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SUPER-NETS AND THEIR HIERARCHY

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Abstract. In this paper we introduce the concept of Super-Net, which subsumes various extensions of Petri nets, proposed earlier in the literature. We consider languages associated with both non-labeled as well as labeled Super-Nets, and use these languages to establish a hierarchy between various types of nets. In the last section we apply this hierarchy to show that Petri net languages are not closed under Kleene star.

1. Introduction

An extensive literature is presently available demonstrating the suitability of Petri nets to the concise and precise modeling of systems involving concurrency (see e.g. [2, 3, 4, 10]). In order to further expand the modeling power of Petri nets, numerous modifications and extensions have been proposed. In particular, we mention the introduction of inhibitor arcs [1, 5, 10], OR-logic transitions [1], Boolean-type places [13, 14, 15], and places with finite capacity [7].

In this paper we introduce the concept of Super-Net which subsumes all the above extensions. Furthermore, Super-Nets also include 'emptying' arcs, which are intended to model the reset-to-zero facility of counting devices.

In modeling, power of various net families can be precisely compared by means of formal languages associated with such nets [5, 6, 10, 16].

In this connection the transitions of a net are frequently considered to be labeled by letters from some finite alphabet Σ [5, 6, 10].

In this paper we consider languages associated with both non-labeled as well as labeled Super-Nets. We define various special types of Super-Nets and use their associated languages in order to establish a hierarchy between them. Some of our theorems reformulate known results. However, we consider most of our results to be new.

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2. Super-Nets

We denote by ω the set of non-negative integers.

Definition 2.1. A *SUP-Net* (Super-Net) is a 4-tuple $N=(P, T, V, K)$, where
(1) P and T are finite sets of *places* and *transitions*, respectively,
(2) $P \cap T = \emptyset$, $P \cup T \neq \emptyset$,
(3) V is a function,

$$V:(P \times T) \cup (T \times P) \to \omega \cup \{I, E, L\};$$

Note. I, E, L are symbols indicating '*Inhibiting*', '*Emptying*', and '*Logical*' arcs, respectively.
(4) $V(T \times P) \subseteq \omega$,
(5) K is a function

$$K:P \to \{\infty\} \cup \omega \times \{A, R\};$$

Note. A, R are symbols indicating '*Absorbing*', and '*Restricting*' places.

If $K(p)=\infty$, we say that the place p has infinite capacity. If $K(p)=(k, A)$ or $K(p)=(k, R)$ we say that the place p has finite capacity $k \in \omega$. We denote by $k(p)$ the capacity (∞ or $k \in \omega$) of the place p.

Definition 2.2. A *marked SUP-Net* is a pair $S=(N, M)$, where N is a SUP-Net and M is a *marking* of N, i.e. a function $M:P \to \omega$, satisfying the condition

$$(\forall p \in P)[k(p) \in \omega \to M(p) \leqslant k(p)].$$

A marked SUP-Net $S=(P, T, V, K, M)$ is represented graphically as follows:
(1) places are represented by circles (○),
(2) each place p is labeled by $p/K(p)$,
(3) a transition t is represented by a bar, labeled by t,
(4) the place $p \in P$ is connected by a directed arc to the transition $t \in T$, iff $V(p, t) \neq 0$; the arc is labeled by $V(p, t)$,
(5) The transition $t \in T$ is connected by a directed arc to the place $p \in P$, iff $V(t, p) > 0$; the arc is labeled by $V(t, p)$,
(6) The integer $m=M(p)$ is written inside the circle representing p; usually, one does not write 0 inside the circle.
An example of a marked SUP-Net is shown in Fig. 1.

In the sequel we need the following:

Definition 3. Let $S=(P, T, V, K, M)$ be a marked SUP-Net. We define a function $W:P \times T \to \omega$ as follows:

$$W(p,t)=\begin{cases} V(p,t) & \text{if } V(p,t)\in\omega, \\ 0 & \text{if } V(p,t)=I, \\ M(p) & \text{if } V(p,t)=E, \\ 1 & \text{if } V(p,t)=L\wedge M(p)>0, \\ 0 & \text{if } V(p,t)=L\wedge M(p)=0. \end{cases}$$

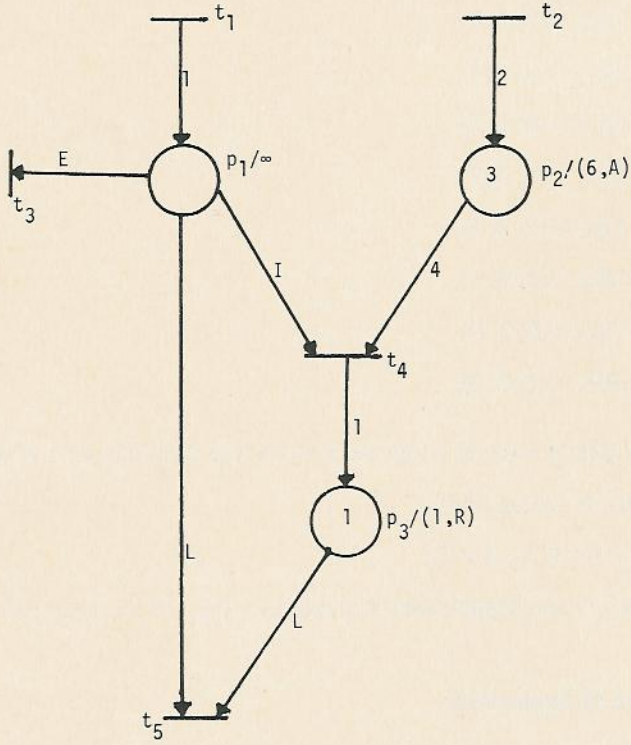


Fig. 1. Example of marked SUP-Net.

Definition 2.4. Let $S=(P, T, V, K, M)$ be a marked SUP-Net. A transition $t\in T$ is *enabled* iff the following conditions are satisfied:

(1) $(\forall p\in P)[V(p,t)\in\omega\rightarrow M(p)\geqslant V(p,t)]$,
(2) $(\forall p\in P)[V(p,t)=I\rightarrow M(p)=0]$,
(3) $(\forall p\in P)[V(p,t)=E\rightarrow M(p)>0]$,
(4) $(\exists p\in P)V(p,t)=L\rightarrow(\exists p\in P)[V(p,t)=L\wedge M(p)>0]$,
(5) $(\forall p\in P)[K(p)\in(\omega,R)\rightarrow M(p)+V(t,p)-W(p,t)\leqslant k(p)]$.

Definition 2.5. Let $S=(P, T, V, K, M)$ be a marked SUP-Net and $t\in T$ an enabled transition of S. We define the marking M' of $N=(P, T, V, K)$ as follows:

$$(\forall p\in P)M'(p)=\min[M(p)+V(t,p)-W(p,t), k(p)].$$

We say that M' is obtained from M by *firing* t (notation: $M[t>M']$). Frequently, it is convenient to represent a marking M by the vector $(M(p_1), M(p_2), \ldots, M(p_n))$, where $P=\{p_1, \ldots, p_n\}$.

For the example of Fig. 1 we obtain the following 'firing sequence',

$$(0, 3, 1)[t_2>(0, 5, 1),$$

$$(0, 5, 1)[t_2>(0, 6, 1),$$

$$(0, 6, 1)[t_5>(0, 6, 0),$$

$$(0, 6, 0)[t_4>(0, 2, 1),$$

$$(0, 2, 1)[t_1>(1, 2, 1),$$

$$(1, 2, 1)[t_1>(2, 2, 1),$$

$$(2, 2, 1)[t_1>(3, 2, 1),$$

$$(3, 2, 1)[t_5>(2, 2, 0),$$

$$(2, 2, 0)[t_3>(0, 2, 0).$$

Definition 2.6. Let $N=(P, T, V, K)$ be a SUP-Net. For every $t \in T$ we set

$$t^{\bullet} \triangleq \{p \in P \mid V(t, p) \neq 0\},$$

$$^{\bullet}t \triangleq \{p \in P \mid V(p, t) \neq 0\}.$$

We call every $p \in t^{\bullet}$ an *output* place of t, and every $p \in {}^{\bullet}t$ an *input* place of t.

3. Classification of Super-Nets

In this section we represent various types of nets as special cases of SUP-Nets.

Definition 3.1. The following table defines various types of nets as special cases of SUP-Nets, by restricting range(V) and range(K).

Type of Net	RANGE(V)	RANGE(K)
GP-Net	ω	$\{\infty\}$
P-Net	$\{0, 1\}$	$\{\infty\}$
I-Net	$\{0, 1\} \cup \{I\}$	$\{\infty\}$
E-Net	$\{0, 1\} \cup \{E\}$	$\{\infty\}$
L-Net	$\{0, 1\} \cup \{L\}$	$\{\infty\}$
A-Net	$\{0, 1\}$	$\omega \times \{A\}$
R-Net	$\{0, 1\}$	$\omega \times \{R\}$

The above defined I-Nets coincide with TN_{com} in [1]. They are called inhibitor nets in [5] and 'Petri nets with inhibitor arcs' in [10]. The above L-Nets coincide

with TN_{log} in [1]. The Boolean-type Petri nets, introduced in [14] coincide with A-Nets, where $RANGE(K) = \{(1, A)\}$.

The above-defined types of Super-Nets can be combined into more complex classes. E.g., the mixed-type nets discussed in [15] are Super-Nets with $RANGE(V) = \{0, 1\}$ and $RANGE(K) = \{\infty\} \cup \{(1, A)\}$. Similarly, the place-transition (P/T) nets of [7] are combinations of GP-type and R-type Super-Nets.

4. Super-Net languages

With a given marked SUP-Net $S = (P, T, V, K, M)$ we associate a language $L(S)$ over the alphabet T.

Definition 4.1. Let $N = (P, T, V, K)$ be a SUP-Net, and let $w \in T^+$, i.e. w is a finite string of transitions $w = t_1 t_2 \cdots t_r$. w is called a firing sequence of the marked SUP-Net $S = (N, M)$ iff there exist marking $M_1, \ldots, M_r$ such that

$$M[t_1 > M_1, M_1[t_2 > M_2, \ldots, M_{r-1}[t_r > M_r.$$

In this case we write $M[w > M_r$, and say that M_r is *reachable* from M. We also write $M[\lambda > M$ for every marking M, where λ denotes the empty sequence.

Definition 4.2. Let $S = (P, T, V, K, M)$ be a marked SUP-Net. We define its language $L(S)$ as follows:

$$L(S) \triangleq \{x \in T^* \mid (\exists M')M[x > M'\}.$$

In this paper we are also concerned with labeled Super-Nets and their languages.

Definition 4.3. A labeled SUP-Net is a triple $\Gamma = (S, \Sigma, \eta)$, where $S = (P, T, V, K, M)$ is a marked SUP-Net, Σ is a finite alphabet and η is a mapping $\eta: T \to \Sigma \cup \{\lambda\}$. Γ is called *λ-free* iff $\eta(T) \subseteq \Sigma$. The language of Γ is defined by

$$L(\Gamma) \triangleq \eta(L(S)) \triangleq \{\eta(x) \mid x \in L(S)\}.$$

If $M[w > M'$ where $w \in T^*$ we also write $M[\eta(w) > M'$, and say that M' is obtained from M by 'firing $\eta(w)$'. Clearly every marked SUP-Net S is also a labeled λ-free SUP-Net $\Gamma = (S, \Sigma, \eta)$ where $\Sigma = T$ and $(\forall t \in T)(\eta(t) = t)$.

Definition 4.4. Let L be a language over some finite alphabet Σ. We say that L is *GP-realizable* iff $L = L(S)$ for some marked GP-Net $S = (P, T, V, K, M)$ with $T = \Sigma$. L is *LGP-realizable* iff $L = L(\Gamma)$ for some labeled λ-free GP-Net $\Gamma = (S, \Sigma, \eta)$. L is *L^λGP-realizable* iff there exists an arbitrary labeled GP-Net Γ with $L = L(\Gamma)$. We denote by GPL, LGPL, L^λGPL the sets of all GP-realizable, LGP-realizable, and L^λGP-realizable languages, respectively.

In a similar way we associate sets of languages with the other types of Super-Nets defined above (Definition 3.1).

Evidently, all the languages defined above are prefix languages. Our language families GPL, LGLP, and L^{λ}GPL correspond to the classes P^f, P, and P^{λ} in [10, p. 157], respectively.

5. The hierarchy of Super-Nets (without labeling)

In this section we study the hierarchy of the following language families: GPL, PL, IL, EL, LL, AL, and RL.

Lemma 5.1. *The languages in* AL, LAL, L^{λ}AL, RL, LRL, L^{λ}RL *are regular.*

Proof. This result follows immediately, since the corresponding SUP-Nets all have finite sets of reachable markings. Hence each such SUP-Net may be viewed as finite automaton with the set of reachable markings as its state set. □

The following lemma is an immediate consequence of a result derived in [10, 11].

Lemma 5.2. *Let L be a language over Σ in either* RL *or* PL *or* IL *or* GPL. *Assume $x\sigma \in L$, where $x \in \Sigma^+$ and $\sigma \in \Sigma$. Furthermore, let y be a permutation of x, with $y \in L$. Then $y\sigma \in L$.*

For two sets A and B, we write $A \subseteq B$ to state that A is a subset of B, and $A \subset B$ to state that $A \subseteq B$ and $A \neq B$.

The following theorem is proven in [11].

Theorem 5.1. (a) PL $\subset$ IL.
(b) PL $\subset$ LL.
(c) IL *and* LL *are not* comparable, *i.e. neither* IL $\subseteq$ LL *nor* LL $\subseteq$ IL *holds.*

Theorem 5.2. (a) RL $\subset$ PL.
(b) RL $\subset$ AL.
(c) AL *and* PL *are not comparable.*

Proof. (a) Let S be a marked R-Net. One easily verifies that there exists a marked P-Net S', which is equivalent to S, i.e. $L(S) = L(S')$. This is illustrated in Fig. 2 (henceforth, we omit the label '1' of arcs). Thus, RL $\subseteq$ PL.

On the other hand, there exist languages in PL which are not regular, e.g. the language of the marked P-Net S_3 of Fig. 3.

Thus, by Lemma 5.1, we have RL $\subset$ PL.

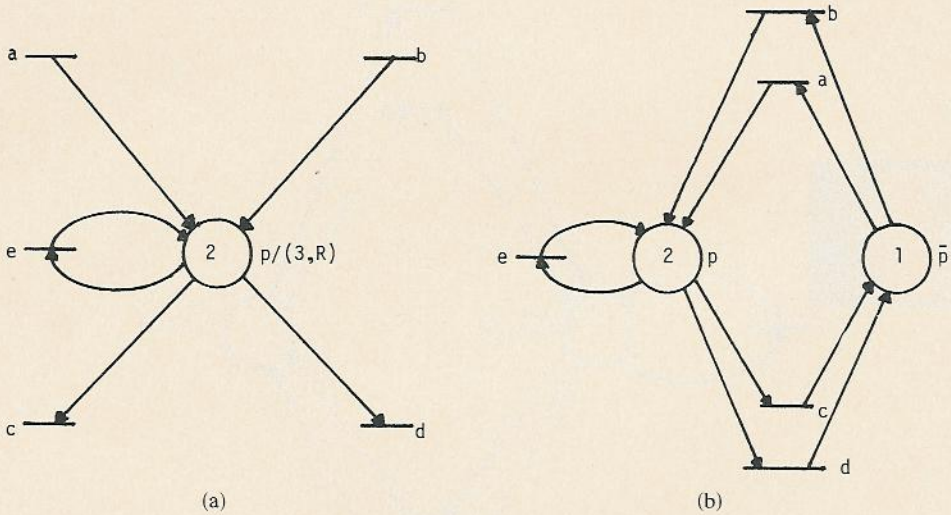


Fig. 2. (a) Example of marked R-Net S_1; (b) An equivalent marked P-Net S_2, i.e. $L(S_1) = L(S_2)$.

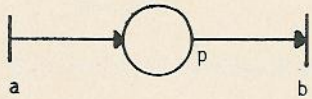


Fig. 3. Example of a marked P-Net S_3.

(b) With every marked R-Net one easily associates an equivalent marked A-Net (see Fig. 4). Hence RL $\subseteq$ AL.

Thus, by Lemma 5.1, we have RL $\subset$ PL.

To show that RL $\subset$ AL, we consider the marked A-Net S_5 of Fig. 5. Clearly $abab \in L(S_5)$ and $aab \in L(S_5)$, but $aabb \notin L(S_5)$. Hence, by Lemma 5.2, $L(S_5) \notin$ RL. Consequently, RL $\subset$ AL.

(c) Consider the marked P-Net S_3 of Fig. 3. Since $L(S_3)$ is not regular, PL $\subseteq$ AL does not hold, in view of Lemma 5.1. The above argument in the proof of (b) about S_5, also shows that $L(S_5) \notin$ PL. Hence AL $\subseteq$ PL does not hold. It follows that AL and PL are not comparable. □

Theorem 5.3. PL $\subset$ EL.

Proof. Clearly, PL $\subseteq$ EL, by Definition 3.1. Consider now the marked E-Net S_6 of Fig. 6. We have $L(S_6) = L(S_5)$. Hence $L(S_6) \notin$ PL. Thus, PL $\subset$ EL. □

Lemma 5.3. *Let L be a language over Σ in either* Al *or* PL *or* EL *or* LL. *Assume $\sigma_1\sigma_1\sigma_2 \in L$, where $\sigma_1 \in \Sigma$ and $\sigma_2 \in \Sigma$. Then $\sigma_1\sigma_2 \in L$.*

Proof. Let $L = L(S)$, where S is the relevant SUP-Net. Assume $\sigma_1\sigma_2 \notin L$, but $\sigma_1\sigma_1\sigma_2 \in L$. After firing σ_1 in S for the first time, there exists a set of input places

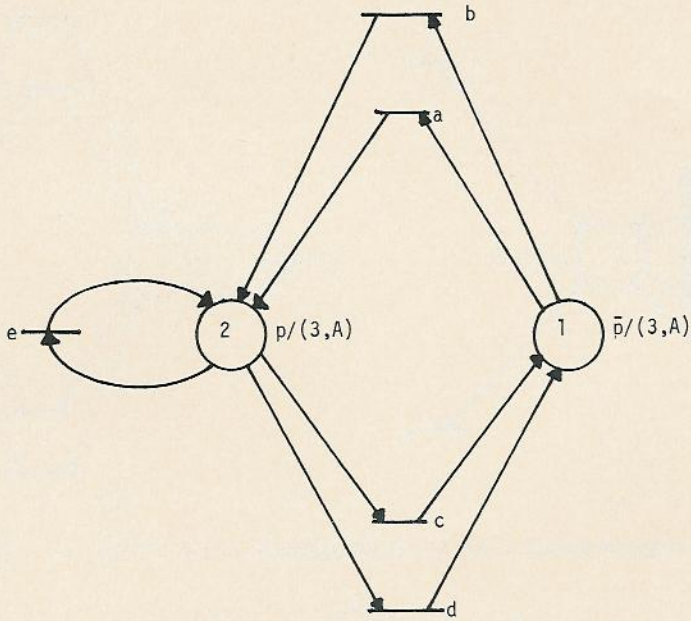


Fig. 4. A marked A-Net S_4 equivalent to the marked R-Net S_1 of Fig. 2(a).

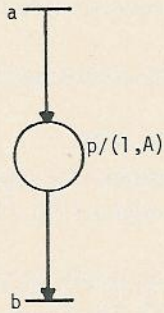


Fig. 5. Example of a marked A-Net S_5.

Fig. 6. Example of a marked E-Net S_6.

of σ_2 which are not marked, preventing σ_2 from firing. But this set of input places of σ_2 remains not marked, after σ_1 is fired a second time, in contradicting with our assumption that $\sigma_1\sigma_1\sigma_2 \in L$. Hence $\sigma_1\sigma_2 \in L$. □

Theorem 5.4. PL ⊂ GPL.

Proof. Evidently, PL ⊆ GPL, by Definition 3.1. We now consider the marked GP-Net S_7 of Fig. 7. We have $ab \notin L(S_7)$, but $aab \in L(S_7)$. Hence, by Lemma 5.3, $L(S_7) \notin$ PL. It follows that PL ⊂ GPL. □

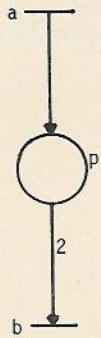


Fig. 7. Example of a marked GP-Net S_7.

Let $S = (P, T, V, K, M)$ be a marked L-Net. For any $t \in T$ we define an equivalence relation E_t on ${}^\bullet t$ as follows:

$$E_t \triangleq \{(p, p') \mid p \in {}^\bullet t \wedge p' \in {}^\bullet t \wedge [p = p' \vee V(p, t) = V(p', t) = L]\}.$$

We denote by Q_t the partition ${}^\bullet t/E_t$ of ${}^\bullet t$. For any $q \in Q_t$, we set

$$M(q) = \max\{M(p) \mid p \in q\}.$$

We say that $q \in Q_t$ is an *output* of t, iff $(\exists p \in q)\, V(t, p) = 1$.

Theorem 5.5. *The language types* EL *and* LL *are not comparable.*

Proof. (a) Consider the marked E-Net S_8 of Fig. 8. We wish to show that $L(S_8) \notin$ LL.

Assume that $L(S_8) = L(S)$, where $S = (N, M)$ is a marked L-Net. We have $a^4b^4 \in L(S)$. Let $M[a^4 > M'$ in S. Then, for every q in Q_b which is not an output of b, we must have $M'(q) \geqslant 4$. Now, $a^4cab \in L(S)$. Let $M'[cab > M''$ in S. Then, for every q in Q_b which is not an output of b, $M''(q) \geqslant 1$. Also, since M'' was obtained by firing b, we must have $M''(q) \geqslant 1$ for every q in Q_b which is an output of b. Hence b is enabled by M''. It follows that $a^4cab^2 \in L(S)$, but $a^4cab^2 \notin L(S_8)$, contradicting our assumption that $L(S_8) = L(S)$. Consequently, $L(S_8) \notin LL$, i.e. EL ⋢ LL.

(b) To show that LL ⋢ EL, we consider the marked L-Net S_9 of Fig. 9. Assume $L(S_9) = L(S)$, where $S = (N, M)$ is marked E-Net. Since $b \in L(S)$ and $b^2 \notin L(S)$, we must have ${}^\bullet b - b^\bullet \neq \emptyset$.

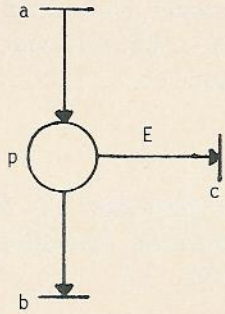


Fig. 8. Example of a marked E-Net S_8.

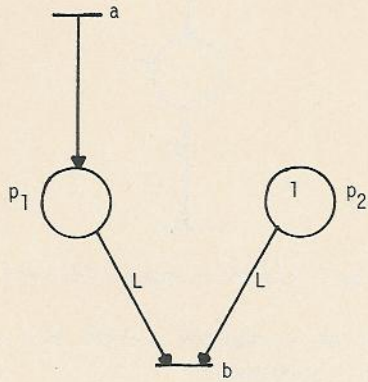


Fig. 9. Example of a marked L-Net S_9.

Let $k = \max\{M(p) \mid p \in {}^\bullet b - b^\bullet\}$. Since $b \in L(S)$, we have $k \geqslant 1$ as well as $(\forall p \in {}^\bullet b) M(p) \geqslant 1$. Now, $a^{k+1} b^{k+1} \in L(S)$. Therefore, $(\forall p \in {}^\bullet b - b^\bullet)(p \in a^\bullet - {}^\bullet a)$. Let $M[a^{k+1} > M'$. It follows that $(\forall p \in {}^\bullet b - b^\bullet)[M'(p) \geqslant k+2]$. Since $a^{k+1} b^2 \in L(S)$, we must have $(\forall p \in {}^\bullet b - b^\bullet) V(p, b) \neq E$. Let $M'[b^{k+1} > M''$. Then $(\forall p \in {}^\bullet b - b^\bullet) M''(p) \geqslant 1$. Since M'' was obtained by firing b, we must also have $(\forall p \in {}^\bullet b \cap b^\bullet) M''(p) \geqslant 1$. It follows that b is enabled in M'', hence $a^{k+1} b^{k+2} \in L(S)$. But $a^{k+1} b^{k+2} \notin L(S_9)$. Consequently, $L(S_9) \notin \mathrm{EL}$. Thus, $\mathrm{LL} \not\subseteq \mathrm{EL}$.

Since $\mathrm{LL} \not\subseteq \mathrm{EL}$ and $\mathrm{EL} \not\subseteq \mathrm{LL}$, Theorem 5.5 is proven. □

Theorem 5.6. AL *is not comparable with either* GPL *or* IL *or* LL *or* EL.

Proof. (a) Consider the marked P-Net S_3 of Fig. 3. S_3 is also a marked GP-Net, I-Net, L-Net and E-Net. Since $L(S_3)$ is not regular, neither $\mathrm{GPL} \subseteq \mathrm{AL}$ nor $\mathrm{IL} \subseteq \mathrm{AL}$ nor $\mathrm{LL} \subseteq \mathrm{AL}$ nor $\mathrm{EL} \subseteq \mathrm{AL}$ can hold, in view of Lemma 5.1.

(b) Consider the marked A-Net S_5 of Fig. 5. The argument in the proof of Theorem 5.2(b) about S_5, also shows that $L(S_5) \notin \mathrm{GPL}$ and $L(S_5) \notin \mathrm{IL}$. Consequently neither $\mathrm{AL} \subseteq \mathrm{GPL}$ nor $\mathrm{AL} \subseteq \mathrm{IL}$ can hold.

(c) Consider the marked A-Net S_{10} of Fig. 10. Assume that $L(S_{10}) = L(S)$, where $S = (N, M)$ is a marked L-Net. We have $a^3b^2 \notin L(S)$. Let $M[a^3 > M'$ and $M'[b^2 > M''$ in S. Since $a^3b^3 \notin L(S)$, there exists an element q of Q_b which is not an output of b, such that $M'(q) = 2$ and $M''(q) = 0$. Since $M'(q) = 2$ we have $(\forall p \in q) \cdot [V(a, p) = 1 \rightarrow V(p, a) \neq 0]$. We also have $a^3b^2a^2 \in L(S)$. Let $M''[a^2 > M'''$. Thus, we have $(\forall p \in q)[M'''(p) \leqslant 1]$. It follows that $a^3b^2a^2b^2 \notin L(S)$, but $a^3b^2a^2b^2 \in L(S_{10})$, contradicting our assumption that $L(S_{10}) = L(S)$. Consequently, AL $\subseteq$ LL does not hold.

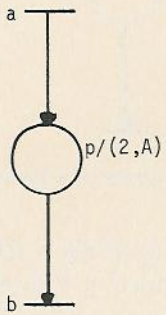


Fig. 10. Example of a marked A-Net S_{10}.

(d) Consider the marked A-Net S_{10} of Fig. 10. Assume that $L(S_{10}) = L(S)$, where $S = (N, M)$ is a marked E-Net. We have $a^3b^2 \in L(S)$. Let $M[a^3 > M'$ and $M'[b^2 > M''$ in S. Since $a^3b^3 \notin L(S)$ we have $(\exists p \in {}^\bullet b - b^\bullet)[V(p, b) \neq E \wedge M'(p) = 2 \wedge M''(p) = 0]$. Let p satisfy this condition. Since $M'(p) = 2$, $V(a, p) = 1$ implies $V(p, a) = 1$.

We distinguish between two cases:

Case I: $V(a, p) = 1 \wedge V(p, a) = 1$. Since $M''(p) = 0$ it follows that $a^3b^2a \notin L(S)$.

Case II: $V(a, p) = 0$. We have $a^3b^2a \in L(S)$. Let $M''[a > M'''$ in S. Hence, $M'''(p) = 0$. It follows that $a^3b^2ab \notin L(S)$.

But, since $a^3b^2ab \in L(S_{10})$, both Cases I and II contradict our assumption that $L(S_{10}) = L(S)$. Consequently, AL $\subseteq$ EL does not hold.

Parts (a), (b), (c) and (d) complete the proof of Theorem 5.6. □

Lemma 5.4. *Let L be a language over Σ in either* EL *or* GPL. *Let $S = (N, M)$ be a marked* E-*Net or a marked* GP-*Net. Let $y_1, y_2 \in L(S)$ and $M[y_1 > M_1$, $M[y_2 > M_2$ in S. If $M_2 \geqslant M_1$ and $y_1y_3 \in L(S)$ then $y_2y_3 \in L(S)$.*

Proof. Since $M_2 \geqslant M_1$ and y_3 is firable in M_1, y_3 is also firable in M_2. □

A similar lemma appears in [11].

Theorem 5.7. IL *is not comparable with either* EL *or* GPL.

Proof. (a) Consider the marked I-Net S_{11} of Fig. 11. Using Lemma 5.4 one easily shows that $L(S_{11}) \notin$ GPL (cf. [1, 9, 11]) and that $L(S_{11}) \notin$ EL.

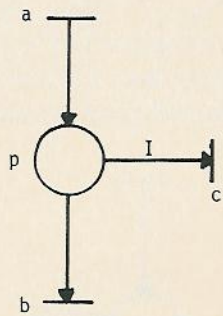


Fig. 11. Example of a marked I-Net S_{11}.

(b) Consider the marked E-Net S_6 of Fig. 6. Clearly $abab \in L(S_6)$ and $aab \in L(S_6)$, but $aabb \notin L(S_6)$. Hence, by Lemma 5.2, $L(S_6) \notin$ IL.

(c) Consider the marked GP-Net S_7 of Fig. 7. Assume that $L(S_7) = L(S)$, where $S = (N, M)$ is a marked I-Net. We have $a \in L(S)$ but $ab \notin L(S)$. Let $M[a > M'$. We have one of the following two cases:

Case I: $(\exists p \in {}^\bullet b)[V(p, b) = 1 \wedge M'(p) = 0]$. Let p denote a place satisfying this condition. Now, $a^2 \in L(S)$, and let $M'[a > M''$. We still have $M''(p) = 0$, hence $a^2 b \notin L(S)$, but $a^2 b \in L(S_7)$, contradicting our assumption that $L(S_7) = L(S)$.

Case II: $(\exists p \in {}^\bullet b)[V(p, b) = I \wedge M'(p) > 0]$. Let again p denote a place satisfying this condition and $M'[a > M''$. Since $a^2 b \in L(S)$ we have $p \in {}^\bullet a$, $V(p, a) = 1$ and $M''(p) = 0$. Hence a is not enabled in M'', but $a^3 \in L(S_7)$, contradicting our assumption that $L(S_7) = L(S)$.

We have thus proved that GPL $\subseteq$ IL does not hold.

Parts (a), (b) and (c) complete the proof of Theorem 5.7. □

Theorem 5.8. GPL *is not comparable with either* LL *or* EL.

Proof. (a) Consider the marked GP-Net S_7 of Fig. 7. We have $ab \notin L(S_7)$, but $aab \in L(S_7)$. Hence, by Lemma 5.3 $L(S_7) \notin$ LL and $L(S_7) \notin$ EL.

(b) Consider the marked L-Net S_9 of Fig. 9. Clearly $bab \in L(S_9)$ and $ab \in L(S_9)$, but $abb \notin L(S_9)$. Hence, By Lemma 5.2 $L(S_9) \notin$ GPL.

(c) Consider the marked E-Net S_6 of Fig. 6. Clearly, $abab \in L(S_6)$ and $aab \in L(S_6)$, but $aabb \notin L(S_6)$. Hence, by Lemma 5.2 $L(S_6) \notin$ GPL.

Parts (a), (b) and (c) complete the proof of Theorem 5.8. □

The theorems of Section 5 yield the hierarchy of SUP-Nets illustrated in Fig. 12, where XL → YL denotes XL ⊂ YL, and XL - - - YL indicates that XL and YL are not comparable.

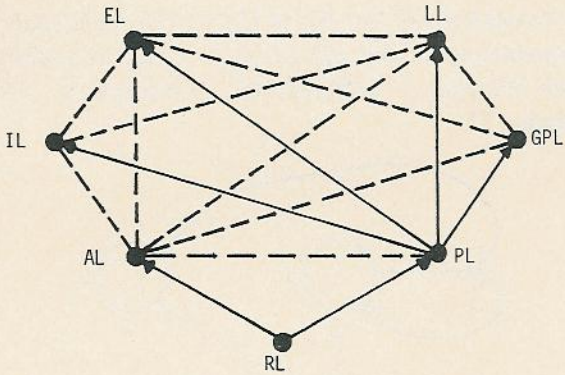


Fig. 12. Hierarchy of SUP-Nets.

6. Some closure properties of GPL

By means of Lemma 5.2, solutions to some open problems mentioned in [10, p. 186] are easily provided.

Theorem 6.1. GPL *is not closed under union.*

Proof. Consider the marked GP-Nets S_{12} and S_{13} of Fig. 13. We have $abb \in L(S_{12}) \cup L(S_{13})$ and $ba \in L(S_{12}) \cup L(S_{13})$ but $bab \notin L(S_{12}) \cup L(S_{13})$. Hence, by Lemma 5.2 $L(S_{12}) \cup L(S_{13}) \notin$ GPL. Consequently, GPL is not closed under union. □

Theorem 6.2. GPL *is not closed under concatenation.*

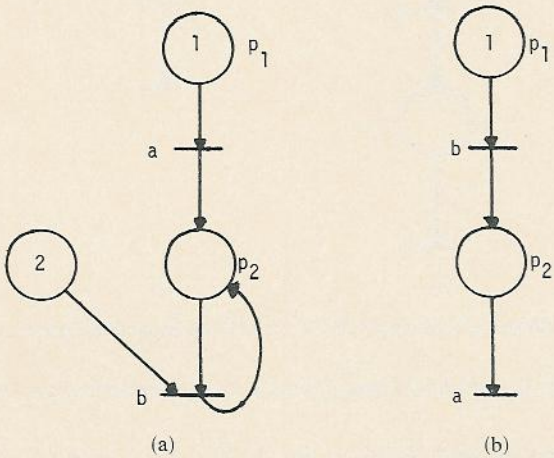


Fig. 13. (a) Example of a marked GP-Net S_{12}; (b) example of a marked GP-Net S_{13}.

Proof. Consider the marked GP-Net S_{14} of Fig. 14. We have $abab \in L(S_{14}) \cdot L(S_{14})$, where $\cdot$ denotes concatenation, and $aab \in L(S_{14}) \cdot L(S_{14})$ but $aabb \notin L(S_{14}) \cdot L(S_{14})$. Hence, by Lemma 5.2 $L(S_{14}) \cdot L(S_{14}) \notin$ GPL. Consequently, GPL is not closed under concatenation. □

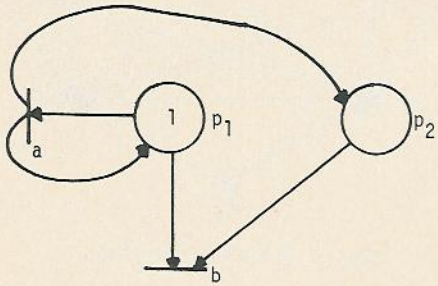


Fig. 14. Example of a marked GP-Net S_{14}.

Theorem 6.3. GPL *is not closed under the concurrency* (*shuffle*) *operator* $\|$ (*see* [10]).

Proof. Consider the marked GP-Nets S_{15} and S_{16} of Fig. 15. Clearly $abc \in L(S_{15})\|L(S_{16})$ and $ba \in L(S_{15})\|L(S_{16})$, but $bac \notin L(S_{15})\|L(S_{16})$. Hence, by Lemma 5.2 $L(S_{15})\|L(S_{16}) \notin$ GPL. Consequently, GPL is not closed under the concurrency operator. □

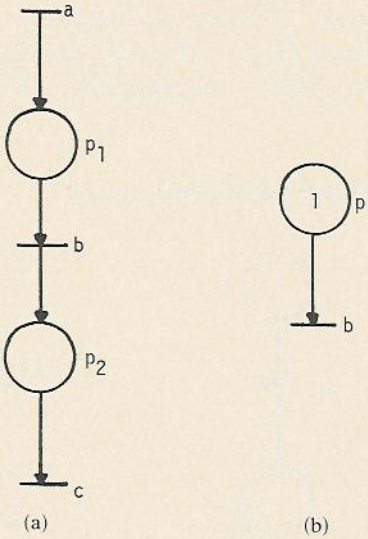


Fig. 15. (a) Example of a marked GP-Net S_{15}; (b) example of a marked GP-Net S_{16}.

Theorem 6.4. GPL *is not closed under prefix regular substitution* (*see* [10]).

Proof. Consider the marked GP-Net S_{17} of Fig. 16. Consider the prefix regular substitution $f(a) = \{\lambda, a, ab, abb, b, ba\}$. We have $abb \in f(L(S_{17}))$ and $ba \in f(L(S_{17}))$

Fig. 16. Example of a marked GP-Net S_{17}.

but $bab \notin f(L(S_{17}))$. Hence, by Lemma 5.2 $f(L(S_{17}) \notin$ GPL. Consequently, GPL is not closed under prefix regular substitution. □

Theorem 6.5. GPL *is not closed under Kleene star* (*iteration*).

Proof. Consider the marked GP-Net S_{14} of Fig. 14. We have $abab \in (L(S_{14}))^*$ and $aab \in (L(S_{14}))^*$ but $aabb \notin (L(S_{14}))^*$. Hence, by Lemma 5.2 $(L(S_{14}))^* \notin$ GPL. Consequently, GPL is not closed under Kleene star □.

In [10] Petri nets with final markings and the corresponding languages are discussed. One easily verifies that the results and proofs of Theorems 6.2, 6.3 and 6.5 also apply to the family of languages denoted by L^f in [10, p. 186].

Similar results concerning L^f are obtained (by a different approach) in [12].

7. Hierarchy of labeled λ-free Super-Nets

In this section we study the hierarchy of the various types of labeled λ-free SUP-Nets, defined in Section 3.

Theorem 7.1. *We have* LRL = LAL = L^λRL = L^λAL. *Each of these sets coincides with the set of all the prefix regular languages.*

Proof. (a) By Lemma 5.1, the languages in sets LRL, LAL, L^λRL, L^λAL are all prefix regular.

(b) Let L be a prefix regular language. Clearly there exists a finite, deterministic automaton A such that $L = L(A)$. By applying a construction similar to that in [5, p. 35], one obtains a labeled A-Net Γ as well as a labeled R-Net Γ such that $L(\Gamma) = L(A) = L$. □

In the sequel we let Γ_i denote the labeled λ-free SUP-Nets corresponding to the marked SUP-Nets S_i defined in Section 5.

Theorem 7.2. LRL ⊂ LPL.

Proof. Similar to proof of Theorem 5.2(a). □

M. Hack [5, p. 64] has shown that any labeled λ-free GP-Net Γ can be transformed into a labeled P-Net Γ' such that $L(\Gamma)=L(\Gamma')$. A stronger result is the following.

Theorem 7.3. LGPL = LPL.

Proof. Clearly LPL $\subseteq$ LGPL.

Consider the labeled λ-free GP-Net $\Gamma=(P, T, V, K, M, \Sigma, \eta)$. We construct a labeled λ-free P-Net $\Gamma'=(P', T', V', K', M', \Sigma, \eta')$ where

$$P'=\{[p, i] \mid p \in P, 0 \leqslant i \leqslant n_p-1\}$$

$$\text{and } n_p=\max(\max_{t \in T} V(p, t), \max_{t \in T} V(t, p)),$$

$$T'=\{[t, i] \mid t \in T, 1 \leqslant i \leqslant n_t\}$$

$$\text{and } n_t=\prod_{\substack{p \in P \\ V(p,t)>0 \\ V(p,t) \neq n_p}} n_p \cdot \prod_{\substack{p \in P \\ V(t,p)>0 \\ V(t,p) \neq n_p \\ V(p,t)=0 \vee V(p,t)=n_p}} n_p.$$

V' has to satisfy the following three conditions, where + indicates addition mod n_p:

I. Assume $V(p, t)=m$.

1. $m=0$: $(\forall i \in\{1, 2 \cdots n_t\})(\forall j \in\{0, 1 \cdots n_p-1\}) V'([p, j],[t, i])=0$.
2. $m=n_p$: $(\forall i \in\{1, 2 \cdots n_t\})(\forall j \in\{0, 1 \cdots n_p-1\}) V'([p, j],[t, i])=1$.
3. $0<m<n_p$: $(\forall i \in\{1, 2 \cdots n_t\})(\exists j(p, t, i) \in\{0, 1 \cdots n_p-1\})$
   $[V'([p, j(p, t, i)],[t, i])=1, V'([p, j(p, t, i)+1],[t, i])=1,$
   $\ldots, V'([p, j(p, t, i)+m-1],[t, i])=1, V'([p, j(p, t, i)+m],[t, i])=0,$
   $\ldots, V'([p, j(p, t, i)-1],[t, i])=0]$.

II. Assume $V(t, p)=k$.

1. $k=0$: $(\forall i \in\{1, 2 \cdots n_t\})(\forall j \in\{0, 1 \cdots n_p-1\}) V'([t, i],[p, j])=0$.
2. $k=n_p$: $(\forall i \in\{1, 2 \cdots n_t\})(\forall j \in\{0, 1 \cdots n_p-1\}) V'([t, i],[p, j])=1$.
3. $0<V(p, t)<n_p$: $(\forall i \in\{1, 2 \cdots n_t\})[V'([t, i],[p, j(p, t, i)])=1,$
   $\ldots, V'([t, i],[p, j(p, t, i)+k-1])=1, V'([t, i],[p, j(p, t, i)+k])=0,$
   $\ldots, V'([t, i],[p, j(p, t, i)-1])=0]$.
4. $V(p, t)=0 \vee V(p, t)=n_p$: $(\forall i \in\{0, 1 \cdots n_t\})(\exists j(p, t, i) \in\{0, 1 \cdots n_p-1\})$
   $[V'([t, i],[p, j(p, t, i)])=1, V'([t, i],[p, j(p, t, i)+1])=1.$
   $\ldots, V'([t, i],[p, j(p, t, i)+k-1])=1, V'([t, i],[p, j(p, t, i)+k])=0$
   $\ldots, V'([t, i],[p, j(p, j(p, t, i)-1])=0]$.

III. $\sim(\exists t \in T)(\exists i, j, i \neq j, 1 \leqslant i, j \leqslant n_t)(\forall p \in P')$
   $[V'([t, i], p)=V'[(t, j], p) \wedge V'(p,[t, i])=V'(p,[t, j])]$.

Furthermore, $(\forall p \in P')(K'(p) = \infty)$

$$(\forall p \in P)(\forall i, 0 \leq i \leq n_p - 1)(M'([p, i])$$
$$= \lfloor M(p)/n_p \rfloor + [i + 1 \leq \mathrm{mod}(M(p), n_p)] \qquad (*)$$

where [Prop] $\triangleq$ if Prop is true then 1 else 0;

$$(\forall t \in T)(\forall i, 1 \leq i \leq n_t)(\eta'([t, i]) = \eta(t)).$$

Fig. 17 illustrates this construction.

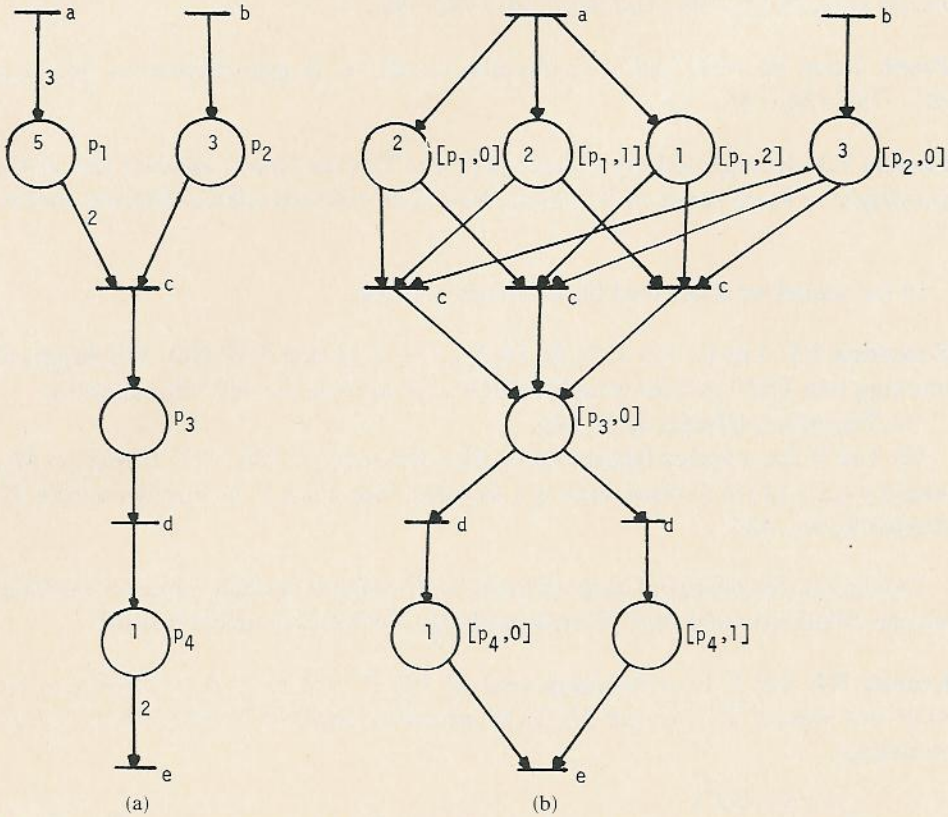


Fig. 17. (a) Example of a labeled λ-free GP-Net Γ_{18}; (b) an equivalent labeled λ-free P-Net Γ_{19}, i.e., $L(\Gamma_{19}) = L(\Gamma_{18})$.

One easily verifies that the validity of the condition $(*)$ can be preserved for any sequence of firings. Hence we have $L(\Gamma') = L(\Gamma)$. Thus, LPL = LGPL. □

The following lemma recalls a well-known result.

Lemma 7.1. *Let $\{a_i\}_{i=1}^{\infty}$ be an infinite sequence of vectors over ω^n. Then there exists an infinite increasing sequence of integers $\{i_j\}_{j=1}^{\infty}$ such that*

$$(\forall m, 1 \leq m \leq n)[(\forall j, k; j < k)(a_{i_j}(m) < a_{i_k}(m)) \vee (\forall j, k)(a_{i_j}(m) = a_{i_k}(m))].$$

Proof. Construct an infinite subsequence of $\{a_i\}_{i=1}^{\infty}$ increasing or non-changing in the first coordinate; construct from this subsequence an infinite subsequence increasing or nonchanging in the second coordinate and so forth. □

The following lemma is an extension of Lemma 5.4.

Lemma 7.2. *Let L be a language over Σ in* L^{λ}PL. *Let $\Gamma=(S,\Sigma,\eta)$ be a labeled* P-*Net where $L(\Gamma)=L$. Let $y_1, y_2 \in L(\Gamma)$ and $M[y_1>M_1, M[y_2>M_2$ in Γ. If $M_2 \geqslant M_1$, $y_1y_3 \in L(\Gamma)$ and $M_1[y_3>M_3$, then there exists a firing sequence $x \in L(S)$ such that $\eta(x)=y_2y_3$, $M_2[y_3>M_3'$ and $M_3'=M_3+M_2-M_1$.*

Proof. Since $M_2 \geqslant M_1$ and y_3 is fireable in M_1, y_3 is also fireable in M_2, and $M_3'-M_3=M_2-M_1$. □

Lemma 7.3 (König's Infinity Lemma [8]). *Let T be an infinite directed tree. If the out-degree of every vertex in T is finite, then there exists an infinite directed path in T.*

In the sequel we shall need the following concept.

Definition 7.1. Let $\Gamma=(P,T,V,K,M,\Sigma,\eta)$ be a labeled SUP-Net. We define its marking tree $\mathrm{Tr}(\Gamma)$ as a directed labeled tree in accordance with the following:

(1) The root is labeled by (λ, M);

(2) Let V_1 be a vertex labeled by (x, M_1), where $M[x>M_1$ in Γ. If $M_1[y>M_2$, where $y \in \Sigma \cup \{\lambda\}$ in Γ, then there is a directed edge from V_1 to another vertex V_2 labeled by (xy, M_2).

Clearly the out-degree of each vertex in $\mathrm{Tr}(\Gamma)$ is finite. Indeed, from any marking we can obtain no more than $|T|$ new markings, by firing a single transition.

Lemma 7.4. *Let L be a language over Σ. Let $\Gamma=(S,\Sigma,\eta)$ be a labeled λ-free* SUP-Net *where $L(\Gamma)=L$. Let $\{y_i\}_{i=0}^{\infty}$ be an infinite sequence of finite words defined as follows:*

$$y_0=\lambda \in L(\Gamma),$$

$$y_j=y_{j-1}\sigma_j \in L(\Gamma), \quad \sigma_j \in \Sigma, \quad j=1,2,\ldots.$$

Then there exists an infinite directed path in $\mathrm{Tr}(\Gamma)$ *the vertices of which represent the words of the sequence $\{y_i\}_{i=0}^{\infty}$.*

Proof. Let us concentrate only on vertices in $\mathrm{Tr}(\Gamma)$ which represent words in the sequence $\{y_i\}_{i=0}^{\infty}$. The corresponding subtree of $\mathrm{Tr}(\Gamma)$ is an infinite tree because the sequence $\{y_i\}_{i=0}^{\infty}$ is infinite. Since the subtree is infinite and the out-degree of each vertex in the tree is finite there exists an infinite directed path in $\mathrm{Tr}(\Gamma)$, passing through some of those vertices by Lemma 7.3. Each directed step along an edge

in this path represents a firing of some $\sigma \in \Sigma$ since Γ is λ-free. Furthermore, all prefixes of any word $y_j \in \{y_i\}_{i=0}^{\infty}$ belong to the sequence $\{y_i\}_{i=0}^{\infty}$; hence, the path's vertices represent all the words of the sequence $\{y_i\}_{i=0}^{\infty}$. $\square$

Theorem 7.4. LPL $\subset$ LLL.

Proof. Clearly LPL $\subseteq$ LLL, by definition. Consider the labeled λ-free L-Net Γ_{20} of Fig. 18.

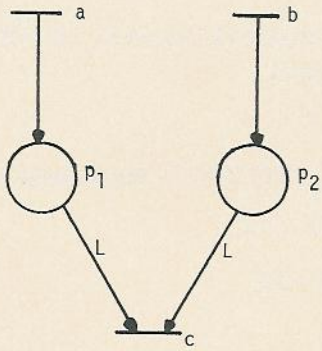


Fig. 18. Example of a labeled λ-free L-Net Γ_{20}.

Assume that $L(\Gamma_{20}) = L(\Gamma)$ where Γ is a labeled λ-free P-Net. By Lemma 7.4 there exists an infinite directed path in $\mathrm{Tr}(\Gamma)$ the vertices of which represent the prefixes of the infinite word

$$acbca^2c^2b^2c^2 \cdots a^ic^ib^ic^i \cdots .$$

We concentrate on markings of this path. Let $\{M_i\}_{i=2}^{\infty}$ be the infinite sequence where $M[acbc \cdots b^{i-1}c^{i-1}a^i > M_i$. Let $\{\bar{M}_i\}_{i=2}^{\infty}$ be the infinite sequence where $M[acbc \cdots a^{i-1}c^{i-1}b^{i-1}c^{i-1} > \bar{M}_i$.

By Lemma 7.1 we construct an infinite increasing sequence $\{i_j\}_{j=1}^{\infty}$ such that $(\forall j, k; j < k)[M_{i_j} \leqslant M_{i_k}]$.

Now, let us look at the subsequence $\{\bar{M}_{i_j}\}_{j=1}^{\infty}$. By Lemma 7.1 we can find two integers l and k such that $l < k$ and $\bar{M}_{i_l} \leqslant \bar{M}_{i_k}$. Let $s = i_l$, $r = i_k$. Then, $M_s \leqslant M_r$ and $\bar{M}_s \leqslant \bar{M}_r$.

Now, by Lemma 7.2 the following markings can be obtained in Γ:

$$M[acbc \cdots b^{r-1}c^{r-1}a^r > M_r,$$

$$M_r[c^sb^sc^s \cdots a^{r-1}c^{r-1}b^{r-1}c^{r-1} > M_x,$$

$$M_x = \bar{M}_r + M_r - M_s \geqslant \bar{M}_r \geqslant \bar{M}_s,$$

$$M_x[a^s > M_y,$$

$$M_y = M_s + (\bar{M}_r + M_r - M_s) - \bar{M}_s = M_r + \bar{M}_r - \bar{M}_s \geqslant M_r,$$

$$M_y[c^r > M_z.$$

But $acbc \cdots b^{r-1}c^{r-1}\underline{a^r c^s b^s c^s} \cdots a^{r-1}c^{r-1}b^{r-1}c^{r-1}\underline{a^s c^r} \notin L(\Gamma_{20})$ since $\max(r, 2s)+s<2s+r$, contradicting our assumption that $L(\Gamma_{20})=L(\Gamma)$. Hence, LPL$\subset$LLL. □

The following theorem compares LPL and LEL. A stronger result requiring a rather complex proof, will be obtained in Section 8 (Theorem 8.2).

Theorem 7.5. LPL$\subset$LEL.

Proof. The proof is similar to that of Theorem 7.4, with Γ_{20} replaced by Γ_8. The infinite word of interest becomes

$$aca^2bca^3b^2c \cdots a^ib^{i-1}c \cdots .$$

The infinite sequences $\{M_i\}_{i=2}^{\infty}$ and $\{\bar{M}_i\}_{i=2}^{\infty}$ are defined by

$$M[ac \cdots a^{i-1}b^{i-2}ca^i > M_i,$$

$$M[ac \cdots a^{i-1}b^{i-2}c > \bar{M}_i,$$

respectively. □

Lemma 7.5. *There exists a language in* IL *which is neither in* L$^\lambda$LL *nor in* L$^\lambda$EL.

Proof. Consider the marked I-Net S_{11} of Fig. 11. By an argument similar to that used in the proof of Theorem 5.7, one easily shows that $L(S_{11}) \notin$ L$^\lambda$LL and $L(S_{11}) \notin$ L$^\lambda$EL. □

Theorem 7.6. LLL$\subset$LIL.

Proof. By a construction, similar to that of [1] one easily proves that LLL$\subseteq$LIL. This construction is illustrated in Fig. 19. In view of Lemma 7.5, we thus have LLL$\subset$LIL. □

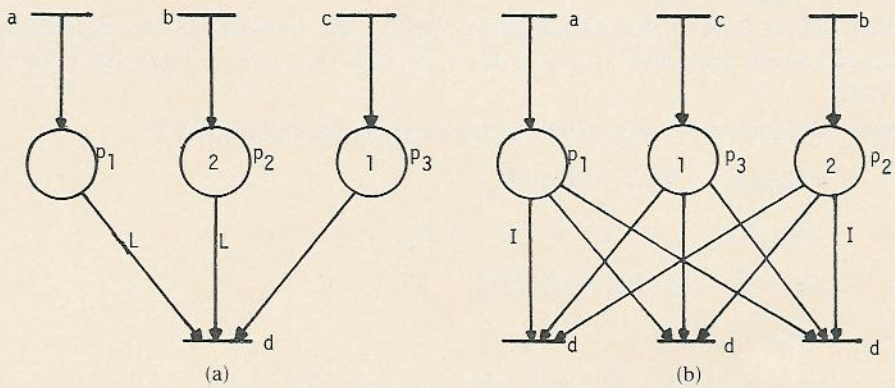


Fig. 19. (a) Example of a labeled λ-free L-Net Γ_{21}; (b) an equivalent labeled λ-free I-Net Γ_{22}, i.e. $L(\Gamma_{22})=L(\Gamma_{21})$.

The theorems of Section 7 yield the hierarchy of labeled λ-free SUP-Nets illustrated in Fig. 20 where XL$\rightarrow$YL denotes XL$\subset$YL, and XL$\leftrightarrow$YL indicates that XL = YL.

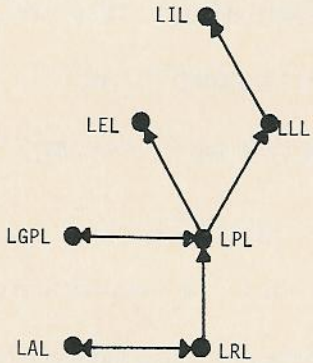


Fig. 20. Hierarchy of labeled λ-free SUP-Nets.

8. Hierarchy of labeled Super-Nets

In this section we study the hierarchy of the various types of labeled SUP-Nets (including λ-labels).

Theorem 8.1. (a) $L^{\lambda}RL \subset L^{\lambda}PL$.
(b) $L^{\lambda}PL = L^{\lambda}GPL$.

Proof. The proofs are similar to those of Theorem 7.2 and Theorem 7.3, respectively. □

Theorem 8.2. $L^{\lambda}PL \subset L^{\lambda}EL$.

Proof. Clearly $L^{\lambda}PL \subseteq L^{\lambda}EL$ by definition. Consider the labeled E-Net Γ_8 of Fig. 8. Assume that $L(\Gamma_8) = L(\Gamma)$ where $\Gamma = (P, T, V, K, M, \Sigma, \eta)$ is a labeled P-Net. Let $|P| = n$. We consider the infinite sequence $\{A_i\}_{i=1}^{\infty}$ of words:

$$A_i = a^i b^{i-1} c a^{i+1} b^i c \cdots a^{i+n-1} b^{i+n-2} c \in L(\Gamma).$$

Now, we construct $2n$ infinite sequences of marking in Γ from the words in $\{A_i\}_{i=1}^{\infty}$:

$$\{M_i^j\}_{i=1}^{\infty}, \qquad j = 1, 2, \ldots, 2n,$$

$$M[a^i b^{i-1} c \cdots a^{i+j-2} b^{i+j-3} c a^{i+j-1} > M_i^{2j-1}, \quad j = 1, 2, \ldots, n,$$

$$M[a^i b^{i-1} c \cdots a^{i+j-1} b^{i+j-2} c > M_i^{2j}, \qquad j = 1, 2, \ldots, n.$$

Next, we extract from $\{M_i^1\}_{i=1}^{\infty}$ an infinite subsequence $\{M_{i_j}^1\}_{j=1}^{\infty}$ according to Lemma 7.1. From $\{M_{i_j}^2\}_{j=1}^{\infty}$ we extract another infinite subsequence according to Lemma 7.1 and so forth, until the $2n$th subsequence.

Now, we have an infinite, increasing subsequence $\{i_j\}_{j=1}^{\infty}$ such that

$$(\forall l, 1 \leq l \leq 2n)(\forall q, 1 \leq q \leq n)$$
$$[(\forall j, k; j<k)(M_{i_j}^{l}(q) < M_{i_k}^{l}(q)) \wedge (\forall j, k)(M_{i_j}^{l}(q) = M_{i_k}^{l}(q))].$$

Claim 1. $(\forall j, k; j<k)(\forall l, 1 \leq l \leq n)[M_{i_j}^{2l-1} < M_{i_k}^{2l-1}$.

Proof. Assume that for some $j<k$ and $1 \leq l \leq n$, $M_{i_j}^{2l-1} = M_{i_k}^{2l-1}$. Thus, by Lemma 7.2,

$$x = a^{i_j} b^{i_j-1} c \cdots a^{i_j+l-2} b^{i_j+l-3} c a^{i_j+l-1} b^{i_k+l-2} c \in L(\Gamma).$$

But, $x \notin L(\Gamma_8)$, since $i_j + l - 1 \leq i_k + l - 2$, contradicting our assumption. □

Claim 2. There exists an infinite increasing sequence $\{t_j\}_{j=1}^{\infty}$, with the properties of $\{i_j\}_{j=1}^{\infty}$ and furthermore

$$(\forall l_1, 1 \leq l_1 \leq n)(\forall q, 1 \leq q \leq n)$$
$$[(\forall j, k)(M_{t_j}^{2l_1-1}(q) = M_{t_k}^{2l_1-1}(q)) \vee (\forall l_2, 1 \leq l_2 \leq n)(\forall j, j>1)$$
$$(M_{t_j}^{2l_2-1}(q) - M_{t_{j-1}}^{2l_2-1}(q) \leq M_{t_{j+1}}^{2l_1-1}(q) - M_{t_j}^{2l_1-1}(q))].$$

Proof. We can construct such a sequence as a subsequence of $\{i_j\}_{j=1}^{\infty}$ in view of the above properties of $\{i_j\}_{j=1}^{\infty}$, as follows:

$$t_1 = i_1,$$

$$t_2 = i_2.$$

$$f_3 = 2 + \max_{\substack{1 \leq q \leq n \\ 1 \leq l \leq n}} (M_{t_2}^{2l-1}(q) - M_{t_1}^{2l-1}(q)),$$

$$t_3 = i_{f_3},$$

$$\vdots$$

$$f_j = f_{j-1} + \max_{\substack{1 \leq q \leq n \\ 1 \leq l \leq n}} (M_{t_{j-1}}^{2l-1}(q) - M_{t_{j-2}}^{2l-1}(q)),$$

$$t_j = i_{f_j}. \quad \square$$

Now, by Lemma 7.2, the following reachable markings can be obtained in Γ:

$$M[a^{t_n} > M_{t_n}^{1},$$
$$M_{t_n}^{1}[b^{t_{n-1}-1} c > \bar{M}_{n-1}^{1},$$
$$\bar{M}_{n-1}^{1} = M_{t_{n-1}}^{2} + M_{t_{n-1}}^{1} - M_{t_{n-1}}^{1} > M_{t_{n-1}}^{2} \geq M_{t_{n-2}}^{2}.$$

Let

$$P_1 = \{p \mid M^1_{t_n}(p) - M^1 t_{n-1}(p) \neq 0\} \quad \text{and}$$

$$Q = \{p \mid M^3_{t_{n-1}}(p) - M^3_{t_{n-2}}(p) \neq 0\}.$$

Clearly, by Claim 1, $1 \leqslant |P_1| \leqslant n$.

We now distinguish between two cases:

Case I: $Q \subseteq P$. By Lemma 7.2, the following reachable markings can be obtained in Γ:

$$\bar{M}^1_{n-1}[a^{t_{n-2}+1} > \bar{M}^2_{n-2},$$

$$\bar{M}^2_{n-2} = M^3_{t_{n-2}} + \bar{M}^1_{n-1} - M^2_{t_{n-2}} = M^3_{t_{n-2}} + M^2_{t_{n-1}} + M^1_{t_n} - M^1_{t_{n-1}} - M^2_{t_{n-2}}$$

$$\geqslant M^3_{t_{n-2}} + M^1_{t_n} - M^1_{t_{n-1}}.$$

By Claim 2 and the fact that $Q \subseteq P_1$ we have

$$M^1_{t_n} - M^1_{t_{n-1}} \geqslant M^3_{t_{n-1}} - M^3_{t_{n-2}} \Rightarrow M^3_{t_{n-2}} + M^1_{t_n} - M^1_{t_{n-1}} \geqslant M^3_{t_{n-1}}.$$

Hence, $\bar{M}^2_{n-2} \geqslant M^3_{t_{n-1}}$, and $\bar{M}^2_{n-2}[b^{t_{n-1}}c > \bar{M}^3_{n-1}$. But $a^{t_n}b^{t_{n-1}-1}c\underbrace{a^{t_{n-2}+1}b^{t_{n-1}}}c \notin L(\Gamma_8)$, since $t_{n-2}+1 \leqslant t_{n-1}$, contradicting our assumption.

Case II: $Q \subseteq P_1$. By Lemma 7.2, the following reachable markings can be obtained in Γ:

$$M^1_{t_n}[b^{t_{n-1}-1}ca^{t_{n-1}+1} > \bar{M}^4_{n-1},$$

$$\bar{M}^4_{n-1} = M^3_{t_{n-1}} + M^1_{t_n} - M^1_{t_{n-1}} > M^3_{t_{n-1}},$$

$$\bar{M}^4_{n-1}[b^{t_{n-2}}c > \bar{M}^1_{n-2},$$

$$\bar{M}^1_{n-2} = M^4_{t_{n-2}} + M^3_{t_{n-1}} + M^1_{t_n} - M^1_{t_{n-1}} - M^3_{t_{n-2}} > M^4_{t_{n-2}} \geqslant M^4_{t_{n-3}}.$$

Let

$$P_2 = \{p \mid M^1_{t_n}(p) - M^1_{t_{n-1}}(p) + M^3_{t_{n-1}}(p) - M^3_{t_{n-2}}(p) \neq 0\}$$

and

$$Q = \{p \mid M^5_{t_{n-2}}(p) - M^5_{t_{n-3}}(p) \neq 0\}.$$

Clearly $P_2 \supset P_1$; hence, by Claim 1, $2 \leqslant |P_2| \leqslant n$. Again, one of the following cases holds: Case I: $Q \subseteq P_2$, Case II: $Q \subseteq P_2$. We handle both cases as before.

After n steps we have $n \leqslant |P_n| \leqslant n$. Therefore, every transition is enabled. Hence, we have $xcc \in L(\Gamma)$, where $x \in \Sigma^*$. But $xcc \notin L(\Gamma_8)$, contradicting our assumption that $L(\Gamma) = L(\Gamma_8)$. Thus, $\mathrm{L}^\lambda\mathrm{PL} \subset \mathrm{L}^\lambda\mathrm{EL}$. □

Theorem 8.3. $\mathrm{L}^\lambda\mathrm{LL} \subset \mathrm{L}^\lambda\mathrm{IL}$.

Proof. The proof is similar to the proof of Theorem 7.6. □

Theorem 8.4. $L^{\lambda}EL \subset L^{\lambda}IL$.

Proof. Consider the labeled E-Net $\Gamma = (P, T, V, K, M, \Sigma, \eta)$. We construct a marked labeled I-Net $\Gamma' = (P', T', V', K', M', \Sigma, \eta')$ where

$$P' = P \cup \{p_e\} \cup \{p_t \mid (t \in T) \wedge (\exists p \in P) V(p, t) = E\},$$

$$T' = T \cup \{\hat{t} \mid t \in T) \wedge (\forall p \in P) V(p, t) = E\}$$

$$\cup \{[t, p] \mid (t \in T) \wedge (p \in P) \wedge V(p, t) = E\}.$$

The function V' is defined as follows:

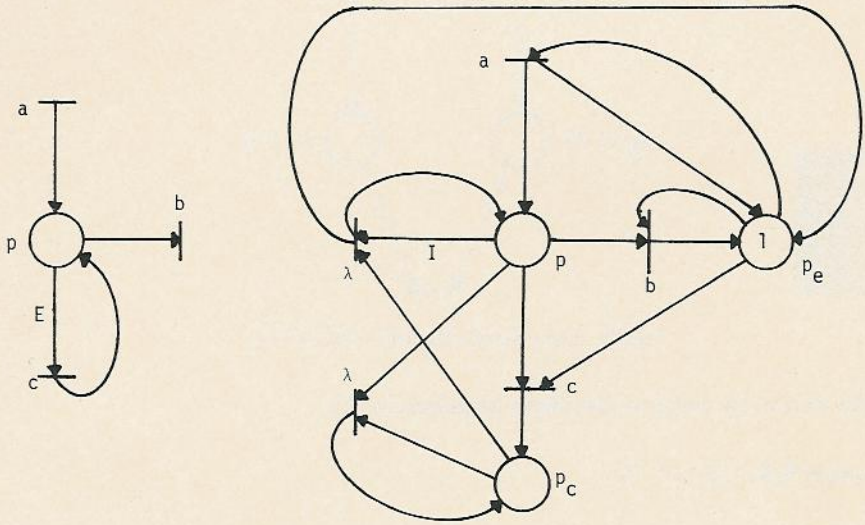
```
I  for all p ∈ P
   do
     for all t ∈ T
     do
       if V(p, t) ≠ 0 then V'(p, t) := 1;
       if ~(∃p' ∈ P) V(p', t) = E then V'(t, p) := V(t, p)
                                  else
                                  do
                                     V'(t̂, p) := V(t, p);
                                     V'(p_t, t̂) := 1;
                                     V'(t, p_t) := 1
                                  end
       if V(p, t) = E then
       do
          V'(p, t̂) := I;
          V'(p, [t, p]) := 1;
          V'(p_t, [t, p]) := 1;
          V'([t, p], p_t) := 1
       end
   end

II for all t ∈ T
   do
     V'(p_e, t) := 1;
     if ~(∃p' ∈ P) V(p', t) = E then V'(t, p_e) := 1
                                else V'(t̂, p_e) := 1,
   end
```

In all other cases $V'(x, y) := 0$. Fig. 21 illustrates this construction.

Clearly $L(\Gamma') = L(\Gamma)$. Hence, $L^{\lambda}EL \subseteq L^{\lambda}IL$. by Lemma 7.5, we have $L^{\lambda}EL \subset L^{\lambda}IL$. □

The theorems of Section 8 and Theorem 7.1 yield the hierarchy of labeled SUP-Nets illustrated in Fig. 22.



(a) (b)

Fig. 21. (a) Example of a labeled E-Net Γ_{23}; (b) an equivalent labeled I-Net Γ_{24}, i.e., $L(\Gamma_{24}) = L(\Gamma_{23})$.

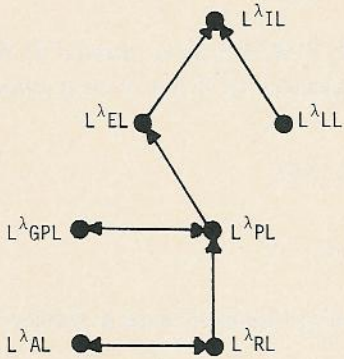


Fig. 22. Hierarchy of labeled SUP-Nets.

9. Further results on the hierarchy of Super-Nets

Theorem 9.1. AL $\subset$ LAL.

Proof. Clearly AL $\subseteq$ LAL.

Consider the labeled λ-free A-Net Γ_{25} of Fig. 23. We have $ab \notin L(\Gamma_{25})$, but $aab \in L(\Gamma_{25})$. Hence by Lemma 5.3, $L(\Gamma_{25}) \notin$ AL. It follows that AL $\subset$ LAL. $\square$

The following result has been obtained in [6].

Theorem 9.2. LPL $\subset$ L^{λ}PL.

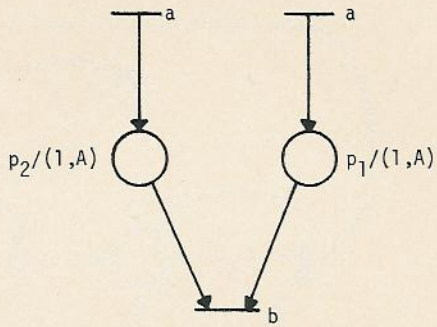


Fig. 23. Example of a labeled λ-free A-Net Γ_{25}.

The following theorem has been obtained in [5].

Theorem 9.3. $\mathrm{LIL} \subset \mathrm{L}^{\lambda}\mathrm{IL}$.

Definition 9.1. Let $\Gamma = (P, T, V, K, M, \Sigma, \eta)$ be a labeled SUP-Net. Assume $w = t_1 t_2 \cdots t_r \in T^+$ and $\eta(t_i) = \sigma \in \Sigma$. We say that i is a σ-index of w. $I(\sigma, w)$ will denote the set of all σ-indices of w.

Lemma 9.1. *Let $S = (P, T, V, K, M)$ be a marked* P-*Net or a marked* I-*Net. Let $x = t_1 t_2 \cdots t_r$ be a firing sequence of S. Let y be a permutation of x. If y is also a firing sequence of S, then*

$$M[x > M' \Leftrightarrow M[y > M'.$$

Proof. This result is evident. □

The proof of the following theorem uses a similar technique as the proof of Theorem 2 in [6].

Theorem 9.4. $\mathrm{LEL} \subset \mathrm{L}^{\lambda}\mathrm{EL}$.

Proof. Clearly $\mathrm{LEL} \subseteq \mathrm{L}^{\lambda}\mathrm{EL}$ by definition. Now, consider the labeled E-Net Γ_{26} of Fig. 24. It will be shown that $L(\Gamma_{26}) \notin \mathrm{LEL}$.

Assume that $L(\Gamma) = L(\Gamma_{26})$ where Γ is a labeled λ-free E-Net. Let $L(\Gamma') = L(\Gamma)$ where $\Gamma' = (P, T, V, K, M, \Sigma, \eta)$ is a labeled I-Net constructed as in Theorem 8.4, and $\Sigma = \{a, b, c\}$. We define the following sets:

$$A = \left\{ x \in (a+b)^* c^i \mid i = \sum_{j \in I(a,x)} 2^j \right\},$$

$$B = \left\{ x \in (a+b)^* c^i \mid i > \sum_{j \in I(a,x)} 2^j \right\}.$$

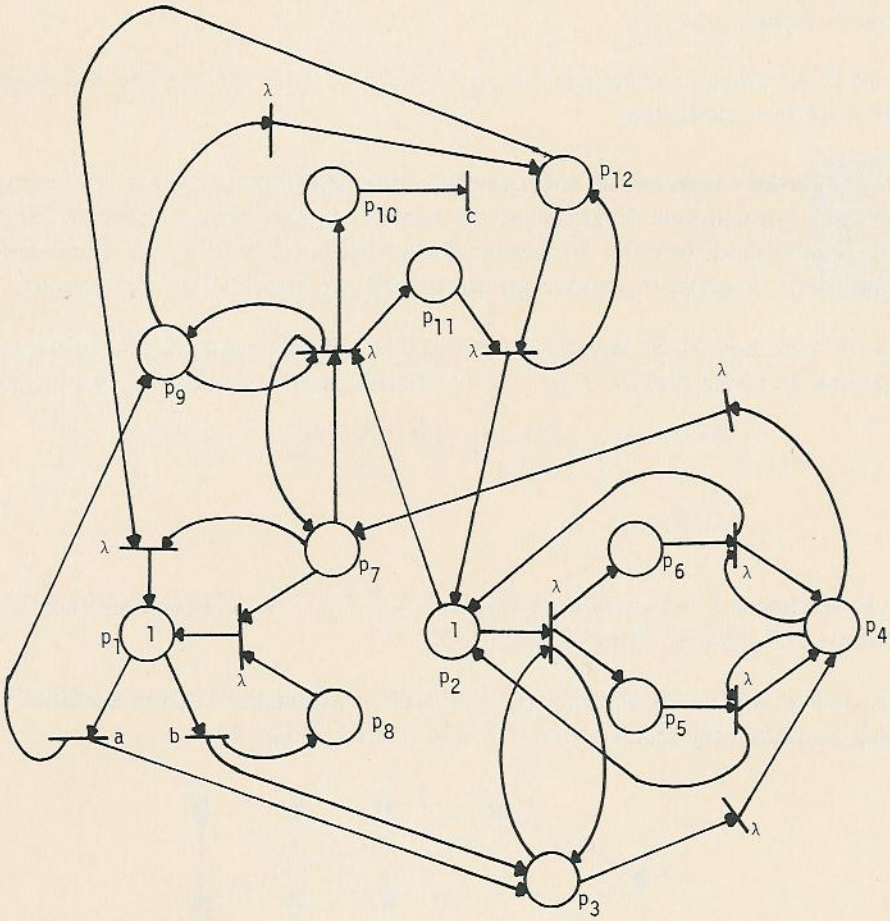


Fig. 24. Example of a labeled E-Net, Γ_{26}.

One easily verifies that $A \subseteq L(\Gamma_{26})$ and $B \cap L(\Gamma_{26}) = \emptyset$.

For any positive integer n, let

$$C_n = \{x \in (a+b)^* \mid \text{length of } x \text{ is } n\},$$

$$D_n = \{M' \mid M[x > M') \wedge (x \in C_n) \wedge (\exists i \in \omega)[(xc^i \in A) \wedge (\exists M'')(M'[c^i > M'')]\}.$$

Claim 1. $|D_n| \geqslant |C_n| = 2^n$.

Proof. Let $i < j$, $x_i c^i \in A$, $x_j c^j \in A$, where $x_i, x_j \in C_n$. Let $M[x_i > M_i$, $M[x_j > M_j$. If $M_i = M_j$, we have $M_i[c^j > M'$ for some M'. Hence $x_i c^j \in L(\Gamma')$. But this contradicts the condition $B \cap L(\Gamma_{26}) = \emptyset$. Therefore, $M_i \neq M_j$. Hence the number of markings in D_n equals to or is greater than the number of words in C_n, which is 2^n. □

Let $m = \max_{p \in P} M(p)$.

Claim 2. In order to generate a word of length n in Γ' we have to fire at most $m|P| + n|P| + n$ transitions.

Proof. In order to remove all tokens of the initial marking we have to fire no more than $m|P|$ λ-transitions. Every place can receive no more than n tokens while the word is generated. In order to remove these tokens we have to fire at most $n|P|$ λ-transitions. In order to remove the word itself, we have to fire n transitions. □

Let $n < \max(m, |P|, 3)$. Hence, $n^3 > n + m|P| + n|P|$. The number of combinations to choose k t's such that $t \in T$ is $\binom{|T|+k-1}{k}$. Therefore, by Lemma 9.1 and Claim 2,

$$|D_n| \leq \binom{|T|+n-1}{n} + \binom{|T|+n}{n+1} + \binom{|T|+n+1}{n+2} + \cdots + \binom{|T|+n^3-1}{n^3}$$

$$< n^3 \binom{|T|+n^3-1}{n^3}.$$

But if we choose n big enough we will have $[n^3\binom{|T|+n^3-1}{n^3} < 2^n]$ contradicting Claim 1. Hence $L(\Gamma_{26}) \notin$ LEL. Thus, L^λEL $\supset$ LEL. □

In view of the results of this section, as well as arguments used in Sections 5, 7, and 8, we obtain the hierarchy of SUP-Nets shown in Fig. 25.

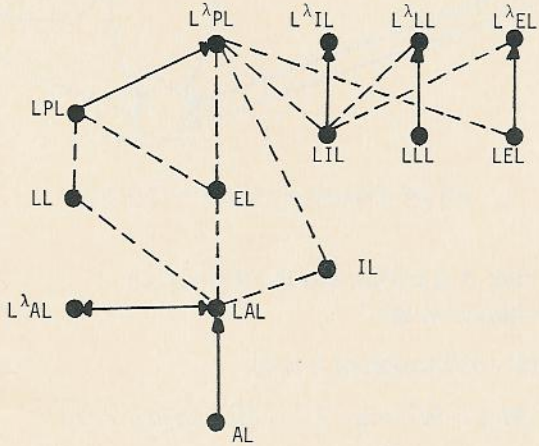


Fig. 25. Hierarchy of SUP-Nets.

The hierarchies derived in Sections 5, 7, 8, 9 are not complete. However, we list the following conjectures, by means of which the hierarchies could be completed.

Conjectures. (a) $L(\Gamma_8) \notin$ LIL.

(b) $L(\Gamma_8) \notin L^\lambda$LL.

(c) $L(\Gamma_{20}) \notin L^\lambda$EL.

10. LGPL and L^{λ}GPL are not closed under Kleene star (iteration)

The following theorem solves an open problem, discussed in [10, p. 186].

Theorem 10.1. LGPL *and* L^{λ}GPL *are not closed under Kleene star* (*iteration*).

Proof. Consider the marked GP-Net S_{27} of Fig. 26.

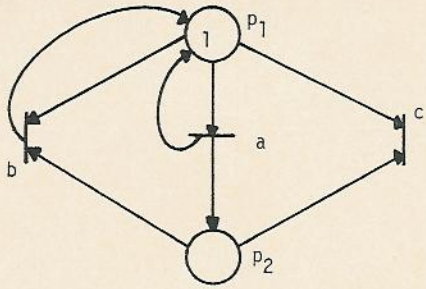


Fig. 26. Example of a marked GP-Net S_{27}.

Consider the marked E-Net S_8 of Fig. 8. One easily verifies that $L(S_8) = (L(S_{27}))^*$. In the proof of the Theorem 8.2 we showed that there is no labeled P-Net Γ such that $L(\Gamma) = L(S_8)$. Since L^{λ}PL = L^{λ}GPL by Theorem 8.1 it follows that there is no labeled GP-Net Γ' such that $L(\Gamma') = L(S_8) = (L(S_{27}))^*$. Hence we proved the theorem. □

References

[1] T. Agerwala and M. Flynn, Comments on capabilities, limitations, and 'correctness' of Petri nets, *Proc. 1st Annual Symposium on Computer Architecture* (ACM, New York, 1973) 81–86.

[2] W. Brauer, Ed., *Net Theory and Applications*, Lecture Notes in Computer Science **84** (Springer, Berlin, 1980).

[3] M. Courvoisier and R. Valette, *Systèmes de Commande en Temps Reel* (Editions SCM, Paris, 1980).

[4] E. Daclin and M. Blanchard, *Synthèse des Systèmes Logiques*, (Editions Cepadues, 1979).

[5] M. Hack, Petri net languages, Laboratory for Computer Science, M.I.T., Technical Report 159 (1976).

[6] M. Jantzen, On the hierarchy of Petri net languages, *R.A.I.R.O. Theoret. Informatique* **13** (1979) 19–30.

[7] M. Jantzen and R. Valk, Formal properties of place/transition nets, in: [2] 165–212.

[8] D. König, *Theorie der endlichen und unendlichen Graphen* (Leipzig 1936; reprinted by Chelsea, 1950).

[9] S. Kosaraju, Limitations of Dijkstra's semaphore primitives and Petri nets, *Operating Systems Rev.* **7** (1973) 122–126.

[10] J.L. Peterson, *Petri Net Theory and the Modeling of Systems* (Prentice-Hall, Englewood Cliffs, NJ, 1981).

[11] S. Porat and M. Yoeli, Towards a hierarchy of nets, Department of Computer Science, Technion, Haifa, Technical Report #224 (1981).

[12] P. Starke, Free Petri net languages, *Proc. 7th Symposium on Mathematical Foundation of Computer Science 1978*, Lecture Notes in Computer Science, **64** (Springer, Berlin, 1978) 506–515.
[13] S. Wendt, Petri-Netze und asynchrone Schaltwerke, *Elektron. Rechenanlagen* **16** (1974) 208–216.
[14] M. Yoeli, Petri nets and asynchronous control networks, Research Report CS-73-07, Department of Computer Science, University of Waterloo (1973).
[15] M. Yoeli and Z. Barzilai, Behavioural descriptions of communication switching systems using extended Petri nets, *Digital Processes* **3** (1977) 307–320.
[16] M. Yeoli and A. Ginzburg, Petri net languages and their applications, Research Report CS-78-45, Department of Computer Science, University of Waterloo (1978).