the same weight distribution.

Let $\bar{C}$ be a ternary self-dual $(n+1,(n+1)/2,3\lfloor(n+1)/12\rfloor+3)$ code and C a punctured $(n,(n+1)/2)$ code. Consider the subcode D of $\bar{C}$ consisting of all vectors in $\bar{C}$ with zero on the deleted coordinate. It is not hard to see that when we puncture D on this coordinate we get $C^\perp$. The number of nonzero A_i, $i\neq 0$, in $\bar{C}$ is $\lfloor(n+1)/3\rfloor-\lfloor(n+1)/12\rfloor$. Let b_i be the number of vectors of weight i in $C^\perp$. Then the number of nonzero b_i, $i\neq 0$, in $C^\perp$ is at most $\lfloor(n+1)/3\rfloor-\lfloor(n+1)/12\rfloor$. Clearly, the mini-

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