SUPER-NETS AND THEIR HIERARCHY

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Abstract. In this paper we introduce the concept of Super-Net, which subsumes various extensions of Petri nets, proposed earlier in the literature. We consider languages associated with both non-labeled as well as labeled Super-Nets, and use these languages to establish a hierarchy between various types of nets. In the last section we apply this hierarchy to show that Petri net languages are not closed under Kleene star.

1. Introduction

An extensive literature is presently available demonstrating the suitability of Petri nets to the concise and precise modeling of systems involving concurrency (see e.g. [2, 3, 4, 10]). In order to further expand the modeling power of Petri nets, numerous modifications and extensions have been proposed. In particular, we mention the introduction of inhibitor arcs [1, 5, 10], OR-logic transitions [1], Boolean-type places [13, 14, 15], and places with finite capacity [7].

In this paper we introduce the concept of Super-Net which subsumes all the above extensions. Furthermore, Super-Nets also include 'emptying' arcs, which are intended to model the reset-to-zero facility of counting devices.

In modeling, power of various net families can be precisely compared by means of formal languages associated with such nets [5, 6, 10, 16].

In this connection the transitions of a net are frequently considered to be labeled by letters from some finite alphabet $\Sigma$ [5, 6, 10].

In this paper we consider languages associated with both non-labeled as well as labeled Super-Nets. We define various special types of Super-Nets and use their associated languages in order to establish a hierarchy between them. Some of our theorems reformulate known results. However, we consider most of our results to be new.

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2. Super-Nets

We denote by $\omega$ the set of non-negative integers.

Definition 2.1. A SUP-Net (Super-Net) is a 4-tuple $N = (P, T, V, K)$, where
(1) $P$ and $T$ are finite sets of places and transitions, respectively,
(2) $P \cap T = \emptyset$, $P \cup T \neq \emptyset$,
(3) $V$ is a function,
\[ V: (P \times T) \cup (T \times P) \rightarrow \omega \cup \{I, E, L\}; \]

Note. $I$, $E$, $L$ are symbols indicating ‘Inhibiting’, ‘Emptying’, and ‘Logical’ arcs, respectively.
(4) $V(T \times P) \subseteq \omega$,
(5) $K$ is a function
\[ K: P \rightarrow \{\infty\} \cup \omega \times \{A, R\}; \]

Note. $A$, $R$ are symbols indicating ‘Absorbing’, and ‘Restricting’ places.
If $K(p) = \infty$, we say that the place $p$ has infinite capacity. If $K(p) = (k, A)$ or $K(p) = (k, R)$ we say that the place $p$ has finite capacity $k \in \omega$. We denote by $k(p)$ the capacity ($\infty$ or $k \in \omega$) of the place $p$.

Definition 2.2. A marked SUP-Net is a pair $S = (N, M)$, where $N$ is a SUP-Net and $M$ is a marking of $N$, i.e. a function $M: P \rightarrow \omega$, satisfying the condition
\[ (\forall p \in P)[k(p) \in \omega \rightarrow M(p) \leq k(p)]. \]

A marked SUP-Net $S = (P, T, V, K, M)$ is represented graphically as follows:
(1) places are represented by circles ($\bigcirc$),
(2) each place $p$ is labeled by $p/K(p)$,
(3) a transition $t$ is represented by a bar, labeled by $t$,
(4) the place $p \in P$ is connected by a directed arc to the transition $t \in T$, iff $V(p, t) \neq 0$; the arc is labeled by $V(p, t)$,
(5) The transition $t \in T$ is connected by a directed arc to the place $p \in P$, iff $V(t, p) > 0$; the arc is labeled by $V(t, p)$,
(6) The integer $m = M(p)$ is written inside the circle representing $p$; usually, one does not write 0 inside the circle.
An example of a marked SUP-Net is shown in Fig. 1.

In the sequel we need the following:

Definition 3. Let $S = (P, T, V, K, M)$ be a marked SUP-Net. We define a function $W: P \times T \rightarrow \omega$ as follows:
\[ W(p, t) = \begin{cases} V(p, t) & \text{if } V(p, t) \in \omega, \\ 0 & \text{if } V(p, t) = I, \\ M(p) & \text{if } V(p, t) = E, \\ 1 & \text{if } V(p, t) = L \wedge M(p) > 0, \\ 0 & \text{if } V(p, t) = L \wedge M(p) = 0. \end{cases} \]

**Fig. 1. Example of marked SUP-Net.**

**Definition 2.4.** Let \( S = (P, T, V, K, M) \) be a marked SUP-Net. A transition \( t \in T \) is enabled iff the following conditions are satisfied:

1. \( \forall p \in P \)[\( V(p, t) \in \omega \rightarrow M(p) \geq V(p, t) \)],
2. \( \forall p \in P \)[\( V(p, t) = I \rightarrow M(p) = 0 \)],
3. \( \forall p \in P \)[\( V(p, t) = E \rightarrow M(p) > 0 \)],
4. \( \exists p \in P \)\( V(p, t) = L \rightarrow (\exists p \in P) \[ V(p, t) = L \wedge M(p) > 0 \],
5. \( \forall p \in P \)[\( K(p) \in (\omega, R) \rightarrow M(p) + V(t, p) - W(p, t) \leq k(p) \)].

**Definition 2.5.** Let \( S = (P, T, V, K, M) \) be a marked SUP-Net and \( t \in T \) an enabled transition of \( S \). We define the marking \( M' \) of \( N = (P, T, V, K) \) as follows:

\( \forall p \in P \)[\( M'(p) = \min[M(p) + V(t, p) - W(p, t), k(p)] \).
We say that $M'$ is obtained from $M$ by firing $t$ (notation: $M[t \rightarrow M']$). Frequently, it is convenient to represent a marking $M$ by the vector $(M(p_1), M(p_2), \ldots, M(p_n))$, where $P = \{p_1, \ldots, p_n\}$.

For the example of Fig. 1 we obtain the following “firing sequence”:

$$(0, 3, 1)[t_2 > (0, 5, 1),$$

$$(0, 5, 1)[t_2 > (0, 6, 1),$$

$$(0, 6, 1)[t_5 > (0, 6, 0),$$

$$(0, 6, 0)[t_4 > (0, 2, 1),$$

$$(0, 2, 1)[t_1 > (1, 2, 1),$$

$$(1, 2, 1)[t_1 > (2, 2, 1),$$

$$(2, 2, 1)[t_1 > (3, 2, 1),$$

$$(3, 2, 1)[t_5 > (2, 2, 0),$$

$$(2, 2, 0)[t_3 > (0, 2, 0).$$

**Definition 2.6.** Let $N = (P, T, V, K)$ be a SUP-Net. For every $t \in T$ we set

$$t^* \triangleq \{p \in P \mid V(t, p) \neq 0\},$$

$$^*t \triangleq \{p \in P \mid V(p, t) \neq 0\}.$$ We call every $p \in t^*$ an output place of $t$, and every $p \in ^*t$ an input place of $t$.

3. Classification of Super-Nets

In this section we represent various types of nets as special cases of SUP-Nets.

**Definition 3.1.** The following table defines various types of nets as special cases of SUP-Nets, by restricting $\text{range}(V)$ and $\text{range}(K)$.

<table>
<thead>
<tr>
<th>Type of Net</th>
<th>RANGE(V)</th>
<th>RANGE(K)</th>
</tr>
</thead>
<tbody>
<tr>
<td>GP-Net</td>
<td>$\omega$</td>
<td>$[\infty]$</td>
</tr>
<tr>
<td>P-Net</td>
<td>${0, 1}$</td>
<td>$[\infty]$</td>
</tr>
<tr>
<td>I-Net</td>
<td>${0, 1} \cup {I}$</td>
<td>$[\infty]$</td>
</tr>
<tr>
<td>E-Net</td>
<td>${0, 1} \cup {E}$</td>
<td>$[\infty]$</td>
</tr>
<tr>
<td>L-Net</td>
<td>${0, 1} \cup {L}$</td>
<td>$[\infty]$</td>
</tr>
<tr>
<td>A-Net</td>
<td>${0, 1}$</td>
<td>$\omega \times {A}$</td>
</tr>
<tr>
<td>R-Net</td>
<td>${0, 1}$</td>
<td>$\omega \times {R}$</td>
</tr>
</tbody>
</table>

The above defined I-Nets coincide with $\text{TN}_{\text{comm}}$ in [1]. They are called inhibitor nets in [5] and ‘Petri nets with inhibitor arcs’ in [10]. The above L-Nets coincide
with TN_{loc} in [1]. The Boolean-type Petri nets, introduced in [14] coincide with A-Nets, where RANGE(K) = \{(1, A)\}.

The above-defined types of Super-Nets can be combined into more complex classes. E.g., the mixed-type nets discussed in [15] are Super-Nets with RANGE(V) = \{0, 1\} and RANGE(K) = \{\infty\} \cup \{(1, A)\}. Similarly, the place-transition (P/T) nets of [7] are combinations of GP-type and R-type Super-Nets.

4. Super-Net languages

With a given marked SUP-Net \( S = (P, T, V, K, M) \) we associate a language \( L(S) \) over the alphabet \( T \).

**Definition 4.1.** Let \( N = (P, T, V, K) \) be a SUP-Net, and let \( w \in T^+ \), i.e. \( w \) is a finite string of transitions \( w = t_1 t_2 \cdots t_n \). \( w \) is called a firing sequence of the marked SUP-Net \( S = (N, M) \) iff there exist marking \( M_1, \ldots, M_r \), such that

\[
M[t_1] > M_1, M_1[t_2] > M_2, \ldots, M_{r-1}[t_r] > M_r.
\]

In this case we write \( M[w] > M_0 \), and say that \( M_r \) is reachable from \( M \). We also write \( M[\lambda] > M \) for every marking \( M \), where \( \lambda \) denotes the empty sequence.

**Definition 4.2.** Let \( S = (P, T, V, K, M) \) be a marked SUP-Net. We define its language \( L(S) \) as follows:

\[
L(S) \triangleq \{x \in T^* | (\exists M') M[x] > M'\}.
\]

In this paper we are also concerned with labeled Super-Nets and their languages.

**Definition 4.3.** A labeled SUP-Net is a triple \( \Gamma = (S, \Sigma, \eta) \), where \( S = (P, T, V, K, M) \) is a marked SUP-Net, \( \Sigma \) is a finite alphabet and \( \eta \) is a mapping \( \eta : T \to \Sigma \cup \{\lambda\} \). \( \Gamma \) is called \( \lambda \)-free iff \( \eta(T) \subseteq \Sigma \). The language of \( \Gamma \) is defined by

\[
L(\Gamma) \triangleq \eta(L(S)) \triangleq \{\eta(x) | x \in L(S)\}.
\]

If \( M[w] > M' \) where \( w \in T^* \) we also write \( M[\eta(w)] > M' \), and say that \( M' \) is obtained from \( M \) by `firing \( \eta(w) \)`. Clearly every marked SUP-Net \( S \) is also a labeled \( \lambda \)-free SUP-Net \( \Gamma = (S, \Sigma, \eta) \) where \( \Sigma = T \) and \( (\forall t \in T)(\eta(t) = t) \).

**Definition 4.4.** Let \( L \) be a language over some finite alphabet \( \Sigma \). We say that \( L \) is GP-realizable iff \( L = L(S) \) for some marked GP-Net \( S = (P, T, V, K, M) \) with \( T = \Sigma \). \( L \) is LGP-realizable iff \( L = L(\Gamma) \) for some labeled \( \lambda \)-free GP-Net \( \Gamma = (S, \Sigma, \eta) \). \( L \) is \( L^A \)GP-realizable iff there exists an arbitrary labeled GP-Net \( \Gamma \) with \( L = L(\Gamma) \). We denote by GPL, LGPL, \( L^A \)GPL the sets of all GP-realizable, LGP-realizable, and \( L^A \)GP-realizable languages, respectively.
In a similar way we associate sets of languages with the other types of Super-Nets defined above (Definition 3.1).

Evidently, all the languages defined above are prefix languages. Our language families GPL, LGLP, and $L^\lambda$GPL correspond to the classes $P^f$, $P$, and $P^\lambda$ in [10, p. 157], respectively.

5. The hierarchy of Super-Nets (without labeling)

In this section we study the hierarchy of the following language families: GPL, PL, IL, EL, LL, AL, and RL.

**Lemma 5.1.** The languages in AL, LAL, $L^\lambda$AL, RL, LRL, $L^\lambda$RL are regular.

**Proof.** This result follows immediately, since the corresponding SUP-Nets all have finite sets of reachable markings. Hence each such SUP-Net may be viewed as finite automaton with the set of reachable markings as its state set. \(\square\)

The following lemma is an immediate consequence of a result derived in [10, 11].

**Lemma 5.2.** Let $L$ be a language over $\Sigma$ in either RL or PL or IL or GPL. Assume $xx \in L$, where $x \in \Sigma^+$ and $x \in \Sigma$. Furthermore, let $y$ be a permutation of $x$, with $y \in L$. Then $yx \in L$.

For two sets $A$ and $B$, we write $A \subseteq B$ to state that $A$ is a subset of $B$, and $A \subset B$ to state that $A \subseteq B$ and $A \neq B$.

The following theorem is proven in [11].

**Theorem 5.1.** (a) PL $\subseteq$ IL.
(b) PL $\subseteq$ LL.
(c) IL and LL are not comparable, i.e. neither IL $\subseteq$ LL nor LL $\subseteq$ IL holds.

**Theorem 5.2.** (a) RL $\subset$ PL.
(b) RL $\subseteq$ AL.
(c) AL and PL are not comparable.

**Proof.** (a) Let $S$ be a marked R-Net. One easily verifies that there exists a marked P-Net $S'$, which is equivalent to $S$, i.e. $L(S) = L(S')$. This is illustrated in Fig. 2 (henceforth, we omit the label ‘1’ of arcs). Thus, RL $\subseteq$ PL.

On the other hand, there exist languages in PL which are not regular, e.g. the language of the marked P-Net $S_3$ of Fig. 3.

Thus, by Lemma 5.1, we have RL $\subseteq$ PL.
(b) With every marked R-Net one easily associates an equivalent marked A-Net (see Fig. 4). Hence $RL \subseteq AL$.

Thus, by Lemma 5.1, we have $RL \subseteq PL$.

To show that $RL \subseteq AL$, we consider the marked A-Net $S_5$ of Fig. 5. Clearly $abab \in L(S_2)$ and $aabb \in L(S_5)$, but $aabb \notin L(S_5)$. Hence, by Lemma 5.2, $L(S_5) \notin RL$. Consequently, $RL \subseteq AL$.

(c) Consider the marked P-Net $S_3$ of Fig. 3. Since $L(S_3)$ is not regular, $PL \subseteq AL$ does not hold, in view of Lemma 5.1. The above argument in the proof of (b) about $S_3$, also shows that $L(S_3) \notin PL$. Hence $AL \subseteq PL$ does not hold. It follows that $AL$ and $PL$ are not comparable. □

**Theorem 5.3.** $PL \subseteq EL$.

**Proof.** Clearly, $PL \subseteq EL$, by Definition 3.1. Consider now the marked E-Net $S_6$ of Fig. 6. We have $L(S_6) = L(S_5)$. Hence $L(S_6) \notin PL$. Thus, $PL \subseteq EL$. □

**Lemma 5.3.** Let $L$ be a language over $\Sigma$ in either $AL$ or $PL$ or $EL$ or LL. Assume $\sigma_1 \sigma_1 \sigma_2 \in L$, where $\sigma_1 \in \Sigma$ and $\sigma_2 \in \Sigma$. Then $\sigma_1 \sigma_2 \in L$.

**Proof.** Let $L = L(S)$, where $S$ is the relevant SUP-Net. Assume $\sigma_1 \sigma_2 \notin L$, but $\sigma_1 \sigma_1 \sigma_2 \in L$. After firing $\sigma_1$ in $S$ for the first time, there exists a set of input places
Fig. 4. A marked A-Net $S_4$ equivalent to the marked R-Net $S_1$ of Fig. 2(a).

Fig. 5. Example of a marked A-Net $S_5$.

Fig. 6. Example of a marked E-Net $S_6$. 
of $\sigma_2$ which are not marked, preventing $\sigma_2$ from firing. But this set of input places of $\sigma_2$ remains not marked, after $\sigma_1$ is fired a second time, in contradicting with our assumption that $\sigma_1\sigma_2 \in L$. Hence $\sigma_1\sigma_2 \in L$. \qed

**Theorem 5.4.** $PL \subset GPL$.

**Proof.** Evidently, $PL \subseteq GPL$, by Definition 3.1. We now consider the marked GP-Net $S_7$ of Fig. 7. We have $ab \notin L(S_7)$, but $aab \in L(S_7)$. Hence, by Lemma 5.3, $L(S_7) \notin PL$. It follows that $PL \subset GPL$. \qed

Fig. 7. Example of a marked GP-Net $S_7$.

Let $S = (P, T, V, K, M)$ be a marked L-Net. For any $t \in T$ we define an equivalence relation $E_t$ on $^*t$ as follows:

$$E_t \triangleq \{(p, p') | p \in ^*t \land p' \in ^*t \land [p = p' \lor V(p, t) = V(p', t) = L]\}.$$

We denote by $Q_t$ the partition $^*t/E_t$ of $^*t$. For any $q \in Q_t$, we set

$$M(q) = \max\{M(p) | p \in q\}.$$

We say that $q \in Q_t$ is an output of $t$, if $(\exists p \in q) V(t, p) = 1$.

**Theorem 5.5.** The language types EL and LL are not comparable.

**Proof.** (a) Consider the marked E-Net $S_8$ of Fig. 8. We wish to show that $L(S_8) \notin LL$.

Assume that $L(S_8) = L(S)$, where $S = (N, M)$ is a marked L-Net. We have $a^4b^4 \in L(S)$. Let $M[a^4 > M']$ in $S$. Then, for every $q$ in $Q_b$ which is not an output of $b$, we must have $M'(q) \geq 4$. Now, $a^4cab \in L(S)$. Let $M'[cab > M''$ in $S$. Then, for every $q$ in $Q_b$ which is not an output of $b$, $M''(q) \geq 1$. Also, since $M''$ was obtained by firing $b$, we must have $M''(q) \geq 1$ for every $q$ in $Q_b$ which is an output of $b$. Hence $b$ is enabled by $M''$. It follows that $a^4cab^2 \in L(S)$, but $a^4cab^2 \notin L(S_8)$, contradicting our assumption that $L(S_8) = L(S)$. Consequently, $L(S_8) \notin LL$, i.e. $EL \notin LL$.

(b) To show that $LL \nsubseteq EL$, we consider the marked L-Net $S_9$ of Fig. 9. Assume $L(S_9) = L(S)$, where $S = (N, M)$ is marked E-Net. Since $b \in L(S)$ and $b^2 \notin L(S)$, we must have $^*b \neq \emptyset$. 
Let \( k = \max \{ M(p) \mid p \in \ast b - b \ast \} \). Since \( b \in L(S) \), we have \( k \geq 1 \) as well as \( (\forall p \in \ast b - b \ast) M(p) \geq 1 \). Now, \( a^{k+1} b^{k+1} \in L(S) \). Therefore, \( (\forall p \in \ast b - b \ast)(p \in a^{-} - a) \). Let \( M'(a^{k+1} b^{k+1}) \). It follows that \( (\forall p \in \ast b - b \ast)(M'(p) \geq k + 2) \). Since \( a^{k+1} b^{2} \in L(S) \), we must have \( (\forall p \in \ast b - b \ast) V(p, b) \neq E \). Let \( M'(a^{k+1} b^{2}) \). Then \( (\forall p \in \ast b - b \ast)M''(p) \geq 1 \). Since \( M'' \) was obtained by firing \( b \), we must also have \( (\forall p \in \ast b \cap b \ast)M''(p) \geq 1 \). It follows that \( b \) is enabled in \( M'' \), hence \( a^{k+1} b^{k+2} \in L(S) \). But \( a^{k+1} b^{k+2} \not\in L(S_{9}) \). Consequently, \( L(S_{9}) \not\in EL \). Thus, \( LL \not\in EL \).

Since \( LL \not\in EL \) and \( EL \not\in LL \), Theorem 5.5 is proven. \( \square \)

**Theorem 5.6.** AL is not comparable with either GPL or IL or LL or EL.

**Proof.** (a) Consider the marked P-Net \( S_{3} \) of Fig. 3. \( S_{3} \) is also a marked GP-Net, I-Net, L-Net and E-Net. Since \( L(S_{3}) \) is not regular, neither GPL \( \not\subseteq AL \) nor IL \( \not\subseteq AL \) nor LL \( \not\subseteq AL \) nor EL \( \not\subseteq AL \) can hold, in view of Lemma 5.1.

(b) Consider the marked A-Net \( S_{5} \) of Fig. 5. The argument in the proof of Theorem 5.2(b) about \( S_{5} \), also shows that \( L(S_{5}) \not\in GPL \) and \( L(S_{5}) \not\in IL \). Consequently, neither \( AL \not\subseteq GPL \) nor \( AL \not\subseteq IL \) can hold.
(c) Consider the marked A-Net $S_{10}$ of Fig. 10. Assume that $L(S_{10}) = L(S)$, where $S = (N, M)$ is a marked L-Net. We have $a^3b^2 \in L(S)$. Let $M'[a^3 > M']$ and $M'[b^2 > M''$ in $S$. Since $a^3b^3 \in L(S)$, there exists an element $q$ of $Q_b$ which is not an output of $b$, such that $M'(q) = 2$ and $M''(q) = 0$. Since $M'(q) = 2$ we have $(\forall p \in q) \cdot [V(a, p) = 1 \iff V(p, a) \neq 0].$ We also have $a^3b^2a^2 \in L(S)$. Let $M''[a > M'$. Thus, we have $(\forall p \in q)[M''(p) \neq 1].$ It follows that $a^3b^2a^2b^2 \in L(S)$, but $a^3b^2a^2b^2 \in L(S_{10})$, contradicting our assumption that $L(S_{10}) = L(S)$. Consequently, AL $\subseteq$ LL does not hold.

(d) Consider the marked A-Net $S_{10}$ of Fig. 10. Assume that $L(S_{10}) = L(S)$, where $S = (N, M)$ is a marked E-Net. We have $a^3b^2 \in L(S)$. Let $M[a^3 > M'$ and $M'[b^2 > M''$ in $S$. Since $a^3b^3 \in L(S)$ we have $(\exists p \in b - b')[V(p, b) \neq E \land M'(p) = 2 \land M''(p) = 0].$ Let $p$ satisfy this condition. Since $M'(p) = 2$, $V(p, p) = 1$ implies $V(p)$,

We distinguish between two cases:

Case I: $V(a, p) = 1 \land V(p, a) = 1$. Since $M''(p) = 0$ it follows that $a^3b^2a \notin L(S)$.

Case II: $V(a, p) = 0$. We have $a^3b^2a \in L(S)$. Let $M''[a > M''$ in $S$. Hence, $M''(p) = 0$. It follows that $a^3b^2ab \in L(S)$.

But, since $a^3b^2ab \in L(S_{10})$, both Cases I and II contradict our assumption that $L(S_{10}) = L(S)$. Consequently, AL $\subseteq$ EL does not hold.

Parts (a), (b), (c) and (d) complete the proof of Theorem 5.6.

\[\square\]

Lemma 5.4. Let $L$ be a language over $\Sigma$ in either EL or GPL. Let $S = (N, M)$ be a marked E-Net or a marked GP-Net. Let $y_1, y_2 \in L(S)$ and $M[y_1 > M_1, M[y_2 > M_2$ in $S$. If $M_2 \geq M_1$ and $y_1y_3 \in L(S)$ then $y_2y_3 \in L(S)$.

Proof. Since $M_2 \geq M_1$ and $y_3$ is fireable in $M_1$, $y_3$ is also fireable in $M_2$.

A similar lemma appears in [11].

Theorem 5.7. IL is not comparable with either EL or GPL.
Proof. (a) Consider the marked I-Net $S_{11}$ of Fig. 11. Using Lemma 5.4 one easily shows that $L(S_{11}) \not\in \text{GPL}$ (cf. [1, 9, 11]) and that $L(S_{11}) \not\in \text{EL}$.

(b) Consider the marked E-Net $S_6$ of Fig. 6. Clearly $abab \in L(S_6)$ and $aab \in L(S_6)$, but $aabb \not\in L(S_6)$. Hence, by Lemma 5.2, $L(S_6) \not\in \text{IL}$.

(c) Consider the marked GP-Net $S_7$ of Fig. 7. Assume that $L(S_7) = L(S)$, where $S = (N, M)$ is a marked I-Net. We have $a \in L(S)$ but $ab \not\in L(S)$. Let $M'[a > M']$. We have one of the following two cases:

Case I: $(\exists p \in \gamma b)[V(p, b) = 1 \land M'(p) = 0]$. Let $p$ denote a place satisfying this condition. Now, $a^2 \in L(S)$, and let $M'[a > M']$. We still have $M''(p) = 0$, hence $a^2b \not\in L(S)$, but $a^2b \in L(S_7)$, contradicting our assumption that $L(S_7) = L(S)$.

Case II: $(\exists p \in \gamma b)[V(p, b) = 1 \land M'(p) > 0]$. Let again $p$ denote a place satisfying this condition and $M''[a > M''$. Since $a^2b \in L(S)$ we have $p \in \gamma a$, $V(p, a) = 1$ and $M''(p) = 0$. Hence $a$ is not enabled in $M''$, but $a^3 \in L(S_7)$, contradicting our assumption that $L(S_7) = L(S)$.

We have thus proved that GPL $\not\subseteq$ IL does not hold.

Parts (a), (b) and (c) complete the proof of Theorem 5.7. □

Theorem 5.8. GPL is not comparable with either LL or EL.

Proof. (a) Consider the marked GP-Net $S_7$ of Fig. 7. We have $ab \not\in L(S_7)$, but $aab \in L(S_7)$. Hence, by Lemma 5.3, $L(S_7) \not\in \text{LL}$ and $L(S_7) \not\in \text{EL}$.

(b) Consider the marked L-Net $S_9$ of Fig. 9. Clearly $bab \in L(S_9)$ and $ab \in L(S_9)$, but $aabh \not\in L(S_9)$. Hence, by Lemma 5.2, $L(S_9) \not\in \text{GPL}$.

(c) Consider the marked E-Net $S_6$ of Fig. 6. Clearly, $abab \in L(S_6)$ and $aab \in L(S_6)$, but $aabb \not\in L(S_6)$. Hence, by Lemma 5.2, $L(S_6) \not\in \text{GPL}$.

Parts (a), (b) and (c) complete the proof of Theorem 5.8. □

The theorems of Section 5 yield the hierarchy of SUP-Nets illustrated in Fig. 12, where XL→YL denotes XL $\subseteq$ YL, and XL $\cdots$ YL indicates that XL and YL are not comparable.
6. Some closure properties of GPL

By means of Lemma 5.2, solutions to some open problems mentioned in [10, p. 186] are easily provided.

**Theorem 6.1.** GPL is not closed under union.

**Proof.** Consider the marked GP-Nets $S_{12}$ and $S_{13}$ of Fig. 13. We have $abh \in L(S_{12}) \cup L(S_{13})$ and $ba \in L(S_{12}) \cup L(S_{13})$ but $bab \not\in L(S_{12}) \cup L(S_{13})$. Hence, by Lemma 5.2 $L(S_{12}) \cup L(S_{13}) \not\in$ GPL. Consequently, GPL is not closed under union. □

**Theorem 6.2.** GPL is not closed under concatenation.

Fig. 13. (a) Example of a marked GP-Net $S_{12}$; (b) example of a marked GP-Net $S_{13}$. 
Proof. Consider the marked GP-Net $S_{14}$ of Fig. 14. We have $abab \in L(S_{14}) \cdot L(S_{14})$, where $\cdot$ denotes concatenation, and $aab \in L(S_{14}) \cdot L(S_{14})$ but $aabb \notin L(S_{14}) \cdot L(S_{14})$. Hence, by Lemma 5.2 $L(S_{14}) \cdot L(S_{14}) \notin \text{GPL}$. Consequently, GPL is not closed under concatenation. \hfill $\square$

![Diagram of marked GP-Net $S_{14}$](image)

Fig. 14. Example of a marked GP-Net $S_{14}$.

**Theorem 6.3.** GPL is not closed under the concurrency (shuffle) operator $\|\|$ (see [10]).

Proof. Consider the marked GP-Nets $S_{15}$ and $S_{16}$ of Fig. 15. Clearly $abc \in L(S_{15})\|L(S_{16})$ and $ba \in L(S_{15})\|L(S_{16})$, but $bac \notin L(S_{15})\|L(S_{16})$. Hence, by Lemma 5.2 $L(S_{15})\|L(S_{16}) \notin \text{GPL}$. Consequently, GPL is not closed under the concurrency operator. \hfill $\square$

![Diagram of marked GP-Nets $S_{15}$ and $S_{16}$](image)

Fig. 15. (a) Example of a marked GP-Net $S_{15}$; (b) example of a marked GP-Net $S_{16}$.

**Theorem 6.4.** GPL is not closed under prefix regular substitution (see [10]).

Proof. Consider the marked GP-Net $S_{17}$ of Fig. 16. Consider the prefix regular substitution $f(a) = \{\lambda, a, ab, abb, b, ba\}$. We have $abb \in f(L(S_{17}))$ and $ba \in f(L(S_{17}))$
but $bab \not\in f(L(S_{17}))$. Hence, by Lemma 5.2 $f(L(S_{17})) \not\subseteq \text{GPL}$. Consequently, GPL is not closed under prefix regular substitution. □

**Theorem 6.5.** GPL is not closed under Kleene star (iteration).

**Proof.** Consider the marked GP-Net $S_{14}$ of Fig. 14. We have $abab \in (L(S_{14}))^*$ and $aab \in (L(S_{14}))^*$ but $aabb \not\in (L(S_{14}))^*$. Hence, by Lemma 5.2 $(L(S_{14}))^* \not\subseteq \text{GPL}$. Consequently, GPL is not closed under Kleene star □.

In [10] Petri nets with final markings and the corresponding languages are discussed. One easily verifies that the results and proofs of Theorems 6.2, 6.3 and 6.5 also apply to the family of languages denoted by $L'$ in [10, p. 186].

Similar results concerning $L'$ are obtained (by a different approach) in [12].

### 7. Hierarchy of labeled $\lambda$-free Super-Nets

In this section we study the hierarchy of the various types of labeled $\lambda$-free SUP-Nets, defined in Section 3.

**Theorem 7.1.** We have $\text{LRL} = \text{LAL} = L^\lambda \text{RL} = L^\lambda \text{AL}$. Each of these sets coincides with the set of all the prefix regular languages.

**Proof.** (a) By Lemma 5.1, the languages in sets $\text{LRL}$, $\text{LAL}$, $L^\lambda \text{RL}$, $L^\lambda \text{AL}$ are all prefix regular.

(b) Let $L$ be a prefix regular language. Clearly there exists a finite, deterministic automaton $A$ such that $L = L(A)$. By applying a construction similar to that in [5, p. 35], one obtains a labeled $A$-Net $\Gamma$ as well as a labeled R-Net $\Gamma$ such that $L(\Gamma) = L(A) = L$. □

In the sequel we let $\Gamma_i$ denote the labeled $\lambda$-free SUP-Nets corresponding to the marked SUP-Nets $S_i$ defined in Section 5.

**Theorem 7.2.** $\text{LRL} \subset \text{LPL}$. 
Proof. Similar to proof of Theorem 5.2(a). □

M. Hack [5, p. 64] has shown that any labeled λ-free GP-Net \( \Gamma \) can be transformed into a labeled P-Net \( \Gamma' \) such that \( L(\Gamma) = L(\Gamma') \). A stronger result is the following.

**Theorem 7.3.** LGPL = LPL.

Proof. Clearly LPL ⊆ LGPL.

Consider the labeled \( \lambda \)-free GP-Net \( \Gamma = (P, T, V, K, M, \Sigma, \eta) \). We construct a labeled \( \lambda \)-free P-Net \( \Gamma' = (P', T', V', K', M', \Sigma, \eta') \) where

\[
P' = \{ [p, i] | p \in P, 0 \leq i \leq n_p - 1 \}
\]

and \( n_p = \max \{ \max_{t \in T} V(p, t), \max_{t \in T} V(t, p) \} \),

\( T' = \{ [t, i] | t \in T, 1 \leq i \leq n_t \} \)

and \( n_t = \prod_{p \in P, V(p, t) > 0} n_p \cdot \prod_{p \in P, V(t, p) > 0} n_p \cdot \prod_{p \in P, V(p, t) = 0} \).

\( V' \) has to satisfy the following three conditions, where \( + \) indicates addition mod \( n_p \):

I. Assume \( V(p, t) = m \).
   1. \( m = 0: (\forall i \in \{1, 2 \cdots n_t\})(\forall j \in \{0, 1 \cdots n_p - 1\}) V'([p, j], [t, i]) = 0 \).
   2. \( m = n_p: (\forall i \in \{1, 2 \cdots n_t\})(\forall j \in \{0, 1 \cdots n_p - 1\}) V'([p, j], [t, i]) = 1 \).
   3. \( 0 < m < n_p: (\forall i \in \{1, 2 \cdots n_t\})(\exists j \in \{0, 1 \cdots n_p - 1\}) V'([p, j], [t, i]) = 1,\]
      \[ V'([p, j(p, t, i)], [t, i]) = 1,\]
      \[ V'([p, j(p, t, i) + 1], [t, i]) = 1,\]
      \[ \ldots, V'([p, j(p, t, i) + m - 1], [t, i]) = 1,\]
      \[ V'([p, j(p, t, i) + m], [t, i]) = 0,\]
      \[ \ldots, V'([p, j(p, t, i) - 1], [t, i]) = 0.\]

II. Assume \( V(t, p) = k \).
   1. \( k = 0: (\forall i \in \{1, 2 \cdots n_t\})(\forall j \in \{0, 1 \cdots n_p - 1\}) V'([t, i], [p, j]) = 0 \).
   2. \( k = n_p: (\forall i \in \{1, 2 \cdots n_t\})(\forall j \in \{0, 1 \cdots n_p - 1\}) V'([t, i], [p, j]) = 1 \).
   3. \( 0 < V(t, p) < n_p: (\forall i \in \{1, 2 \cdots n_t\}) V'([t, i], [p, j(p, t, i)]) = 1,\]
      \[ V'([t, i], [p, j(p, t, i) + k - 1]) = 1,\]
      \[ V'([t, i], [p, j(p, t, i) + k]) = 0,\]
      \[ \ldots, V'([t, i], [p, j(p, t, i) - 1]) = 0.\]

III. \( \sim (\exists t \in T)(\exists i, j \neq j, 1 \leq i, j \leq n_t)(\forall p \in P')\)

\[
V'([t, i], [p]) = V'([t, j], [p]) \wedge V'([t, [i]], [p]) = V'([t, [j]], [p])
\].
Furthermore, \((\forall p \in P')(K'(p) = \infty)\)
\[
(\forall p \in P)(\forall i, 0 \leq i \leq n_p - 1)(M'(\lfloor p, i \rfloor) = [M(p)/n_p] + [i + 1 \leq \text{mod}(M(p), n_p)])
\]

where \([\text{Prop}] \triangleq \text{if Prop is true then 1 else 0};\)
\[
(\forall t \in T)(\forall i, 1 \leq i \leq n_t)(\eta'(\lfloor t, i \rfloor) = \eta(t)).
\]

Fig. 17 illustrates this construction.

![Diagram](image)

Fig. 17. (a) Example of a labeled \(\lambda\)-free GP-Net \(\Gamma_{18}\); (b) an equivalent labeled \(\lambda\)-free P-Net \(\Gamma_{19}\), i.e.,
\[
L(\Gamma_{19}) = L(\Gamma_{18}).
\]

One easily verifies that the validity of the condition \((\ast)\) can be preserved for any sequence of firings. Hence we have \(L(\Gamma') = L(\Gamma)\). Thus, \(LPL = LGPL\). \(\square\)

The following lemma recalls a well-known result.

**Lemma 7.1.** Let \(\{a_i\}_{i=1}^{\infty}\) be an infinite sequence of vectors over \(\omega\). Then there exists an infinite increasing sequence of integers \(\{i_k\}_{k=1}^{\infty}\) such that
\[
(\forall m, 1 \leq m \leq n)[(\forall j, k; j < k)(a_{i_j}(m) < a_{i_k}(m)) \vee (\forall j, k)(a_{i_j}(m) = a_{i_k}(m))].
\]
Proof. Construct an infinite subsequence of \( \{a_i\}_{i=1}^{\infty} \) increasing or non-changing in the first coordinate; construct from this subsequence an infinite subsequence increasing or non-changing in the second coordinate and so forth. □

The following lemma is an extension of Lemma 5.4.

Lemma 7.2. Let \( L \) be a language over \( \Sigma \) in \( L^{PL} \). Let \( \Gamma = (S, \Sigma, \eta) \) be a labeled P-Net where \( L(\Gamma) = L \). Let \( y_1, y_2 \in L(\Gamma) \) and \( M[y_1 > M_1, M[y_2 > M_2] \in \Gamma \). If \( M_2 > M_1, y_1y_3 \in L(\Gamma) \) and \( M_1[y_3 > M_3, \) then there exists a firing sequence \( x \in L(S) \) such that \( \eta(x) = y_2y_3, M_2[y_3 > M_3' \) and \( M_3 = M_3 + M_2 - M_1 \).

Proof. Since \( M_2 > M_1 \) and \( y_3 \) is fireable in \( M_1, y_3 \) is also fireable in \( M_2, \) and \( M_3' = M_2 - M_1 \). □

Lemma 7.3 (König’s Infinity Lemma [8]). Let \( T \) be an infinite directed tree. If the out-degree of every vertex in \( T \) is finite, then there exists an infinite directed path in \( T \).

In the sequel we shall need the following concept.

Definition 7.1. Let \( \Gamma = (P, T, V, K, M, \Sigma, \eta) \) be a labeled SUP-Net. We define its marking tree \( \Tr(\Gamma) \) as a directed labeled tree in accordance with the following:

1. The root is labeled by \( (\lambda, M) \);
2. Let \( V_1 \) be a vertex labeled by \( (x, M_1) \), where \( M[x > M_1] \) in \( \Gamma \). If \( M_1[y > M_2, \) where \( y \in \Sigma \cup \{\lambda\} \) in \( \Gamma \), then there is a directed edge from \( V_1 \) to another vertex \( V_2 \) labeled by \( (xy, M_2) \).

Clearly the out-degree of each vertex in \( \Tr(\Gamma) \) is finite. Indeed, from any marking we can obtain no more than \( |T| \) new markings, by firing a single transition.

Lemma 7.4. Let \( L \) be a language over \( \Sigma \). Let \( \Gamma = (S, \Sigma, \eta) \) be a labeled \( \lambda \)-free SUP-Net where \( L(\Gamma) = L \). Let \( \{y_i\}_{i=0}^{\infty} \) be an infinite sequence of finite words defined as follows:

\[
y_0 = \lambda \in L(\Gamma), \quad y_i = y_{i-1}\sigma_i \in L(\Gamma), \quad \sigma_i \in \Sigma, \quad j = 1, 2, \ldots \]

Then there exists an infinite directed path in \( \Tr(\Gamma) \) the vertices of which represent the words of the sequence \( \{y_i\}_{i=0}^{\infty} \).

Proof. Let us concentrate only on vertices in \( \Tr(\Gamma) \) which represent words in the sequence \( \{y_i\}_{i=0}^{\infty} \). The corresponding subtree of \( \Tr(\Gamma) \) is an infinite tree because the sequence \( \{y_i\}_{i=0}^{\infty} \) is infinite. Since the subtree is infinite and the out-degree of each vertex in the tree is finite there exists an infinite directed path in \( \Tr(\Gamma) \), passing through some of those vertices by Lemma 7.3. Each directed step along an edge
in this path represents a firing of some \( \sigma \in \Sigma \) since \( \Gamma \) is \( \lambda \)-free. Furthermore, all prefixes of any word \( y_i \in \{y_i\}_{i=0}^{\infty} \) belong to the sequence \( \{y_i\}_{i=0}^{\infty} \); hence, the path's vertices represent all the words of the sequence \( \{y_i\}_{i=0}^{\infty} \).

**Theorem 7.4.** \( \text{LPL} \subseteq \text{LLL} \).

**Proof.** Clearly \( \text{LPL} \subseteq \text{LLL} \), by definition. Consider the labeled \( \lambda \)-free L-Net \( \Gamma_{20} \) of Fig. 18.

![Diagram](image)

**Fig. 18.** Example of a labeled \( \lambda \)-free L-Net \( \Gamma_{20} \).

Assume that \( L(\Gamma_{20}) = L(\Gamma') \) where \( \Gamma \) is a labeled \( \lambda \)-free P-Net. By Lemma 7.4 there exists an infinite directed path in \( \text{Tr}(\Gamma) \) the vertices of which represent the prefixes of the infinite word

\[
abcba^2 c^2 b^2 c^2 \cdots a^i c^j b^i c^j \cdots .
\]

We concentrate on markings of this path. Let \( \{M_i\}_{i=2}^{\infty} \) be the infinite sequence where \( M[a^i c^j b^i c^j] > M_i \). Let \( \{\tilde{M}_i\}_{i=2}^{\infty} \) be the infinite sequence where \( M[a^i c^j b^i c^j] > \tilde{M}_i \).

By Lemma 7.1 we construct an infinite increasing sequence \( \{i_j\}_{j=1}^{\infty} \) such that \( (\forall j, k; j < k)[M_{i_j} \leq M_{i_k}] \).

Now, let us look at the subsequence \( \{\tilde{M}_{i_j}\}_{j=1}^{\infty} \). By Lemma 7.1 we can find two integers \( l \) and \( k \) such that \( l < k \) and \( \tilde{M}_l \leq M_l \). Let \( s = i_l, r = i_k \). Then, \( M_s \leq M_r \) and \( \tilde{M}_s \leq \tilde{M}_r \).

Now, by Lemma 7.2 the following markings can be obtained in \( \Gamma \):

\[
M[a^i c^j b^i c^j] > M_{i_j},
\]

\[
M_{i_j}[c^i b^i c^i] > M_{i_j},
\]

\[
M_{i_j} = \tilde{M}_l + M_r - M_s \geq M_r \geq \tilde{M}_s,
\]

\[
M_{i_j}[a^i] > M_{i_j},
\]

\[
M_{i_j} = M_{i_j} + (\tilde{M}_l + M_r - M_s) - \tilde{M}_s = M_r + M_r - M_s \geq M_r,
\]

\[
M_{i_j}[c^i] > M_{i_j}.
\]
But \( acbc \cdots b^{r-1}c^{r-1}a^r c^r b^r c^r \cdots a^{r-1}c^{r-1}b^{r-1}c^{r-1}a^{r-1}c^{r-1} \notin L(\Gamma_{20}) \) since \( \max(r,2s) + s < 2s + r \), contradicting our assumption that \( L(\Gamma_{20}) = L(\Gamma) \). Hence, \( \text{LPL} \subseteq \text{LLL} \).

The following theorem compares LPL and LEL. A stronger result requiring a rather complex proof, will be obtained in Section 8 (Theorem 8.2).

**Theorem 7.5.** \( \text{LPL} \subseteq \text{LEL} \).

**Proof.** The proof is similar to that of Theorem 7.4, with \( \Gamma_{20} \) replaced by \( \Gamma_8 \). The infinite word of interest becomes

\[ aca^2bca^3b^2c \cdots a^i b^{i-1}c \cdots. \]

The infinite sequences \( \{M_i\}_{i=2}^\infty \) and \( \{M_i\}_{i=2}^\infty \) are defined by

\[
M[ac \cdots a^{i-1}b^{i-2}ca^i] > M_i
\]

\[
M[ac \cdots a^{i-1}b^{i-2}c] > M_i
\]

respectively.

**Lemma 7.5.** There exists a language in IL which is neither in \( \text{L}^\lambda \text{LL} \) nor in \( \text{L}^\lambda \text{EL} \).

**Proof.** Consider the marked I-Net \( S_{11} \) of Fig. 11. By an argument similar to that used in the proof of Theorem 5.7, one easily shows that \( L(S_{11}) \notin \text{L}^\lambda \text{LL} \) and \( L(S_{11}) \notin \text{L}^\lambda \text{EL} \).

**Theorem 7.6.** \( \text{LLL} \subseteq \text{LIL} \).

**Proof.** By a construction, similar to that of [1] one easily proves that \( \text{LLL} \subseteq \text{LIL} \). This construction is illustrated in Fig. 19. In view of Lemma 7.5, we thus have \( \text{LLL} \subseteq \text{LIL} \).

Fig. 19. (a) Example of a labeled \( \lambda \)-free L-Net \( \Gamma_{21} \); (b) an equivalent labeled \( \lambda \)-free I-Net \( \Gamma_{22} \), i.e., \( L(\Gamma_{22}) = L(\Gamma_{21}) \).
The theorems of Section 7 yield the hierarchy of labeled λ-free SUP-Nets illustrated in Fig. 20 where XL→YL denotes XL⊂YL, and XL↔YL indicates that XL=YL.

8. Hierarchy of labeled Super-Nets

In this section we study the hierarchy of the various types of labeled SUP-Nets (including λ-labels).

Theorem 8.1. (a) $L^\lambda RL ⊆ L^\lambda PL$.
(b) $L^\lambda PL = L^\lambda GPL$.

Proof. The proofs are similar to those of Theorem 7.2 and Theorem 7.3, respectively. □

Theorem 8.2. $L^\lambda PL ⊆ L^\lambda EL$.

Proof. Clearly $L^\lambda PL ⊆ L^\lambda EL$ by definition. Consider the labeled E-Net $Γ_8$ of Fig. 8. Assume that $L(Γ_8)=L(Γ)$ where $Γ=(P, T, V, K, M, Σ, η)$ is a labeled P-Net. Let $|P|=n$. We consider the infinite sequence $\{A_i\}_{i=1}^∞$ of words:

$$A_i = a^i b^{i-1} c a^{i+1} b^i c \cdots a^{i+n-1} b^{i+n-2} c \in L(Γ).$$

Now, we construct $2n$ infinite sequences of marking in $Γ$ from the words in $\{A_i\}_{i=1}^∞$:

$$\{M^i_j\}_{j=1}^∞, \quad \text{for } j = 1, 2, \ldots, 2n,$$

$$M[a^i b^{i-1} c \cdots a^{i-j-2} b^{i-j-3} c a^{i-j-1} > M^2_{i-1}], \quad \text{for } j = 1, 2, \ldots, n,$$

$$M[a^i b^{i-1} c \cdots a^{i+j-1} b^{i+j-2} c > M^2_j], \quad \text{for } j = 1, 2, \ldots, n.$$

Next, we extract from $\{M^1_i\}_{i=1}^∞$ an infinite subsequence $\{M^1_j\}_{j=1}^∞$ according to Lemma 7.1. From $\{M^2_j\}_{j=1}^∞$ we extract another infinite subsequence according to Lemma 7.1 and so forth, until the $2n$th subsequence.
Now, we have an infinite, increasing subsequence $\{i_j\}_{j=1}^\infty$ such that

$$(\forall l, 1 \leq l \leq 2n)(\forall q, 1 \leq q \leq n)
[(\forall j, k; j < k)(M_{i_j}^l(q) < M_{i_k}^l(q)) \land (\forall j, k)(M_{i_j}^l(q) = M_{i_k}^l(q))].$$

Claim 1. $(\forall j, k; j < k)(\forall l, 1 \leq l \leq n)[M_{i_j}^{2l-1} < M_{i_k}^{2l-1}].$

Proof. Assume that for some $j < k$ and $1 \leq l \leq n$, $M_{i_j}^{2l-1} = M_{i_k}^{2l-1}$. Thus, by Lemma 7.2,

$$x = a^{l_1}b^{l_1-1}c \cdots a^{l_i+1}b^{l_i+2}ca^{l_i+i-3}b^{l_i+i-2}c \in \mathcal{L}(\Gamma).$$

But, $x \notin \mathcal{L}(\Gamma_8)$, since $i_j + l - 1 \leq i_k + l - 2$, contradicting our assumption. □

Claim 2. There exists an infinite increasing sequence $\{i_j\}_{j=1}^\infty$, with the properties of $\{i_j\}_{j=1}^\infty$ and furthermore

$$(\forall l_1, 1 \leq l_1 \leq n)(\forall q, 1 \leq q \leq n)
[(\forall j, k)(M_{i_j}^{2l_1-1}(q) = M_{i_k}^{2l_1-1}(q)) \lor (\forall l_2, 1 \leq l_2 \leq n)(\forall j, j > 1)
(M_{i_j}^{2l_2-1}(q) - M_{i_{j-1}}^{2l_2-1}(q) \leq M_{i_{j-1}}^{2l_2-1}(q) - M_{i_j}^{2l_2-1}(q)).$$

Proof. We can construct such a sequence as a subsequence of $\{i_j\}_{j=1}^\infty$ in view of the above properties of $\{i_j\}_{j=1}^\infty$, as follows:

$$t_1 = i_1,$n

$$t_2 = i_2.$n

$$f_3 = 2 + \max_{1 \leq q \leq n, 1 \leq l \leq n}(M_{i_2}^{2l-1}(q) - M_{i_1}^{2l-1}(q)),$n

$$t_3 = i_{f_3},$$n

$$\vdots$$n

$$f_j = f_{j-1} + \max_{1 \leq q \leq n, 1 \leq l \leq n}(M_{i_{j-1}}^{2l-1}(q) - M_{i_{j-2}}^{2l-1}(q)),$n

$$t_j = i_{f_j}. □$$n

Now, by Lemma 7.2, the following reachable markings can be obtained in $\Gamma$:

$$M[a^{l_n} > M_{i_n}^1],$$n

$$M_{i_n}^1[b^{l_n-1}c > M_{i_n-1}^1],$$n

$$M_{i_n-1} = M_{i_{n-1}}^2 + M_{i_{n-1}}^1 - M_{i_{n-1}}^1 > M_{i_{n-1}}^2 \geq M_{i_{n-2}}^2.$$
Let
\[ P_1 = \{ p | M_{n}^1(p) - M_{n-1}^1(p) \neq 0 \} \quad \text{and} \quad Q = \{ p | M_{n-1}^3(p) - M_{n-2}^3(p) \neq 0 \}. \]

Clearly, by Claim 1, \( 1 \leq |P_1| \leq n \).

We now distinguish between two cases:

Case I: \( Q \subseteq P \). By Lemma 7.2, the following reachable markings can be obtained in \( \Gamma \):
\[
\begin{align*}
\overline{M}_{n-2}^1 &= M_{n-2}^1, \\
\overline{M}_{n-2}^2 &= M_{n-2}^2 + \overline{M}_{n-1}^1 - M_{n-2}^1 = M_{n-2}^3 + M_{n-2}^2 + M_{n-1}^1 - M_{n-2}^3 + M_{n-2}^2 \\
&= M_{n-2}^3 + M_{n-1}^1 - M_{n-2}^3.
\end{align*}
\]

By Claim 2 and the fact that \( Q \subseteq P \), we have
\[ M_{n-1}^1 - M_{n-2}^1 \geq M_{n-2}^3 + M_{n-1}^1 - M_{n-2}^1 \geq M_{n-1}^3. \]

Hence, \( \overline{M}_{n-2}^2 \geq M_{n-1}^3 \), and \( \overline{M}_{n-2}^2 [b^{n-1}c | a^{n-1}b^{n-1}] c \notin L(\Gamma_8) \), since \( t_{n-2} + 1 \leq t_{n-1} \), contradicting our assumption.

Case II: \( Q \subseteq P_1 \). By Lemma 7.2, the following reachable markings can be obtained in \( \Gamma \):
\[
\begin{align*}
\overline{M}_{n-1}^1 &= M_{n-1}^1 + M_{n-1}^1 - M_{n-1}^1 > M_{n-1}^1, \\
\overline{M}_{n-2}^1 &= M_{n-2}^1 + M_{n-1}^1 - M_{n-2}^1 > M_{n-2}^1, \\
\overline{M}_{n-3}^1 &= M_{n-3}^1 + M_{n-1}^1 - M_{n-2}^1 > M_{n-3}^1. \\
\end{align*}
\]

Let
\[ P_2 = \{ p | M_{n}^1(p) - M_{n-1}^1(p) + M_{n-1}^2(p) - M_{n-2}^3(p) \neq 0 \} \]
and
\[ Q = \{ p | M_{n-2}^5(p) - M_{n-3}^5(p) \neq 0 \}. \]

Clearly \( P_2 = P_1 \); hence, by Claim 1, \( 2 \leq |P_2| \leq n \). Again, one of the following cases holds: Case I: \( Q \subseteq P_2 \), Case II: \( Q \subseteq P_2 \). We handle both cases as before.

After \( n \) steps we have \( n \leq |P_n| \leq n \). Therefore, every transition is enabled. Hence, we have \( xcc \in L(\Gamma) \), where \( x \in \Sigma^* \). But \( xcc \notin L(\Gamma_8) \), contradicting our assumption that \( L(\Gamma) = L(\Gamma_8) \). Thus, \( L^*PL \subset L^*EL \). \( \square \)

**Theorem 8.3.** \( L^*L \subset L^*IL \).

**Proof.** The proof is similar to the proof of Theorem 7.6. \( \square \)
Theorem 8.4. $L^\Lambda EL \subseteq L^\Lambda IL$.

Proof. Consider the labeled E-Net $\Gamma = (P, T, V, K, M, \Sigma, \eta)$. We construct a marked labeled I-Net $\Gamma' = (P', T', V', K', M', \Sigma, \eta')$ where

$$P' = P \cup \{p_e\} \cup \{p_t | (t \in T) \land (\exists p \in P) V(p, t) = E\}$$

$$T' = T \cup \{\hat{t} | (t \in T) \land (\forall p \in P) V(p, t) = E\}$$

$$\cup \{[t, p] | (t \in T) \land (p \in P) \land V(p, t) = E\}.$$

The function $V'$ is defined as follows:

I. For all $p \in P$
   do
     for all $t \in T$
       do
         if $V(p, t) \neq 0$ then $V'(p, t) := 1$;
         if $\neg(\exists p' \in P) V(p', t) = E$ then $V'(t, p) := V(t, p)$
         else
           do
             $V'(\hat{t}, p) := V(t, p)$;
             $V'(p_o, \hat{t}) := 1$;
             $V'(t, p_t) := 1$
           end
         end
     if $V(p, t) = E$ then
       do
         $V'(p, \hat{t}) := 1$;
         $V'(p, [t, p]) := 1$;
         $V'(p_o, [t, p]) := 1$;
         $V'([t, p], p_t) := 1$
       end
   end

II. For all $t \in T$
   do
     $V'(p_o, t) := 1$;
     if $\neg(\exists p' \in P) V(p', t) = E$ then $V'(t, p_o) := 1$
     else $V'(\hat{t}, p_o) := 1$,
   end

In all other cases $V'(x, y) := 0$. Fig. 21 illustrates this construction.

Clearly $L(\Gamma') = L(\Gamma)$. Hence, $L^\Lambda EL \subseteq L^\Lambda IL$. by Lemma 7.5, we have $L^\Lambda EL \subseteq L^\Lambda IL$. \(\square\)

The theorems of Section 8 and Theorem 7.1 yield the hierarchy of labeled SUP-Nets illustrated in Fig. 22.
Fig. 21. (a) Example of a labeled E-Net $I_{23}$; (b) an equivalent labeled I-Net $I'_{23}$, i.e., $L(I_{23}) = L(I'_{23})$.

Fig. 22. Hierarchy of labeled SUP-Nets.

9. Further results on the hierarchy of Super-Nets

Theorem 9.1. $AL \subseteq LAL$.

Proof. Clearly $AL \subseteq LAL$.

Consider the labeled $\lambda$-free A-Net $I_{25}$ of Fig. 23. We have $ab \notin L(I_{25})$, but $aab \in L(I_{25})$. Hence by Lemma 5.3, $L(I_{25}) \notin AL$. It follows that $AL \subseteq LAL$. □

The following result has been obtained in [6].

Theorem 9.2. $LPL \subseteq L^\lambda PL$. 
The following theorem has been obtained in [5].

**Theorem 9.3.** \( LIL \subseteq L^4IL \).

**Definition 9.1.** Let \( \Gamma = (P, T, V, K, M, \Sigma, \eta) \) be a labeled \( \Sigma \)-free \( \Sigma \)-Net. Assume \( w = t_1t_2 \cdots t, \in T^+ \) and \( \eta(t_i) = \sigma \in \Sigma \). We say that \( i \) is a \( \sigma \)-index of \( w \). \( I(\sigma, w) \) will denote the set of all \( \sigma \)-indices of \( w \).

**Lemma 9.1.** Let \( S = (P, T, V, K, M) \) be a marked \( \Sigma \)-Net or a marked \( \Sigma \)-Net. Let \( x = t_1t_2 \cdots t, \) be a firing sequence of \( S \). Let \( y \) be a permutation of \( x \). If \( y \) is also a firing sequence of \( S \), then
\[
M[x > M' \iff M[y > M'].
\]

**Proof.** This result is evident. \( \square \)

The proof of the following theorem uses a similar technique as the proof of Theorem 2 in [6].

**Theorem 9.4.** \( LEL \subseteq L^4EL \).

**Proof.** Clearly \( LEL \subseteq L^4EL \) by definition. Now, consider the labeled E-Net \( \Gamma_26 \) of Fig. 24. It will be shown that \( L(\Gamma_26) \notin LEL \).

Assume that \( L(\Gamma') = L(\Gamma_26) \) where \( \Gamma' \) is a labeled \( \lambda \)-free E-Net. Let \( L(\Gamma'') = L(\Gamma') \) where \( \Gamma'' = (P, T, V, K, M, \Sigma, \eta) \) is a labeled \( \Sigma \)-Net constructed as in Theorem 8.4, and \( \Sigma = \{a, b, c\} \). We define the following sets:

\[
A = \left\{ x \in (a+b)^*c^i \mid i = \sum_{j \in I(a,c)} 2^j \right\},
\]

\[
B = \left\{ x \in (a+b)^*c^i \mid i > \sum_{j \in I(a,c)} 2^j \right\}.
\]
One easily verifies that $A \subseteq L(\Gamma_{26})$ and $B \cap L(\Gamma_{26}) = \emptyset$.

For any positive integer $n$, let

$$C_n = \{ x \in (a + b)^* | \text{length of } x \text{ is } n \},$$

$$D_n = \{ M' | M[x > M'] \land (x \in C_n) \land (\exists i \in \omega) [(x c^i \in A) \land (\exists M'')(M'[c^i > M''])] \}.$$

Claim 1. $|D_n| \geq |C_n| = 2^n$.

Proof. Let $i < j$, $x c^i \in A$, $x c^j \in A$, where $x_i, x_j \in C_n$. Let $M[x_i > M_n, M[x_j > M_j$. If $M_i = M_j$, we have $M_i[c^i > M'$ for some $M'$. Hence $x c^i \in L(\Gamma')$. But this contradicts the condition $B \cap L(\Gamma_{26}) = \emptyset$. Therefore, $M_i \neq M_j$. Hence the number of markings in $D_n$ equals to or is greater than the number of words in $C_n$, which is $2^n$. \qed
Let \( m = \max_{p \in P} M(p) \).

**Claim 2.** In order to generate a word of length \( n \) in \( \Gamma' \) we have to fire at most \( m|P| + n|P| + n \) transitions.

**Proof.** In order to remove all tokens of the initial marking we have to fire no more than \( m|P| \lambda \)-transitions. Every place can receive no more than \( n \) tokens while the word is generated. In order to remove these tokens we have to fire at most \( n|P| \lambda \)-transitions. In order to remove the word itself, we have to fire \( n \) transitions. \( \square \)

Let \( n < \max(m, |P|, 3) \). Hence, \( n^3 > n + m|P| + n|P| \). The number of combinations to choose \( k \) \( \tau \)'s such that \( t \in T \) is \( \binom{|T| + k - 1}{k} \). Therefore, by Lemma 9.1 and Claim 2,

\[
|D_n| \leq \binom{|T| + n - 1}{n} + \binom{|T| + n}{n + 1} + \binom{|T| + n + 1}{n + 2} + \ldots + \binom{|T| + n^3 - 1}{n^3}
\]

\[
< n^3 \binom{|T| + n^3 - 1}{n^3}.
\]

But if we choose \( n \) big enough we will have \([n^3 \binom{|T| + n^3 - 1}{n^3} < 2^n]\) contradicting Claim 1. Hence \( L(\Gamma_{26}) \notin LEL \). Thus, \( L^\lambda EL \Rightarrow LEL \). \( \square \)

In view of the results of this section, as well as arguments used in Sections 5, 7, and 8, we obtain the hierarchy of SUP-Nets shown in Fig. 25.

![Fig. 25. Hierarchy of SUP-Nets.](image)

The hierarchies derived in Sections 5, 7, 8, 9 are not complete. However, we list the following conjectures, by means of which the hierarchies could be completed.

**Conjectures.**

(a) \( L(\Gamma_8) \notin L LL \).

(b) \( L(\Gamma_8) \notin L^\lambda LL \).

(c) \( L(\Gamma_{20}) \notin L^\lambda EL \).
10. LGPL and L^AGPL are not closed under Kleene star (iteration)

The following theorem solves an open problem, discussed in [10, p. 186].

**Theorem 10.1.** LGPL and L^AGPL are not closed under Kleene star (iteration).

**Proof.** Consider the marked GP-Net $S_{27}$ of Fig. 26.

![Fig. 26. Example of a marked GP-Net $S_{27}$.](image)

Consider the marked E-Net $S_8$ of Fig. 8. One easily verifies that $L(S_8) = (L(S_{27}))^*$. In the proof of the Theorem 8.2 we showed that there is no labeled P-Net $I'$ such that $L(I') = L(S_8)$. Since $L^A_{PL} = L^A_{GPL}$ by Theorem 8.1 it follows that there is no labeled GP-Net $I''$ such that $L(I'') = L(S_8) = (L(S_{27}))^*$. Hence we proved the theorem. □

**References**


